Proceedings of the Edinburgh Mathematical Society (2011) **54**, 249–271
DOI:10.1017/S0013091509000236 First published online 30 November 2010

SOLVABILITY CONDITIONS FOR SOME NON-FREDHOLM OPERATORS

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(Received 17 February 2009)

Abstract We obtain solvability conditions for some elliptic equations involving non-Fredholm operators with the methods of spectral theory and scattering theory for Schrödinger-type operators. One of the main results of the paper concerns solvability conditions for the equation $-\Delta u + V(x)u - au = f$ where $a \geqslant 0$. The conditions are formulated in terms of orthogonality of the function f to the solutions of the homogeneous adjoint equation.

Keywords: solvability conditions; non-Fredholm operators; elliptic problems

2010 *Mathematics subject classification:* Primary 35J10; 35P10; 35P25

1. Introduction

Linear elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if and only if the ellipticity condition, proper ellipticity and the Lopatinskii conditions are satisfied. The Fredholm property implies the solvability conditions: the non-homogeneous operator equation $Lu = f$ is solvable if and only if the right-hand side f is orthogonal to all solutions of the homogeneous adjoint problem $L^*v = 0$. Orthogonality is understood in the sense of duality in the corresponding spaces.

In the case of unbounded domains, one more condition should be imposed in order to preserve the Fredholm property. This condition can be formulated in terms of limiting operators and requires that all limiting operators be invertible or that the only bounded solution of limiting problems is trivial [**16**]. Limiting operators are the operators with limiting values of the coefficients at infinity, if such limiting values exist. Otherwise, limiting coefficients are determined by means of sequences of shifted coefficients and locally convergent subsequences.

If we consider, for example, the operator $Lu = -\Delta u - au$ in $\mathbb{R}^n$, where a is a positive constant, then its only limiting operator coincides with the operator L. Since the limiting

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equation $Lu = 0$ has a non-zero bounded solution, then the operator L, considered in Sobolev or Hölder spaces, does not satisfy the Fredholm property. Therefore, the solvability conditions are not applicable. However, the particular form of the equation $-\Delta u - au = f$ in $\mathbb{R}^n$ allows us to apply the Fourier transform and to find its solution. It can easily be verified that it has a solution $u \in L^2(\mathbb{R}^n)$ if and only if $\hat{f}(\xi)/(\xi^2 - a) \in L^2(\mathbb{R}^n)$, where the hat denotes the Fourier transform. In other words, the solvability conditions are given by the equality

$$\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi x} f(x)\,\mathrm{d}x = 0 \tag{1.1}$$

for any $\xi \in \mathbb{R}^n$ such that $|\xi|^2 = a$. This means that formally we obtain solvability conditions similar to those for Fredholm operators: the right-hand side is orthogonal to all solutions of the homogeneous formally adjoint problem.

It should be noted that the left-hand side of (1.1) is not a bounded functional over $L^2(\mathbb{R}^n)$. Therefore, these orthogonality conditions do not imply that the range of the operator is closed. Indeed, we can construct a sequence $f_n \in L^2(\mathbb{R}^n)$ such that it converges in $L^2(\mathbb{R}^n)$ to some f_0 and all functions f_n satisfy the solvability conditions, while f_0 does not satisfy them. In order to construct such a sequence, we consider the Fourier transforms $\hat{f}_n(\xi)$ and assume that they vanish at $|\xi|^2 = a$. These functions can converge in $L^2(\mathbb{R}^n)$ to a function that does not vanish on this sphere. Thus, the range of the operator is not closed and the similarity with Fredholm solvability conditions is only formal.

In this example, we are able to obtain solvability conditions due to the fact that the operator has constant coefficients and we can apply the Fourier transform. In general, the question about solvability conditions for non-Fredholm operators is open and represents one of the major challenges in the theory of elliptic problems. Some classes of reaction–diffusion operators without the Fredholm property can be studied by the introduction of weighted spaces [**16**] or by reducing them to integro-differential operators [**5**, **6**]. Solvability conditions different from the usual orthogonality conditions are obtained for some second-order operators on the real axis or in cylinders [**8**]. Some elliptic problems in $\mathbb{R}^2$ are studied in [**17**], where the solvability conditions are obtained with the help of space decomposition of the operators.

A special class of elliptic operators in $\mathbb{R}^n$, $A = A_\infty + A_0$, where A_∞ is a homogeneous operator with constant coefficients and A_0 is an operator with rapidly decaying coefficients, is studied in specially chosen spaces with a polynomial weight. The finiteness of the kernel is proved in [**12**, **18**] and the Fredholm property of this class of operators was proved in [**10**, **11**, **19**] in the case of weighted Sobolev spaces and in [**2**, **3**] for weighted Hölder spaces. The Fredholm property and the index of such operators are determined by their principal part A_∞. The operator A_0 does not change them due to the rapid decay of the coefficients. The Laplace operator in exterior domains is studied in [**1**].

In this work we consider two classes of non-Fredholm operators and establish the solvability conditions for the equations involving them. The methods cited above are not applicable here and we develop some new approaches. In the first case we study the operator H_a on $L^2(\mathbb{R}^3)$, such that

$$H_a u = -\Delta u + V(x)u - au,$$

where $a \geqslant 0$ is a parameter and the potential $V(x)$ decays to zero as $x \to \infty$. We investigate the conditions on the function $f \in L^2(\mathbb{R}^3)$ under which the equations

$$H_a u = f \tag{1.2}$$

and

$$H_0 u = f, \tag{1.3}$$

the second one being the limiting case of the first as $a \to 0$, have the unique solution in $L^2(\mathbb{R}^3)$. Since the potential equals zero at infinity, the operator H_a has a unique limiting operator $Lu = -\Delta u - au$, which is the same as that discussed above. The limiting problem $Lu = 0$ has non-zero bounded solutions. Therefore, the operator H_a, $a \geqslant 0$, does not satisfy the Fredholm property, and the solvability of (1.2) and (1.3) is not known. The coefficients of the operators are no longer constant and we cannot simply apply the Fourier transform as in the example above. We will use the spectral decomposition of self-adjoint operators.

We note that in the case where $a = 0$ and the potential is rapidly decaying at infinity, the operator H_0 belongs to the class of operators $A_\infty + A_0$ discussed above. The results of this work differ from the results in the cited papers. We do not work in the weighted spaces and we obtain solvability conditions without proving the Fredholm property, which may not hold. However, a more important difference is that we also consider the case $a > 0$. This is a significant difference and the previous methods are not applicable. To the best of our knowledge, solvability conditions for (1.2) with $a > 0$ and $n \geqslant 2$ have not previously been obtained. The solvability conditions are formulated in terms of the orthogonality of the right-hand side f to all solutions of the homogeneous adjoint equation $H_a v = 0$ (the operator is self-adjoint).

For a function $\psi(x)$ belonging to an $L^p(\mathbb{R}^d)$ space with $1 \leqslant p \leqslant \infty$, $d \in \mathbb{N}$, its norm is denoted as $\|\psi\|_{L^p(\mathbb{R}^d)}$. We will use technical tools for estimating the appropriate norms of the functions, in particular the Young's inequality

$$\|f_1 * f_2\|_{L^\infty(\mathbb{R}^3)} \leqslant \|f_1\|_{L^4(\mathbb{R}^3)} \|f_2\|_{L^{4/3}(\mathbb{R}^3)}, \quad f_1 \in L^4(\mathbb{R}^3),\ f_2 \in L^{4/3}(\mathbb{R}^3),$$

where '$*$' stands for the convolution, and the Hardy–Littlewood–Sobolev inequality

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x) f_1(y)}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y \right| \leqslant c_{\mathrm{HLS}} \|f_1\|^2_{L^{3/2}(\mathbb{R}^3)}, \quad f_1 \in L^{3/2}(\mathbb{R}^3),$$

with the constant c_{HLS} given on p. 98 of [**9**]. In our notation,

$$(f_1(x), f_2(x))_{L^2(\mathbb{R}^3)} := \int_{\mathbb{R}^3} f_1(x) \bar{f}_2(x) \, \mathrm{d}x$$

and, for some $A(x) = (A_1(x), A_2(x), A_3(x))$, the inner product $(f_1(x), A(x))_{L^2(\mathbb{R}^3)}$ is the vector with the coordinates

$$\int_{\mathbb{R}^3} f_1(x) \bar{A}_i(x) \, \mathrm{d}x, \quad i = 1, 2, 3.$$

Note that with slight abuse the same notation can be used even if the functions above are not square integrable, as in the case of the so-called perturbed plane waves $\varphi_k(x)$, which are normalized to a delta function (see (2.1)). We make the following technical assumption.

Assumption 1.1. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the bound*

$$|V(x)| \leqslant \frac{C}{1+|x|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ and $x \in \mathbb{R}^3$ a.e. such that

$$4^{1/9}\tfrac{9}{8}(4\pi)^{-2/3}\|V\|_{L^\infty(\mathbb{R}^3)}^{1/9}\|V\|_{L^{4/3}(\mathbb{R}^3)}^{8/9} < 1 \quad \text{and} \quad \sqrt{c_{\mathrm{HLS}}}\|V\|_{L^{3/2}(\mathbb{R}^3)} < 4\pi.$$

The function $f(x) \in L^2(\mathbb{R}^3)$ and $|x|f(x) \in L^1(\mathbb{R}^3)$.

