On a Differential Equation and the Construction of Milner's Lamp.

By Professor CAYLEY.

What sort of an equation is

$$b^3\cos(\alpha+\theta)=a\cos\theta\int_\theta^\beta r^2 d\theta-\frac{2}{3}\left\{\cos\theta\int_\theta^\beta r^3\cos\theta d\theta+\sin\theta\int_\theta^\beta r^3\sin\theta d\theta\right\}?\quad(1)$$

Write $$X=\int_\theta^\beta r^2 d\theta,\ Y=\int_\theta^\beta r^3\cos\theta d\theta,\ Z=\int_\theta^\beta r^3\sin\theta d\theta,\quad(2)$$

and start with the equations

$$d\theta=\frac{dX}{-r^2}=\frac{dY}{-r^3\cos\theta}=\frac{dZ}{-r^3\sin\theta}\quad(3)$$

$$\left(\frac{d^2}{d\theta^2}+1\right)\left\{a\cos\theta.X-\frac{2}{3}(Y\cos\theta+Z\sin\theta)\right\}=0.\quad(4)$$

This last gives $$(r-a\cos\theta)dr+ar\sin\theta.d\theta=0,\quad(5)$$
and the system thus is

$$d\theta=\frac{dX}{-r^2}=\frac{dY}{-r^3\cos\theta}=\frac{dZ}{-r^3\sin\theta}=\frac{(r-a\cos\theta)dr}{-ar\sin\theta},\quad(6)$$

viz., this is a system of ordinary differential equations between the five variables θ, r, X, Y, Z: the system can therefore be integrated with 4 arbitrary constants, and these may be so determined that for the value β of θ, X, Y, Z shall be each $=0$; and r shall have the value r_0.

But this being so, from the assumed equations (3) and (4) we have

$$X=\int_\theta^\beta r^2 d\theta,\ Y=\int_\theta^\beta r^3\cos\theta d\theta,\ Z=\int_\theta^\beta r^3\sin\theta d\theta$$

and further (by integration of 4)

$$L\cos\theta+M\sin\theta=a\cos\theta.X-\frac{2}{3}(Y\cos\theta+Z\sin\theta).$$

Where L and M denote properly determined constants: viz., the conclusion is that r, X, Y, Z admit of being determined as functions of θ and of an arbitrary constant r_0, in such wise that

$$a\cos\theta.X-\frac{2}{3}(Y\cos\theta+Z\sin\theta)$$

shall be a function of θ, of the proper form $L\cos\theta+M\sin\theta$, but not so that it shall be the precise function $b^3\cos(\alpha+\theta)$. To make it have

https://doi.org/10.1017/S0013091500001504 Published online by Cambridge University Press

this value we must have $L = b^3\cos a$, $M = -b^3\sin a$ (where L, M are given functions of a, β, r_0), *i.e.*, we must have *two* given relations between a, b, a, β, r_0: or treating r_0 as a disposable constant we must have *one* given relation between a, b, a, β.

The equation $d\theta = \frac{r - a\cos\theta}{-ar\sin\theta}dr$ gives $r^2 - 2ar\cos\theta = C$. ($C = r_0^2 - 2ar_0\cos\beta$). There would be considerable difficulty in working the question out with r_0 arbitrary, but we may do it easily enough for the particular value $r_0 = 0$ or $r_0 = 2a\cos\beta$, giving $C = 0$ and $\therefore r = 2a\cos\theta$: and we ought in this case to be able to satisfy the given equation not in general but with *two* determinate relations between the constants a, b, α, β.

We have

$$\int\cos^2\theta d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$$

$$\int\cos^4\theta d\theta = \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta$$

$$\int\cos^3\theta\sin\theta d\theta = -\frac{1}{4}\cos^4\theta$$

And thence

$$a\cos\theta . X - \frac{2}{3}(Y\cos\theta + Z\sin\theta)$$

$$= 4a^3\cos\theta\left\{\frac{1}{2}(\beta - \theta) + \frac{1}{4}(\sin 2\beta - \sin 2\theta)\right\}$$

$$-\frac{16}{3}a^3\cos\theta\left\{\frac{3}{8}(\beta - \theta) + \frac{1}{4}(\sin 2\beta - \sin 2\theta) + \frac{1}{32}(\sin 4\beta - \sin 4\theta)\right\}$$

$$-\frac{16}{3}a^3(\sin\theta\left\{\qquad\qquad -\frac{1}{4}(\cos^4\beta - \cos^4\theta)\right\}$$

$$= -\frac{1}{3}a^3\cos\theta(\sin 2\beta - \sin 2\theta)$$

$$-\frac{1}{6}a^3\cos\theta(\sin 4\beta - \sin 4\theta)$$

$$+\frac{4}{3}a^3\sin\theta(\cos^4\beta - \cos^4\theta)$$

Where the terms containing β are readily reduced to $\frac{4}{3}a^3\cos^3\beta \sin(\theta - \beta)$; hence also the terms without β disappear of themselves: and we have

$$a\cos\theta . X - \frac{2}{3}(Y\cos\theta + Z\sin\theta) = \frac{4}{3}a^3\cos^3\beta . \sin(\theta - \beta),$$

which may be put $= b^3\cos(\theta + a)$ ·

https://doi.org/10.1017/S0013091500001504 Published online by Cambridge University Press

viz., this will be so if we have the *two* relations

$$a = \frac{\pi}{2} - \beta\ ;\ \text{and}\ b^3 = -\frac{4}{3}a^3\cos^2\beta.$$

I make (see fig. 84) Milner's lamp, with a circular section, β arbitrary, but a segment AM ($\angle$ SAM $=\beta$) made solid. G in the line SG at right angles to AM is the C.G. of the lamp, and G′ the C.G. of the oil.

And this seems to be the *only* form—for the pole of r must, it seems to me, be *on* the bounding circle—viz., in the equation $r^2 - 2ar\cos\theta = C$, we must have $C = 0$.

An Exercise on Logarithmic Tables.

By Professor TAIT.

In reducing some experiments, I noticed that the logarithm of 237 is about 2.37 ... . Hence it occurred to me to find in what cases the figures of a number and of its common logarithm are identical :—*i.e.*, to solve the equation

$$\log_{10}x = x/10^m,$$

where m is any positive integer.

It is easy to see that, in all cases, there are two solutions ; one greater than, the other less than, ϵ. This follows at once from the position of the maximum ordinate of the curve

$$y = (\log x)/x.$$

The smaller root is, for

$$m = 1,\ x = 1{\cdot}371288\ \ldots\ \ldots$$

$$m = 2,\ x = 1{\cdot}023855\ \ldots\ \ldots$$

For higher values of m, it differs but little from 1, and the excess may be calculated approximately from

$$y - y^2/2 + \ \ldots\ = (1+y)\log_\epsilon 10/10^m.$$

Ultimately, therefore, the value of the smaller root is

$$1.00\ \ldots\ \ldots\ 0230258\ \ldots\ \ldots$$

where the number of cyphers following the decimal point is $m-1$.

The greater root must have $m+p$ places of figures before the decimal point ; p being unit till $m = 9$, thenceforth 2 till $m = 98$, 3 till $m = 997$, &c. Thus, for example, if $m > 8 < 98$ we may assume

$$x = (m+1)10^m + y$$

https://doi.org/10.1017/S0013091500001504 Published online by Cambridge University Press