Here and throughout C stands for a finite positive constant. Since, under our assumptions on the potential, the essential spectrum $\sigma_{\mathrm{ess}}(H_a)$ of the Schrödinger-type operator $H_a = H_0 - a$ fills the interval $[-a, \infty)$, the Fredholm Alternative Theorem fails to work in this case. The problem can easily be handled by the method of the Fourier transform in the absence of the potential term $V(x)$. We show that this method can be generalized in the presence of a shallow, short-range $V(x)$ by means of replacing the Fourier harmonics by the functions $\varphi_k(x)$, $k \in \mathbb{R}^3$, of the continuous spectrum of the operator H_0, which are the solutions of the Lippmann–Schwinger Equation (see (2.1) and the explicit formula (2.2)). Note that the condition $|x|f(x) \in L^1(\mathbb{R}^3)$ of Assumption 1.1 is used here to show the regularity of the gradient of the generalized Fourier transform with respect to $\varphi_k(x)$, $k \in \mathbb{R}^3$ (see Lemma 2.4). This is similar to the standard Fourier transform, where this condition stipulates that its derivative belongs to L^∞.

While the wave vector k attains all the possible values in $\mathbb{R}^3$, the function $\varphi_0(x)$ corresponds to $k = 0$ in the formulae (2.1) and (2.2). The sphere of radius r in $\mathbb{R}^d$, $d \in \mathbb{N}$, centred at the origin is denoted S_r^d, the unit sphere is denoted S^d and $|S^d|$ stands for its Lebesgue measure. Our first main result is as follows.

Theorem 1.2. *Let Assumption 1.1 hold. Then*

(a) *the problem (1.2) admits a unique solution $u \in L^2(\mathbb{R}^3)$ if and only if*

$$(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0 \quad \text{for } k \in S^3_{\sqrt{a}} \text{ a.e.};$$

(b) *the problem (1.3) has a unique solution $u \in L^2(\mathbb{R}^3)$ if and only if*

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0.$$

As in the case of the example with zero potential that was considered at the start of the introduction, solvability conditions are formulated here in the form of orthogonality to the solutions of the homogeneous adjoint equation, which is similar to the usual Fredholm solvability conditions. As above, we stress that this similarity is only formal because the operator does not satisfy the Fredholm property and its range is not closed.

In the second part of the paper we consider the operator $\mathcal{L} = -\Delta_x - \Delta_y + \mathcal{V}(y)$ on $L^2(\mathbb{R}^{n+m})$ with the Laplacian operators Δ_x and Δ_y in $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$ and prove the necessary and sufficient conditions for the solvability in $L^2(\mathbb{R}^{n+m})$ of the inhomogeneous problem

$$\mathcal{L}u = g(x, y), \tag{1.4}$$

where $g(x, y) \in L^2(\mathbb{R}^{n+m})$. We assume the following.

Assumption 1.3. *The function $\mathcal{V}(y) : \mathbb{R}^m \to \mathbb{R}$ is continuous and $\lim_{y\to\infty} \mathcal{V}(y) = \mathcal{V}_+ > 0$.*

Thus for the operator $h := -\Delta_y + \mathcal{V}(y)$ the essential spectrum $\sigma_{\text{ess}}(h) = [\mathcal{V}_+, \infty)$. Let us denote the eigenvalues of the operator h located below $\mathcal{V}_+$ as $e_j, e_j < e_{j+1}$, $j \geqslant 1$, and the corresponding elements of the orthonormal set of eigenfunctions as φ_j^k, such that $h\varphi_j^k = e_j\varphi_j^k$, $1 \leqslant k \leqslant m_j$, $(\varphi_i^k, \varphi_j^l)_{L^2(\mathbb{R}^m)} = \delta_{i,j}\delta_{k,l}$, where m_j stands for the eigenvalue multiplicity, which is finite since the essential spectrum starts only at $\mathcal{V}_+$, and $\delta_{i,j}$ stands for the Kronecker symbol. We make the following key assumption on the discrete spectrum of the operator h relevant to the problem (1.4).

Assumption 1.4. *The eigenvalues $e_j < 0$ for all $1 \leqslant j \leqslant N-1$ and $e_N = 0$.*

Thus, under our assumptions the operator $\mathcal{L}$ is not Fredholm. Zero is the lower limit of the essential spectrum of the operator $-\Delta_x$ and h has the square-integrable zero modes. Moreover, the operator h has the negative eigenvalues e_j, $j = 1, \dots, N-1$, and $-\Delta_x$ has the Fourier harmonics $\mathrm{e}^{\mathrm{i}px}/(2\pi)^{n/2}$, such that $p \in S^n_{\sqrt{-e_j}}$. However, (1.4) can be solved on the proper subspace and the orthogonality conditions will strongly depend on the dimensions of the problem.

Let us introduce the following subspace weighted in the first variable for the right-hand side of (1.4):

$$L^2_{\alpha,x} = \{g(x, y) : g(x, y) \in L^2(\mathbb{R}^{n+m}) \text{ and } |x|^{\alpha/2} g(x, y) \in L^2(\mathbb{R}^{n+m})\}, \quad \alpha > 0. \tag{1.5}$$

Our second main result is as follows.

Theorem 1.5. *Let Assumptions 1.3 and 1.4 hold. Then, for (1.4), we have the following.*

(I) *When $n = 1$ and $g(x, y) \in L^2_{\alpha,x}$ for some $\alpha > 5$ there exists a unique solution $u \in L^2(\mathbb{R}^{1+m})$ if and only if*

$$\left.\begin{aligned} (g(x, y), \varphi_N^k(y))_{L^2(\mathbb{R}^{1+m})} &= 0, \\ (g(x, y), x\varphi_N^k(y))_{L^2(\mathbb{R}^{1+m})} &= 0, \end{aligned}\right\} \quad 1 \leqslant k \leqslant m_N,$$

and

$$\left(g(x, y), \frac{\mathrm{e}^{\pm\mathrm{i}\sqrt{-e_j}x}}{\sqrt{2\pi}} \varphi_j^k(y)\right)_{L^2(\mathbb{R}^{1+m})} = 0, \quad 1 \leqslant j \leqslant N-1,\ 1 \leqslant k \leqslant m_j.$$

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(II) *When* $n = 2$ *such that* $x = (x_1, x_2) \in \mathbb{R}^2$ *and* $g(x, y) \in L^2_{\alpha,x}$ *for some* $\alpha > 6$ *there exists a unique solution* $u \in L^2(\mathbb{R}^{2+m})$ *if and only if*

$$\left.\begin{aligned}(g(x,y),\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})}&=0,\\(g(x,y),x_i\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})}&=0,\end{aligned}\right\}\quad i=1,2,\ 1\leqslant k\leqslant m_N,$$

and

$$\left(g(x,y),\frac{\mathrm{e}^{\mathrm{i}px}}{2\pi}\varphi_j^k(y)\right)_{L^2(\mathbb{R}^{2+m})}=0,\quad \text{a.e. } p\in S^2_{\sqrt{-e_j}},\ 1\leqslant j\leqslant N-1,\ 1\leqslant k\leqslant m_j.$$

(III) *When* $n = 3, 4$ *and* $g(x, y) \in L^2_{\alpha,x}$ *for some* $\alpha > n+2$ *there exists a unique solution* $u \in L^2(\mathbb{R}^{n+m})$ *if and only if*

$$(g(x,y),\varphi_N^k(y))_{L^2(\mathbb{R}^{n+m})}=0,\quad 1\leqslant k\leqslant m_N,$$

and

$$\left(g(x,y),\frac{\mathrm{e}^{\mathrm{i}px}}{(2\pi)^{n/2}}\varphi_j^k(y)\right)_{L^2(\mathbb{R}^{n+m})}=0,$$
$$\text{a.e. } p\in S^n_{\sqrt{-e_j}},\ 1\leqslant j\leqslant N-1,\ 1\leqslant k\leqslant m_j.$$

(IV) *When* $n \geqslant 5$ *and* $g(x, y) \in L^2_{\alpha,x}$ *for some* $\alpha > n + 2$ *there exists a unique solution* $u \in L^2(\mathbb{R}^{n+m})$ *if and only if*

$$\left(g(x,y),\frac{\mathrm{e}^{\mathrm{i}px}}{(2\pi)^{n/2}}\varphi_j^k(y)\right)_{L^2(\mathbb{R}^{n+m})}=0,$$
$$\text{a.e. } p\in S^n_{\sqrt{-e_j}},\ 1\leqslant j\leqslant N-1,\ 1\leqslant k\leqslant m_j.$$

Proving solvability conditions for linear elliptic problems with non-Fredholm operators plays a crucial role in various applications, including those to travelling wave solutions of reaction–diffusion systems (see [**17**]). Let us first establish several important properties for the functions of the spectrum of the Schrödinder operator in the left-hand side of (1.2) and for the related quantities.

2. Spectral properties of the operator H_0 and the proof of Theorem 1.2

The functions of the continuous spectrum satisfy the Lippmann–Schwinger Equation (see, for example, [**13**, p. 98])

$$\varphi_k(x)=\frac{\mathrm{e}^{\mathrm{i}kx}}{(2\pi)^{3/2}}-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{\mathrm{i}|k|\,|x-y|}}{|x-y|}(V\varphi_k)(y)\,\mathrm{d}y \tag{2.1}$$

and the orthogonality relations $(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k - q)$, $k, q \in \mathbb{R}^3$. We define the integral operator

$$(Q\varphi)(x):=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\mathrm{e}^{\mathrm{i}|k|\,|x-y|}}{|x-y|}(V\varphi)(y)\,\mathrm{d}y,\quad \varphi\in L^\infty(\mathbb{R}^3).$$

Let us show that the norm of the operator $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$, denoted by $\|Q\|_\infty$, is small when the potential $V(x)$ satisfies our assumptions. We prove the following lemma.

Lemma 2.1. *Let Assumption 1.1 hold. Then* $\|Q\|_\infty < 1$.

Proof. Clearly,

$$\|Q\|_\infty \leqslant \sup_{x\in\mathbb{R}^3} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\,\mathrm{d}y.$$

The expression involved in the right-hand side of the inequality above can be written as

$$\frac{1}{4\pi}\left(\chi_{\{|x|\leqslant R\}}\frac{1}{|x|}\right) * |V(x)| + \frac{1}{4\pi}\left(\chi_{\{|x|>R\}}\frac{1}{|x|}\right) * |V(x)|$$

with some $R > 0$ and χ denoting the characteristic function of the correspondent set. This can be estimated above using the Young inequality as

$$\frac{1}{4\pi}\|V\|_{L^\infty(\mathbb{R}^3)}\int_0^R 4\pi r\,\mathrm{d}r + \frac{1}{4\pi}\left\|\chi_{\{|x|>R\}}\frac{1}{|x|}\right\|_{L^4(\mathbb{R}^3)}\|V\|_{L^{4/3}(\mathbb{R}^3)}$$
$$= \tfrac{1}{2}\|V\|_{L^\infty(\mathbb{R}^3)}R^2 + \frac{1}{(4\pi)^{3/4}}\|V\|_{L^{4/3}(\mathbb{R}^3)}R^{-1/4}.$$

We optimize the right-hand side of the equality above over R. The minimum occurs when

$$R = \left\{\frac{\|V\|_{L^\infty(\mathbb{R}^3)}(4\pi)^{3/4}4}{\|V\|_{L^{4/3}(\mathbb{R}^3)}}\right\}^{-4/9},$$

such that

$$\|Q\|_\infty \leqslant 4^{1/9}\tfrac{9}{8}(4\pi)^{-2/3}\|V\|^{1/9}_{L^\infty(\mathbb{R}^3)}\|V\|^{8/9}_{L^{4/3}(\mathbb{R}^3)},$$

which is k-independent. Assumption 1.1 yields the statement of the lemma. Note that $V \in L^{4/3}(\mathbb{R}^3)$, which is guaranteed by its rate of decay, which is given explicitly in Assumption 1.1. □

Corollary 2.2. *Let Assumption 1.1 hold. The functions of the continuous spectrum of the operator* H_0 *are then* $\varphi_k(x) \in L^\infty(\mathbb{R}^3)$ *for all* $k \in \mathbb{R}^3$*, such that*

$$\|\varphi_k(x)\|_{L^\infty(\mathbb{R}^3)} \leqslant \frac{1}{1-\|Q\|_\infty}\frac{1}{(2\pi)^{3/2}}, \quad k \in \mathbb{R}^3.$$

Proof. By means of the Lippmann–Schwinger Equation (2.1) and the fact that $\|Q\|_\infty < 1$, the functions can be expressed as

$$\varphi_k(x) = (I-Q)^{-1}\frac{\mathrm{e}^{\mathrm{i}kx}}{(2\pi)^{3/2}}, \quad k \in \mathbb{R}^3. \tag{2.2}$$

Lemma 2.1 yields the bound on the operator norm

$$\|(I-Q)^{-1}\|_\infty \leqslant \frac{1}{1-\|Q\|_\infty}.$$

□

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The following elementary lemma shows that in our problem the operator H_0 possesses the spectrum analogous to the one of the minus Laplacian and therefore only the functions $\varphi_k(x)$, $k \in \mathbb{R}^3$, need to be taken into consideration.

Lemma 2.3. *Let Assumption 1.1 be true. Then the operator* H_0 *is unitarily equivalent to* $-\Delta$ *on* $L^2(\mathbb{R}^3)$.

Proof. By means of the Hardy–Littlewood–Sobolev inequality (see, for example, [**9**, p. 98]) and Assumption 1.1 we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)|\,|V(y)|}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \leqslant c_{\mathrm{HLS}}\|V\|^2_{L^{3/2}(\mathbb{R}^3)} < (4\pi)^2.$$

The left-hand side of the inequality above is usually referred to as the Rollnik norm (see, for example, [**15**]) and the upper bound we obtained on it is the sufficient condition for the operator $H_0 = -\Delta + V(x)$ on $L^2(\mathbb{R}^3)$ to be self-adjoint and unitarily equivalent to $-\Delta$ via the wave operators (see, for example, [**7**, **14**]) given by

$$\Omega^{\pm} := s - \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}t(-\Delta+V)}\mathrm{e}^{\mathrm{i}t\Delta},$$

where the limit is understood in the strong L^2 sense (see, for example, [**13**, p. 34] and [**4**, p. 90]). □

By means of the spectral theorem for the self-adjoint operator H_0, any function $\psi(x) \in L^2(\mathbb{R}^3)$ can be expanded through the functions $\varphi_k(x)$, $k \in \mathbb{R}^3$, forming the complete system in $L^2(\mathbb{R}^3)$. The generalized Fourier transform with respect to these functions is denoted by

$$\tilde{\psi}(k) := (\psi(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3. \tag{2.3}$$

We prove the following technical estimate concerning the above-mentioned generalized Fourier transform for the right-hand sides of (1.2) and (1.3).

Lemma 2.4. *Let Assumption 1.1 hold. Then*

$$\nabla_k \tilde{f}(k) \in L^\infty(\mathbb{R}^3).$$

Proof. Obviously, $\nabla_k \tilde{f}(k) = (f(x), \nabla_k\varphi_k(x))_{L^2(\mathbb{R}^3)}$. From the Lippmann–Schwinger Equation (2.1) we easily obtain

$$\nabla_k\varphi_k = \frac{\mathrm{e}^{\mathrm{i}kx}}{(2\pi)^{3/2}}\mathrm{i}x + (I-Q)^{-1}Q\frac{\mathrm{e}^{\mathrm{i}kx}}{(2\pi)^{3/2}}\mathrm{i}x + (I-Q)^{-1}(\nabla_kQ)(I-Q)^{-1}\frac{\mathrm{e}^{\mathrm{i}kx}}{(2\pi)^{3/2}}, \tag{2.4}$$

where $\nabla_kQ : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3;\mathbb{C}^3)$ stands for the operator with the integral kernel

$$\nabla_kQ(x,y,k) = -\frac{\mathrm{i}}{4\pi}\mathrm{e}^{\mathrm{i}|k|\,|x-y|}\frac{k}{|k|}V(y).$$

An elementary computation shows that its norm

$$\|\nabla_kQ\|_\infty \leqslant \frac{1}{4\pi}\|V\|_{L^1(\mathbb{R}^3)} < \infty$$

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due to the rate of decay of the potential $V(x)$, which is given explicitly by Assumption 1.1. It is clear from the identity (2.4) that we need to demonstrate the boundedness of the three terms in the k-space. The first one is

$$T_1(k) := \left(f(x), \frac{\mathrm{e}^{\mathrm{i}kx}}{(2\pi)^{3/2}}\mathrm{i}x\right)_{L^2(\mathbb{R}^3)},$$

such that

$$|T_1(k)| \leqslant \frac{1}{(2\pi)^{3/3}}\|xf\|_{L^1(\mathbb{R}^3)} < +\infty$$

by Assumption 1.1. The second term to be estimated is

$$T_2(k) := \left(f(x), (I-Q)^{-1}Q\frac{\mathrm{e}^{\mathrm{i}kx}}{(2\pi)^{3/2}}\mathrm{i}x\right)_{L^2(\mathbb{R}^3)}.$$

Thus

$$|T_2(k)| \leqslant \frac{1}{(2\pi)^{3/3}}\|f\|_{L^1(\mathbb{R}^3)}\frac{1}{1-\|Q\|_\infty}\|Q\mathrm{e}^{\mathrm{i}kx}x\|_{L^\infty(\mathbb{R}^3)}.$$

Note that $f(x) \in L^1(\mathbb{R}^3)$ by means of Assumption 1.1 and Fact 1 in the appendix. Using the definition of the operator Q along with the Young inequality we have the upper bound

$$\begin{aligned}
|Q\mathrm{e}^{\mathrm{i}kx}x| &\leqslant \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|V(y)|\,|y|}{|x-y|}\,\mathrm{d}y \\
&= \frac{1}{4\pi}\left\{\left(\chi_{\{|x|\leqslant 1\}}\frac{1}{|x|}\right) * |V(x)|\,|x| + \left(\chi_{\{|x|>1\}}\frac{1}{|x|}\right) * |V(x)|\,|x|\right\} \\
&\leqslant \frac{1}{4\pi}\left\{\|V(y)y\|_{L^\infty(\mathbb{R}^3)}\int_0^1 4\pi r\,\mathrm{d}r + \left\|\chi_{\{|x|>1\}}\frac{1}{|x|}\right\|_{L^4(\mathbb{R}^3)}\|V(x)x\|_{L^{4/3}(\mathbb{R}^3)}\right\} \\
&< +\infty
\end{aligned}$$

independent of k, since $V(x)x \in L^\infty(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$ due to the explicit rate of decay of the potential $V(x)$ stated in Assumption 1.1. Therefore, $T_2(k) \in L^\infty(\mathbb{R}^3)$. We complete the proof of the lemma with the estimate of the remaining term

$$T_3(k) := \left(f(x), (I-Q)^{-1}(\nabla_k Q)(I-Q)^{-1}\frac{\mathrm{e}^{\mathrm{i}kx}}{(2\pi)^{3/2}}\right)_{L^2(\mathbb{R}^3)},$$

such that we easily arrive at the k-independent upper bound

$$|T_3(k)| \leqslant \frac{1}{4\pi(2\pi)^{3/2}}\|f\|_{L^1(\mathbb{R}^3)}\frac{1}{(1-\|Q\|_\infty)^2}\|V\|_{L^1(\mathbb{R}^3)} < \infty.$$

□

Armed with the auxiliary lemmas established above, we proceed to prove the first theorem.

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Proof of Theorem 1.2. First of all, if (1.2) admits two solutions $u_1(x), u_2(x) \in L^2(\mathbb{R}^3)$, their difference $v(x) := u_1(x) - u_2(x)$ would satisfy the homogeneous problem $H_a v = 0$. Since the operator H_a possesses no non-trivial square-integrable zero modes, $v(x)$ will vanish a.e. The analogous argument holds for the solutions of (1.3).

From (1.2), by applying the transform (2.3), we obtain

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{k^2 - a}, \quad k \in \mathbb{R}^3.$$

We write this expression as a sum of the singular and non-singular parts:

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{k^2 - a}\chi_{A_\sigma} + \frac{\tilde{f}(k)}{k^2 - a}\chi_{A_\sigma^c}, \tag{2.5}$$

where χ_{A_σ} is the characteristic function of the spherical layer

$$A_\sigma := \{k \in \mathbb{R}^3 : \sqrt{a} - \sigma \leqslant |k| \leqslant \sqrt{a} + \sigma\}, \quad 0 < \sigma < \sqrt{a},$$

and $\chi_{A_\sigma^c}$ of the layer's complement in the three-dimensional k-space. For the second term on the right-hand side of the identity (2.5),

$$\left|\frac{\tilde{f}(k)}{k^2 - a}\chi_{A_\sigma^c}\right| \leqslant \frac{|\tilde{f}(k)|}{\sqrt{a}\sigma} \in L^2(\mathbb{R}^3).$$

To estimate the remaining term we will make use of the identity

$$\tilde{f}(k) = \tilde{f}(\sqrt{a}, \omega) + \int_{\sqrt{a}}^{|k|} \frac{\partial \tilde{f}(|s|, \omega)}{\partial |s|} \, \mathrm{d}|s|.$$

Here and below ω denotes the angle variable on the sphere. Thus we can split the first term on the right-hand side of (2.5) as $\tilde{u}_1(k) + \tilde{u}_2(k)$, where

$$\tilde{u}_1(k) = \frac{\int_{\sqrt{a}}^{|k|}(\partial \tilde{f}(|s|, \omega)/\partial |s|) \, \mathrm{d}|s|}{k^2 - a}\chi_{A_\sigma}, \quad \tilde{u}_2(k) = \frac{\tilde{f}(\sqrt{a}, \omega)}{k^2 - a}\chi_{A_\sigma}. \tag{2.6}$$

Clearly, we have the bound

$$|\tilde{u}_1(k)| \leqslant \frac{\|\nabla_k \tilde{f}(k)\|_{L^\infty(\mathbb{R}^3)}}{|k| + \sqrt{a}}\chi_{A_\sigma} \in L^2(\mathbb{R}^3)$$

by means of Lemma 2.4. We complete the proof of the part (a) of the theorem by estimating the norm

$$\|\tilde{u}_2(k)\|^2_{L^2(\mathbb{R}^3)} = \int_{\sqrt{a}-\sigma}^{\sqrt{a}+\sigma} \mathrm{d}|k| \, \frac{|k|^2}{(|k| - \sqrt{a})^2(|k| + \sqrt{a})^2} \int_{S^3} \mathrm{d}\omega \, |\tilde{f}(\sqrt{a}, \omega)|^2 < \infty$$

if and only if $(f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0$ for k a.e. on the sphere $S^3_{\sqrt{a}}$. We then turn our attention to (1.3) by applying to it the generalized Fourier transform with respect to the eigenfunctions of the continuous spectrum of the operator H_0, which yields

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{k^2} = \frac{\tilde{f}(k)}{k^2}\chi_{\{|k| \leqslant 1\}} + \frac{\tilde{f}(k)}{k^2}\chi_{\{|k| > 1\}}.$$

Clearly,

$$\left|\frac{\tilde{f}(k)}{k^2}\chi_{\{|k|>1\}}\right| \leqslant |\tilde{f}(k)| \in L^2(\mathbb{R}^3).$$

We use the formula

$$\tilde{f}(k) = \tilde{f}(0) + \int_0^{|k|} \frac{\partial \tilde{f}(|s|, \omega)}{\partial |s|}\,\mathrm{d}|s|$$

with $\tilde{f}(0) = (f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)}$ and $\varphi_0(x)$ is given by (2.2) with $k = 0$. Hence

$$\left|\frac{\int_0^{|k|}(\partial \tilde{f}(|s|, \omega)/\partial |s|)\,\mathrm{d}|s|}{k^2}\chi_{\{|k|\leqslant 1\}}\right| \leqslant \|\nabla_k \tilde{f}(k)\|_{L^\infty(\mathbb{R}^3)} \frac{\chi_{\{|k|\leqslant 1\}}}{|k|} \in L^2(\mathbb{R}^3)$$

via Lemma 2.4. Therefore, it remains to estimate the norm

$$\left\|\frac{\tilde{f}(0)}{k^2}\chi_{\{|k|\leqslant 1\}}\right\|^2_{L^2(\mathbb{R}^3)} = 4\pi \int_0^1 \mathrm{d}|k|\, \frac{|\tilde{f}(0)|^2}{|k|^2} < \infty$$

if and only if $(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0$, which completes the proof of the theorem. □

Note that if we let the potential function $V(x)$ in the statement of Theorem 1.2 vanish, we obtain precisely the usual orthogonality conditions in terms of the Fourier harmonics.

In the next section we prove Theorem 1.5. In contrast to Theorem 1.2, the dimensions of the problem are no longer fixed and we show how robust the dependence of the solvability conditions on these dimensions can be.

3. Spectral properties of the operator $\mathcal{L}$ and the proof of Theorem 1.5

Let $P_\pm$ and P_0 be the orthogonal projections onto the positive, negative and zero subspaces of the operator h. Applying these operators to both sides of (1.4) via the spectral theorem we relate the problem to the equivalent system of the following three equations:

$$\mathcal{L}_+ u_+ = g_+, \tag{3.1}$$

$$\mathcal{L}_- u_- = g_-, \tag{3.2}$$

$$\mathcal{L}_0 u_0 = g_0, \tag{3.3}$$

where the operators $\mathcal{L}_\pm = P_\pm \mathcal{L} P_\pm$ and $\mathcal{L}_0 = P_0 \mathcal{L} P_0$ act on the functions $u_\pm = P_\pm u$ and $u_0 = P_0 u$, respectively, and the right-hand sides of the equations are given by $g_\pm = P_\pm g$ and $g_0 = P_0 g$. Without loss of generality we can assume that

$$g_0(x, y) = v_0(x)\varphi_N^1(y), \tag{3.4}$$

where $v_0(x) = (g_0, \varphi_N^1)_{L^2(\mathbb{R}^m)} = (g, \varphi_N^1)_{L^2(\mathbb{R}^m)}$. Let us first turn our attention to (3.1). We have the following lemma.

Lemma 3.1. *Equation (3.1) possesses a solution $u_+ \in L^2(\mathbb{R}^{n+m})$, $n \in \mathbb{N}$, $m \in \mathbb{N}$.*

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Proof. By means of the orthogonal decomposition of the right-hand side of (1.4), $g = g_+ + g_0 + g_-$, we have the estimate

$$\|g_+\|_{L^2(\mathbb{R}^{n+m})} \leqslant \|g\|_{L^2(\mathbb{R}^{n+m})}.$$

The lower bound in the sense of the quadratic forms

$$\mathcal{L}_+ \geqslant P_+ h P_+ \geqslant e_{N+1} > 0,$$

where e_{N+1}, is either the bottom of the essential spectrum $\mathcal{V}_+$ of the operator h or its lowest positive eigenvalue, whichever is smaller. Thus $\mathcal{L}_+$ is the self-adjoint operator on the product of spaces $L^2(\mathbb{R}^n)$ and the range $\operatorname{Ran}(P_+)$ such that the bottom of its spectrum is located above zero. Therefore, it is invertible and the norm of the inverse $\mathcal{L}_+^{-1} : L^2(\mathbb{R}^n) \otimes \operatorname{Ran}(P_+) \to L^2(\mathbb{R}^{n+m})$ is bounded above by $1/e_{N+1}$. Thus (3.1) has the solution $u_+ = \mathcal{L}_+^{-1} g_+$ and its norm can be estimated as follows:

$$\|u_+\|_{L^2(\mathbb{R}^{n+m})} \leqslant \frac{1}{e_{N+1}} \|g\|_{L^2(\mathbb{R}^{n+m})} < \infty.$$

□

Let us turn our attention to the analysis of the solvability conditions for (3.3). This equation is equivalent to

$$(-\Delta_x) u_0 = g_0. \tag{3.5}$$

The solution of this Poisson equation can be expressed as

$$\hat{u}_0 = \frac{\hat{g}_0}{p^2} \chi_1 + \frac{\hat{g}_0}{p^2} \chi_{1^c}, \tag{3.6}$$

where χ_1 denotes the characteristic function of the unit ball in the Fourier space centred at the origin and χ_{1^c} denotes the characteristic function of its complement. Here and below the hat denotes a Fourier transform in the first variable, such that

$$\hat{\psi}(p) := \frac{1}{(2\pi)^{1/2} n} \int_{\mathbb{R}^n} \psi(x) \mathrm{e}^{-\mathrm{i}px} \, \mathrm{d}x.$$

The second term on the right-hand side of (3.6) is square integrable for all dimensions $n, m \in \mathbb{N}$ since $\hat{g}_0 \in L^2(\mathbb{R}^{n+m})$ and $1/p^2$ is bounded away from the origin. Thus it remains to analyse the first term. We have the following lemma when the dimension $n = 1$.

Lemma 3.2. *Let the assumptions of Theorem 1.5 hold. Equation (3.5) then possesses a solution $u_0 \in L^2(\mathbb{R}^{1+m})$, $m \in \mathbb{N}$, if and only if*

$$(g(x,y), \varphi_N^k(y))_{L^2(\mathbb{R}^{1+m})} = 0, \quad (g(x,y), \varphi_N^k(y) x)_{L^2(\mathbb{R}^{1+m})} = 0, \quad 1 \leqslant k \leqslant m_N.$$

Proof. We will make use of the following representation:

$$\hat{g}_0(p,y) = \hat{g}_0(0,y) + \frac{\partial}{\partial p} \hat{g}_0(0,y) p + \int_0^p \left(\int_0^s \frac{\partial^2}{\partial q^2} \hat{g}_0(q,y) \, \mathrm{d}q \right) \mathrm{d}s,$$

where

$$\frac{\partial^2}{\partial p^2}\hat{g}_0(p,y) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g_0(x,y)\mathrm{e}^{-\mathrm{i}px}x^2\,\mathrm{d}x.$$

The first term on the right-hand side of (3.6) in our case is therefore equal to

$$\frac{\hat{g}_0(0,y)}{p^2}\chi_1 + \frac{\partial}{\partial p}\hat{g}_0(0,y)\frac{\chi_1}{p} + \int_0^p \left(\int_0^s \frac{\partial^2}{\partial q^2}\hat{g}_0(q,y)\,\mathrm{d}q\right)\mathrm{d}s\frac{\chi_1}{p^2}. \tag{3.7}$$

Clearly, we have the upper bound

$$\left|\frac{\partial^2}{\partial q^2}\hat{g}_0(q,y)\right| \leqslant \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} |g_0(x,y)|x^2\,\mathrm{d}x.$$

By means of the Schwarz inequality and (3.4) we have the estimate

$$|g_0(x,y)| \leqslant \sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z}\,|\varphi_N^1(y)|, \quad x\in\mathbb{R}^n,\ y\in\mathbb{R}^m,\ n,m\geqslant 1, \tag{3.8}$$

which is valid in a space of arbitrary dimensions, and this yields

$$\left|\frac{\partial^2}{\partial q^2}\hat{g}_0(q,y)\right| \leqslant \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{d}x\,\frac{x^2}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m}|g(x,s)|^2\,\mathrm{d}s}\,|\varphi_N^1(y)|$$

with $\alpha > 5$ such that $g(x,y)\in L^2_{\alpha,x}$. The Schwarz inequality yields the upper bound

$$\frac{1}{\sqrt{2\pi}}\sqrt{\int_{-\infty}^{+\infty}\mathrm{d}x\,\frac{x^4}{1+|x|^\alpha}}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{1+m})} + \|\,|x|^{\alpha/2}g\|^2_{L^2(\mathbb{R}^{1+m})}}|\varphi_N^1(y)| = C|\varphi_N^1(y)|.$$

Therefore, for the last term in (3.7) we obtain

$$\left|\int_0^p\left(\int_0^s\frac{\partial^2}{\partial q^2}\hat{g}_0(q,y)\,\mathrm{d}q\right)\mathrm{d}s\frac{\chi_1}{p^2}\right| \leqslant \tfrac{1}{2}C|\varphi_N^1(y)|\chi_1 \in L^2(\mathbb{R}^{1+m}).$$

Because of the behaviour of the first two terms in the Fourier space, (3.7) belongs to $L^2(\mathbb{R}^{1+m})$ if and only if

$$\hat{g}_0(0,y) = 0, \quad \frac{\partial}{\partial p}\hat{g}_0(0,y) = 0 \quad \text{a.e.},$$

which is equivalent to

$$(g(x,y),\varphi_N^k(y))_{L^2(\mathbb{R}^{1+m})} = 0, \quad (g(x,y),\varphi_N^k(y)x)_{L^2(\mathbb{R}^{1+m})} = 0, \quad 1\leqslant k\leqslant m_N.$$

□

When the dimension $n = 2$ we come up with the following analogous statement.

Lemma 3.3. *Let the assumptions of Theorem 1.5 hold. Equation (3.5) then possesses a solution* $u_0 \in L^2(\mathbb{R}^{2+m})$, $m \in \mathbb{N}$, *if and only if*

$$(g(x,y), \varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})} = 0, \quad (g(x,y), \varphi_N^k(y)x_i)_{L^2(\mathbb{R}^{2+m})} = 0, \quad i = 1, 2, \ 1 \leqslant k \leqslant m_N.$$

Proof. Let us use an expansion analogous to the one we had for proving the previous lemma:

$$\hat{g}_0(p,y) = \hat{g}_0(0,y) + \frac{\partial}{\partial |p|}\hat{g}_0(0,\theta_p,y)|p| + \int_0^{|p|}\left(\int_0^s \frac{\partial^2}{\partial |q|^2}\hat{g}_0(|q|,\theta_p,y)\,\mathrm{d}|q|\right)\mathrm{d}s,$$

with

$$\hat{g}_0(|p|,\theta_p,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} g_0(x,y)\mathrm{e}^{-\mathrm{i}|p|\,|x|\cos\theta}\,\mathrm{d}x$$

and where the angle between the $p = (|p|,\theta_p)$ and $x = (|x|,\theta_x)$ vectors on the plane is $\theta = \theta_p - \theta_x$. Therefore, the first term on the right-hand side of (3.6) when $n = 2$ is equal to

$$\frac{\hat{g}_0(0,y)}{p^2}\chi_1 + \frac{\partial}{\partial |p|}\hat{g}_0(0,\theta_p,y)\frac{\chi_1}{|p|} + \int_0^{|p|}\left(\int_0^s \frac{\partial^2}{\partial |q|^2}\hat{g}_0(|q|,\theta_p,y)\,\mathrm{d}|q|\right)\mathrm{d}s\frac{\chi_1}{p^2}. \tag{3.9}$$

Obviously,

$$\left|\frac{\partial^2}{\partial |q|^2}\hat{g}_0\right| \leqslant \frac{1}{2\pi}\int_{\mathbb{R}^2}|g_0(x,y)|\,|x|^2\,\mathrm{d}x.$$

Using the estimate (3.8) we arrive at the upper bound for the right-hand side of this inequality:

$$\frac{1}{2\pi}|\varphi_N^1(y)|\int_{\mathbb{R}^2}\mathrm{d}x\frac{|x|^2}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z}$$

with $\alpha > 6$ such that $g(x,y) \in L^2_{\alpha,x}$. By means of the Schwarz inequality we estimate this from above as

$$\frac{1}{\sqrt{2\pi}}|\varphi_N^1(y)|\sqrt{\int_0^\infty \mathrm{d}|x|\,\frac{|x|^5}{1+|x|^\alpha}}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{2+m})} + \|\,|x|^{\alpha/2}g\|^2_{L^2(\mathbb{R}^{2+m})}} = C|\varphi_N^1(y)|.$$

Therefore, for the last term in (3.9) we arrive at

$$\frac{\chi_1}{p^2}\left|\int_0^{|p|}\left(\int_0^s \frac{\partial^2}{\partial |q|^2}\hat{g}_0(|q|,\theta_p,y)\,\mathrm{d}|q|\right)\mathrm{d}s\right| \leqslant \tfrac{1}{2}C\chi_1|\varphi_N^1(y)| \in L^2(\mathbb{R}^{2+m}).$$

A simple computation using the Fourier transform yields

$$\frac{\partial}{\partial |p|}\hat{g}_0(0,\theta_p,y) = -\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^2} g_0(x,y)|x|\cos\theta\,\mathrm{d}x = Q_1(y)\cos\theta_p + Q_2(y)\sin\theta_p,$$

where

$$Q_1(y) := -\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^2} g_0(x,y)x_1\,\mathrm{d}x, \qquad Q_2(y) := -\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^2} g_0(x,y)x_2\,\mathrm{d}x$$

https://doi.org/10.1017/S0013091509000236 Published online by Cambridge University Press

and $x = (x_1, x_2) \in \mathbb{R}^2$. Computing the square of the $L^2(\mathbb{R}^{2+m})$ norm of the first two terms of (3.9) we arrive at

$$2\pi \int_0^1 \frac{\mathrm{d}|p|}{|p|^3} \int_{\mathbb{R}^m} \mathrm{d}y\, |\hat{g}_0(0,y)|^2 + \pi \int_0^1 \frac{\mathrm{d}|p|}{|p|} \int_{\mathbb{R}^m} (|Q_1(y)|^2 + |Q_2(y)|^2)\, \mathrm{d}y,$$

which is finite if and only if the quantities $\hat{g}_0(0,y)$, $Q_1(y)$ and $Q_2(y)$ vanish a.e. This is equivalent to the orthogonality conditions

$$\begin{aligned}(g(x,y), \varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})} &= 0,\\ (g(x,y), x_1\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})} &= 0,\\ (g(x,y), x_2\varphi_N^k(y))_{L^2(\mathbb{R}^{2+m})} &= 0,\end{aligned}$$

with $1 \leqslant k \leqslant m_N$. □

Let us investigate how the situation with solvability conditions differs in dimensions $n = 3, 4$.

Lemma 3.4. *Let the assumptions of Theorem 1.5 hold. Equation (3.5) then possesses a solution $u_0 \in L^2(\mathbb{R}^{n+m})$, $n = 3, 4$, $m \in \mathbb{N}$, if and only if*

$$(g(x,y), \varphi_N^k(y))_{L^2(\mathbb{R}^{n+m})} = 0, \quad n = 3, 4,\ 1 \leqslant k \leqslant m_N.$$

Proof. Let us use the following equality:

$$\hat{g}_0(p,y) = \hat{g}_0(0,y) + \int_0^{|p|} \frac{\partial}{\partial |s|} \hat{g}_0(|s|, \omega, y)\, \mathrm{d}|s|.$$

Thus by means of (3.6) we need to estimate

$$\frac{\chi_1}{p^2}\biggl[\hat{g}_0(0,y) + \int_0^{|p|} \frac{\partial}{\partial |s|} \hat{g}_0(|s|, \omega, y)\, \mathrm{d}|s|\biggr]. \tag{3.10}$$

By means of the Fourier transform,

$$\frac{\partial}{\partial |p|} \hat{g}_0(|p|, \omega, y) = \frac{-\mathrm{i}}{(2\pi)^{1/2} n} \int_{\mathbb{R}^n} g_0(x,y) \mathrm{e}^{-\mathrm{i}|p|\,|x|\cos\theta} |x| \cos\theta\, \mathrm{d}x,$$

where θ is the angle between p and x in $\mathbb{R}^n$. Using (3.8) along with the Schwarz inequality and $\alpha > n + 2$ such that $g(x,y) \in L^2_{\alpha,x}$ we easily obtain

$$\begin{aligned}\left|\frac{\partial}{\partial |s|}\hat{g}_0\right| &\leqslant \frac{1}{(2\pi)^{1/2} n} \int_{\mathbb{R}^n} \mathrm{d}x\, |x| \sqrt{\int_{\mathbb{R}^m} |g(x,z)|^2\, \mathrm{d}z}\, |\varphi_N^1(y)|\\ &\leqslant \frac{1}{(2\pi)^{1/2} n} \sqrt{\int_0^\infty |S^n| \frac{|x|^{n+1}}{1 + |x|^\alpha}\, \mathrm{d}|x|} \sqrt{\|g\|^2_{L^2(\mathbb{R}^{n+m})} + \|\,|x|^{\alpha/2} g\|^2_{L^2(\mathbb{R}^{n+m})}}\, |\varphi_N^1(y)|\\ &= C|\varphi_N^1(y)|,\end{aligned}$$

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which implies the bound

$$\left|\frac{\chi_1}{p^2}\int_0^{|p|}\frac{\partial}{\partial|s|}\hat{g}_0(|s|,\omega,y)\,\mathrm{d}|s|\right| \leqslant C\frac{\chi_1}{|p|}|\varphi_N^1(y)| \in L^2(\mathbb{R}^{n+m}), \quad n=3,4.$$

We finalize the proof of the lemma by estimating the square of the L^2 norm of the first term in (3.10):

$$|S^n|\int_{\mathbb{R}^m}\mathrm{d}y\,|\hat{g}_0(0,y)|^2\int_0^1\mathrm{d}|p|\,|p|^{n-5} < \infty, \quad n=3,4,$$

if and only if $\hat{g}_0(0,y)=0$ a.e., which is equivalent to

$$(g(x,y),\varphi_N^k(y))_{L^2(\mathbb{R}^{n+m})}=0, \quad n=3,4,\ 1\leqslant k\leqslant m_N.$$

□

Thus it only remains to establish the orthogonality conditions in dimensions 5 and higher in the x variable under which (3.5) admits a square-integrable solution.

Lemma 3.5. *Let the assumptions of Theorem 1.5 hold. Equation (3.5) then possesses a solution* $u_0 \in L^2(\mathbb{R}^{n+m})$*,* $n \geqslant 5$*,* $m \in \mathbb{N}$*.*

Proof. We estimate the Fourier transform using the bound (3.8) along with the Schwarz inequality and $\alpha > n+2$ such that $g(x,y) \in L^2_{\alpha,x}$:

$$\begin{aligned}
|\hat{g}_0(p,y)| &\leqslant \frac{1}{(2\pi)^{1/2}n}\int_{\mathbb{R}^n}|g_0(x,y)|\,\mathrm{d}x \\
&\leqslant \frac{|\varphi_N^1(y)|}{(2\pi)^{1/2}n}\int_{\mathbb{R}^n}\mathrm{d}x\sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z} \\
&\leqslant \frac{|\varphi_N^1(y)|}{(2\pi)^{1/2}n}\sqrt{\int_{\mathbb{R}^n}\frac{\mathrm{d}x}{1+|x|^\alpha}}\sqrt{\int_{\mathbb{R}^n}\mathrm{d}x(1+|x|^\alpha)\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z} \\
&= \frac{|\varphi_N^1(y)|}{(2\pi)^{1/2}n}\sqrt{\int_0^\infty\mathrm{d}|x|\,\frac{|x|^{n-1}}{1+|x|^\alpha}|S^n|}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{n+m})}+\|\,|x|^{\alpha/2}g\|^2_{L^2(\mathbb{R}^{n+m})}} \\
&= C|\varphi_N^1(y)|, \quad n\geqslant 5,\ m\in\mathbb{N}.
\end{aligned}$$

This enables us to obtain the bound on the square of the L^2 norm of the first term on the right-hand side of (3.6):

$$\int_{\mathbb{R}^n}\mathrm{d}p\int_{\mathbb{R}^m}\mathrm{d}y\,\frac{|\hat{g}_0|^2}{|p|^4}\chi_1 \leqslant C\int_0^1\mathrm{d}|p|\,|p|^{n-5}|S^n|\int_{\mathbb{R}^m}|\varphi_N^1(y)|^2\,\mathrm{d}y < \infty,$$

which completes the proof of the lemma. □

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We proceed with establishing the conditions under which (3.2) admits a square-integrable solution. Let $\{P_j^-\}_{j=1}^{N-1}$ be the orthogonal projections onto the subspaces correspondent to $\{e_j\}_{j=1}^{N-1}$, the negative eigenvalues of the operator h, such that

$$P_- = \sum_{j=1}^{N-1} P_j^-, \quad P_j^- P_k^- = P_j^- \delta_{j,k}, \quad 1 \leqslant j, k \leqslant N-1.$$

Applying these projection operators to both sides of (3.2) and using the orthogonal decompositions

$$u_- = \sum_{j=1}^{N-1} u_j^- \quad \text{and} \quad g_- = \sum_{j=1}^{N-1} g_j^-$$

with $P_j^- u_- = u_j^-$ and $P_j^- g_- = g_j^-$, we easily obtain the system of equations equivalent to (3.2):

$$[-\Delta_x - \Delta_y + \mathcal{V}(y)]u_j^- = g_j^-, \quad 1 \leqslant j \leqslant N-1. \tag{3.11}$$

Without loss of generality we can assume that

$$g_j^-(x, y) = v_j(x)\varphi_j^1(y), \quad 1 \leqslant j \leqslant N-1, \tag{3.12}$$

where

$$v_j(x) := (g_j^-, \varphi_j^1)_{L^2(\mathbb{R}^m)} = (g, \varphi_j^1)_{L^2(\mathbb{R}^m)}.$$

Using the Schwarz inequality we find that

$$|v_j(x)| \leqslant \sqrt{\int_{\mathbb{R}^m} |g(x, z)|^2 \, \mathrm{d}z}, \quad x \in \mathbb{R}^n. \tag{3.13}$$

Hence the goal is to establish the conditions under which an equation such as (3.11) possesses a square-integrable solution. We make the Fourier transform in the x variable and, using the fact that the operator $-\Delta_x$ does not have positive eigenvalues on $L^2(\mathbb{R}^n)$, we obtain the expression for a solution of (3.11) as

$$\hat{u}_j^-(p, y) = \frac{\hat{v}_j(p)}{p^2 + e_j}\varphi_j^1(y), \quad 1 \leqslant j \leqslant N-1.$$

We differentiate between the two cases according to the dimension of the problem in the first variable.

Lemma 3.6. *Let the assumptions of Theorem 1.5 hold. Equation (3.11) then possesses a solution $u_j^-(x, y) \in L^2(\mathbb{R}^{1+m})$, $m \in \mathbb{N}$, if and only if*

$$\left(g(x, y), \frac{\mathrm{e}^{\pm \mathrm{i}\sqrt{-e_j}x}}{\sqrt{2\pi}}\varphi_j^k(y)\right)_{L^2(\mathbb{R}^{1+m})} = 0, \quad 1 \leqslant k \leqslant m_j, \ 1 \leqslant j \leqslant N-1.$$

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Proof. We express a solution of (3.11) as the sum of its regular and singular components,

$$\hat{u}_j^-(p,y) = \frac{\hat{v}_j(p)\chi_{\Omega_\delta^c}}{p^2+e_j}\varphi_j^1(y) + \frac{\hat{v}_j(p)\chi_{\Omega_\delta}}{p^2+e_j}\varphi_j^1(y). \tag{3.14}$$

Here Ω_δ is a set in the Fourier space

$$\Omega_\delta := [\sqrt{-e_j}-\delta, \sqrt{-e_j}+\delta] \cup [-\sqrt{-e_j}-\delta, -\sqrt{-e_j}+\delta] = \Omega_\delta^+ \cup \Omega_\delta^-$$

where $0<\delta<\sqrt{-e_j}$ and where Ω_δ^c is its complement, χ_{Ω_δ} and $\chi_{\Omega_\delta^c}$ are their characteristic functions. It is trivial to estimate the first term on the right-hand side of (3.14) since we are away from the positive and negative singularities $\pm\sqrt{-e_j}$. Thus

$$\left|\frac{\hat{v}_j(p)\chi_{\Omega_\delta^c}}{p^2+e_j}\varphi_j^1(y)\right| \leqslant C|\varphi_j^1(y)|\,|\hat{v}_j(p)|\chi_{\Omega_\delta^c},$$

which along with (3.13) enables us to estimate the square of its L^2 norm:

$$\int_{-\infty}^{+\infty} \mathrm{d}p \int_{\mathbb{R}^m} \mathrm{d}y\, |\varphi_j^1(y)|^2 |\hat{v}_j(p)|^2 \chi_{\Omega_\delta^c} \leqslant \|v_j\|^2_{L^2(\mathbb{R})} \leqslant \|g\|^2_{L^2(\mathbb{R}^{1+m})} < \infty.$$

To obtain the conditions under which the remaining term in (3.14) is square integrable we first study its behaviour near its negative singularity using the formula

$$\hat{v}_j(p) = \int_{-\sqrt{-e_j}}^{p} \frac{\mathrm{d}\hat{v}_j(s)}{\mathrm{d}s}\,\mathrm{d}s + \hat{v}_j(-\sqrt{-e_j}).$$

Thus one needs to estimate

$$\frac{\hat{v}_j(-\sqrt{-e_j}) + \int_{-\sqrt{-e_j}}^{p}(\mathrm{d}\hat{v}_j(s)/\mathrm{d}s)\,\mathrm{d}s}{p^2+e_j}\chi_{\Omega_\delta^-}\varphi_j^1(y). \tag{3.15}$$

We derive the upper bound for the derivative using (3.13) along with the Schwarz inequality with $\alpha>5$ such that $g(x,y)\in L^2_{\alpha,x}$:

$$\begin{aligned}
\left|\frac{\mathrm{d}\hat{v}_j(p)}{\mathrm{d}p}\right| &\leqslant \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x|x|\,|v_j(x)| \\
&\leqslant \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\frac{|x|}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z} \\
&\leqslant \frac{1}{\sqrt{2\pi}}\sqrt{\int_{-\infty}^{+\infty}\mathrm{d}x\,\frac{x^2}{1+|x|^\alpha}}\sqrt{\int_{-\infty}^{\infty}\mathrm{d}x(1+|x|^\alpha)\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z} \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{\int_{-\infty}^{+\infty}\mathrm{d}x\,\frac{x^2}{1+|x|^\alpha}}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{1+m})} + \|\,|x|^{\alpha/2}g\|^2_{L^2(\mathbb{R}^{1+m})}} \\
&= C < \infty.
\end{aligned}$$

This enables us to prove the square integrability for the second term in (3.15):

$$\left|\frac{\int_{-\sqrt{-e_j}}^{p}(\mathrm{d}\hat{v}_j(s)/\mathrm{d}s)\,\mathrm{d}s}{p^2+e_j}\chi_{\Omega_\delta^-}\varphi_j^1(y)\right| \leqslant \frac{C}{|p-\sqrt{-e_j}|}\chi_{\Omega_\delta^-}|\varphi_j^1(y)| \leqslant \frac{C}{2\sqrt{-e_j}-\delta}\chi_{\Omega_\delta^-}|\varphi_j^1(y)| \in L^2(\mathbb{R}^{1+m}).$$

Near the positive singularity we use the identity

$$\hat{v}_j(p) = \int_{\sqrt{-e_j}}^{p}\frac{\mathrm{d}\hat{v}_j(s)}{\mathrm{d}s}\,\mathrm{d}s + \hat{v}_j(\sqrt{-e_j})$$

to study the conditions of the square integrability of the term

$$\frac{\hat{v}_j(\sqrt{-e_j}) + \int_{\sqrt{-e_j}}^{p}(\mathrm{d}\hat{v}_j(s)/\mathrm{d}s)\,\mathrm{d}s}{p^2+e_j}\chi_{\Omega_\delta^+}\varphi_j^1(y). \tag{3.16}$$

As we did for the situation at the negative singularity, we prove the square integrability of the second term in (3.16) using the bound on the derivative involved in it. Hence

$$\left|\frac{\int_{\sqrt{-e_j}}^{p}(\mathrm{d}\hat{v}_j(s)/\mathrm{d}s)\,\mathrm{d}s}{p^2+e_j}\chi_{\Omega_\delta^+}\varphi_j^1(y)\right| \leqslant \frac{C}{|p+\sqrt{-e_j}|}\chi_{\Omega_\delta^+}|\varphi_j^1(y)| \leqslant \frac{C}{2\sqrt{-e_j}-\delta}\chi_{\Omega_\delta^+}|\varphi_j^1(y)| \in L^2(\mathbb{R}^{1+m}).$$

Thus it remains to derive the conditions under which the first term in (3.15) and the first term in (3.16) are square integrable. Estimating the square of the $L^2(\mathbb{R}^{1+m})$ norm of

$$\frac{\hat{v}_j(-\sqrt{-e_j})}{p^2+e_j}\chi_{\Omega_\delta^-}\varphi_j^1(y) + \frac{\hat{v}_j(\sqrt{-e_j})}{p^2+e_j}\chi_{\Omega_\delta^+}\varphi_j^1(y)$$

we easily arrive at

$$\int_{-\sqrt{-e_j}-\delta}^{-\sqrt{-e_j}+\delta}\mathrm{d}p\,\frac{|\hat{v}_j(-\sqrt{-e_j})|^2}{(p^2+e_j)^2} + \int_{\sqrt{-e_j}-\delta}^{\sqrt{-e_j}+\delta}\mathrm{d}p\,\frac{|\hat{v}_j(\sqrt{-e_j})|^2}{(p^2+e_j)^2},$$

which can be bounded below by

$$\frac{|\hat{v}_j(-\sqrt{-e_j})|^2}{(2\sqrt{-e_j}+\delta)^2}\int_{-\delta}^{\delta}\frac{\mathrm{d}s}{s^2} + \frac{|\hat{v}_j(\sqrt{-e_j})|^2}{(2\sqrt{-e_j}+\delta)^2}\int_{-\delta}^{\delta}\frac{\mathrm{d}s}{s^2}.$$

This bound implies that the necessary and sufficient conditions for the existence of $u_j^-(x,y) \in L^2(\mathbb{R}^{1+m})$ solving (3.11) are

$$\hat{v}_j(\sqrt{-e_j}) = 0, \qquad \hat{v}_j(-\sqrt{-e_j}) = 0,$$

which by means of the definition of the functions $v_j(x)$ is equivalent to

$$\left(g(x,y), \frac{\mathrm{e}^{\pm\mathrm{i}\sqrt{-e_j}x}}{\sqrt{2\pi}}\varphi_j^k(y)\right)_{L^2(\mathbb{R}^{1+m})} = 0, \quad 1 \leqslant k \leqslant m_j,\ 1 \leqslant j \leqslant N-1.$$

□

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After establishing the solvability conditions for (3.11) when the situation is one dimensional in the first variable we turn our attention to the cases of dimensions 2 and higher.

Lemma 3.7. *Let the assumptions of Theorem 1.5 hold. Equation (3.11) then possesses a solution* $u_j^-(x,y)\in L^2(\mathbb{R}^{n+m})$, $n\geqslant 2$, $m\in\mathbb{N}$, *if and only if*

$$\left(g(x,y),\frac{\mathrm{e}^{\mathrm{i}px}}{(2\pi)^{1/2}n}\varphi_j^k(y)\right)_{L^2(\mathbb{R}^{n+m})}=0,\quad \text{a.e. } p\in S^n_{\sqrt{-e_j}},\ 1\leqslant k\leqslant m_j,\ 1\leqslant j\leqslant N-1.$$

Proof. It is convenient to represent a solution of (3.11) as the sum of the singular and regular parts:

$$\hat{u}_j^-(p,y)=\frac{\hat{v}_j(p)\chi_{A_\delta}}{p^2+e_j}\varphi_j^1(y)+\frac{\hat{v}_j(p)\chi_{A_\delta^c}}{p^2+e_j}\varphi_j^1(y),\tag{3.17}$$

where the spherical layer in Fourier space $A_\delta:=\{p\in\mathbb{R}^n:\sqrt{-e_j}-\delta\leqslant|p|\leqslant\sqrt{-e_j}+\delta\}$, its complement in $\mathbb{R}^n$ is A_δ^c. The characteristic functions of these sets are χ_{A_δ} and $\chi_{A_\delta^c}$, respectively, and $0<\delta<\sqrt{-e_j}$. Clearly, for the second term on the right-hand side of (3.17) we have the upper bound

$$\left|\frac{\hat{v}_j(p)\chi_{A_\delta^c}\varphi_j^1(y)}{p^2+e_j}\right|\leqslant\frac{|\hat{v}_j(p)|\,|\varphi_j^1(y)|}{\delta\sqrt{-e_j}},$$

such that, via (3.13),

$$\int_{\mathbb{R}^n}|\hat{v}_j(p)|^2\,\mathrm{d}p\int_{\mathbb{R}^m}|\varphi_j^1(y)|^2\,\mathrm{d}y=\|v_j\|^2_{L^2(\mathbb{R}^n)}\leqslant\|g\|^2_{L^2(\mathbb{R}^{n+m})}<\infty.$$

Hence the first term on the right-hand side of (3.17) will play a crucial role in establishing the solvability conditions for (3.11). We will make use of the formula

$$\hat{v}_j(p)=\int_{\sqrt{-e_j}}^{|p|}\frac{\partial\hat{v}_j}{\partial|s|}(|s|,\omega)\,\mathrm{d}|s|+\hat{v}_j(\sqrt{-e_j},\omega)$$

to get the estimate for

$$\frac{\int_{\sqrt{-e_j}}^{|p|}(\partial\hat{v}_j/\partial|s|)(|s|,\omega)\,\mathrm{d}|s|+\hat{v}_j(\sqrt{-e_j},\omega)}{p^2+e_j}\chi_{A_\delta}\varphi_j^1(y).$$

Let us derive the upper bound for the derivative of the Fourier transform involved in it using (3.13) along with the Schwarz inequality, $\alpha>6$ for $n=2$ and $\alpha>n+2$ for $n\geqslant 3$ such that $g(x,y)\in L^2_{\alpha,x}$:

$$\begin{aligned}\left|\frac{\partial\hat{v}_j}{\partial|p|}\right|&\leqslant\frac{1}{(2\pi)^{1/2}n}\int_{\mathbb{R}^n}|v_j(x)|\,|x|\,\mathrm{d}x\\&\leqslant\frac{1}{(2\pi)^{1/2}n}\int_{\mathbb{R}^n}\mathrm{d}x\,|x|\sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z}\\&=\frac{1}{(2\pi)^{1/2}n}\int_{\mathbb{R}^n}\mathrm{d}x\,\frac{|x|}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z}\end{aligned}$$

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$$\leqslant \frac{1}{(2\pi)^{1/2}n}\sqrt{\int_{\mathbb{R}^n} \mathrm{d}x\, \frac{|x|^2}{1+|x|^\alpha}}\sqrt{\int_{\mathbb{R}^n} \mathrm{d}x\,(1+|x|^\alpha)\int_{\mathbb{R}^m}|g(x,z)|^2\,\mathrm{d}z}$$
$$= \frac{1}{(2\pi)^{1/2}n}\sqrt{\int_0^\infty \mathrm{d}|x|\,|S^n|\frac{|x|^{n+1}}{1+|x|^\alpha}}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{n+m})}+\|\,|x|^{\alpha/2}g\|^2_{L^2(\mathbb{R}^{n+m})}}$$
$$= C < \infty.$$

Therefore,

$$\left|\frac{\int_{\sqrt{-e_j}}^{|p|}(\partial\hat{v}_j/\partial|s|)(|s|,\omega)\,\mathrm{d}|s|}{p^2+e_j}\chi_{A_\delta}\varphi_j^1(y)\right| \leqslant \frac{C}{\sqrt{-e_j}}\chi_{A_\delta}|\varphi_j^1(y)| \in L^2(\mathbb{R}^{n+m})$$

and it remains to estimate from below the square of the L^2 norm of the term

$$\frac{\hat{v}_j(\sqrt{-e_j},\omega)}{p^2+e_j}\chi_{A_\delta}\varphi_j^1(y).$$

Thus

$$\int_{\mathbb{R}^n}\mathrm{d}p\int_{\mathbb{R}^m}\mathrm{d}y\,\frac{|\hat{v}_j(\sqrt{-e_j},\omega)|^2}{(p^2+e_j)^2}\chi_{A_\delta}|\varphi_j^1(y)|^2$$
$$\geqslant \int_{\sqrt{-e_j}-\delta}^{\sqrt{-e_j}+\delta}\frac{\mathrm{d}|p|\,|p|^{n-1}}{(|p|-\sqrt{-e_j})^2(2\sqrt{-e_j}+\delta)^2}\int_{S^n}\mathrm{d}\omega\,|\hat{v}_j(\sqrt{-e_j},\omega)|^2$$
$$\geqslant \frac{(\sqrt{-e_j}-\delta)^{n-1}}{(2\sqrt{-e_j}+\delta)^2}\int_{S^n}\mathrm{d}\omega\,|\hat{v}_j(\sqrt{-e_j},\omega)|^2\int_{-\delta}^{\delta}\frac{\mathrm{d}s}{s^2},$$

which yields the necessary and sufficient conditions of solvability of (3.11) in $L^2(\mathbb{R}^{n+m})$, $n \geqslant 2$: namely $\hat{v}_j(\sqrt{-e_j},\omega) = 0$ a.e. on the sphere $S^n_{\sqrt{-e_j}}$. Using the definition of the functions $v_j(x)$, we easily arrive at

$$\left(g(x,y),\frac{\mathrm{e}^{\mathrm{i}px}}{(2\pi)^{1/2}n}\varphi_j^k(y)\right)_{L^2(\mathbb{R}^{n+m})} = 0, \quad \text{a.e. } p \in S^n_{\sqrt{-e_j}},\ 1 \leqslant k \leqslant m_j,\ 1 \leqslant j \leqslant N-1.$$

□

Having established the orthogonality conditions in the lemmas above, which guarantee the existence of square-integrable solutions for our equations, we conclude the proof of Theorem 1.5.

Proof of Theorem 1.5. We construct the solution of (1.4) as

$$u := u_+ + u_0 + \sum_{j=1}^{N-1} u_j^-,$$

where the existence of $u_+ \in L^2(\mathbb{R}^{n+m})$ is guaranteed by Lemma 3.1, the existence of $u_0 \in L^2(\mathbb{R}^{n+m})$ is guaranteed by Lemmas 3.2–3.5, and the existence of $\{u_j^-\}_{j=1}^{N-1} \in L^2(\mathbb{R}^{n+m})$ is guaranteed by Lemmas 3.6 and 3.7.

Suppose that (1.4) admits two solutions $u_1, u_2 \in L^2(\mathbb{R}^{n+m})$. Their difference $w := u_1 - u_2 \in L^2(\mathbb{R}^{n+m})$ then solves the homogeneous problem with separation of variables

$$\mathcal{L}w = 0,$$

which admits two types of solution: the first ones are of the form $\gamma(x)\varphi_N^k(y)$, $1 \leqslant k \leqslant m_N$, with $\gamma(x)$ harmonic; the second ones are of the form

$$\frac{\mathrm{e}^{\mathrm{i}px}}{(2\pi)^{n/2}}\varphi_j^k(y)$$

with $p \in S^n_{\sqrt{-e_j}}$, $1 \leqslant j \leqslant N-1$, $1 \leqslant k \leqslant m_j$. In both cases, they belong to the space $L^2(\mathbb{R}^{n+m})$ only if they vanish. □

Appendix A

Fact 1. *Let $f(x) \in L^2(\mathbb{R}^3)$ and $|x|f(x) \in L^1(\mathbb{R}^3)$. Then $f(x) \in L^1(\mathbb{R}^3)$.*

Proof. The norm $\|f\|_{L^1(\mathbb{R}^3)}$ is estimated from above by means of the Schwarz inequality as

$$\sqrt{\int_{|x|\leqslant 1} |f(x)|^2\,\mathrm{d}x}\sqrt{\int_{|x|\leqslant 1} \mathrm{d}x} + \int_{|x|>1} |x|\,|f(x)|\,\mathrm{d}x \leqslant \|f\|_{L^2(\mathbb{R}^3)}\sqrt{\frac{4\pi}{3}} + \||x|f\|_{L^1(\mathbb{R}^3)} < \infty.$$

□

Acknowledgements. The first author thanks I. M. Sigal for partial support via NSERC Grant 7901.

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