NOTES ON NUMERICAL ANALYSIS IV. ON ACCELERATING ITERATION PROCEDURES WITH SUPERLINEAR CONVERGENCE

Wesley Kotzé

(received January 13, 1964)

In the study of algorithms for the iterative solution of an arbitrary analytic equation $f(z) = 0$, acceleration procedures are of importance in practice and of considerable interest in the theory of the subject. Let

$$z_n = F(z_{n-1}) = G(z_{n-1}, f(z_{n-1}), f'(z_{n-1}), \ldots, f^{(s)}(z_{n-1}))$$

$$s \geq 1, \quad n = 1, 2, \ldots$$

be an iteration formula which has a zero ξ of $f(z)$ as attractive fixed point. An algorithm of this type is said to converge towards a root ξ of $f(z) = 0$ for all initial approximations $z = z_0$ in a vicinity of ξ, of order $k > 0$, when

$$(1) \qquad F(z) - \xi = O(|z-\xi|^k), \quad z \to \xi .$$

This type of algorithm includes, of course, the classical Newton-Raphson method ($k=2$) and its numerous modifications (see e. g. [4]). The two best known modifications of third order convergence are Frame's method [2] and an algorithm probably due to E. Schröder [3], but lately erroneously ascribed to Tchebycheff by some authors. These two algorithms are respectively given by equations (5) and (6) below. There exist several other workable algorithms of second or third order convergence, some of which are of more importance for certain types of functions $f(z)$ than others. The author suggested several such new algorithms in [1]. Algorithms of convergence order higher than three are of little practical importance in comparison with those of second and third order

Canad. Math. Bull. vol. 7, no. 3, July 1964.

convergence. Two acceleration methods that can be used in combination with any of these algorithms are given by the following two theorems.

THEOREM 1. Given an algorithm of order $k > 2$ which yields, after n applications,

$$(2) \qquad \frac{|z_n - \xi|}{|z_{n-1} - \xi|^k} = A + O(|z_{n-1} - \xi|), \quad A \neq 0, \quad n \to \infty .$$

Then the approximation to ξ will be improved if z_n is replaced by

$$(3) \qquad Z_n^* = z_n - A|z_{n-1} - z_n|^k \operatorname{sgn}(z_n - \xi) ,$$

where $\operatorname{sgn} a = \frac{a}{|a|}$.

Proof. Clearly

$$\frac{z_{n-1} - z_n}{z_{n-1} - \xi} = 1 - \frac{z_n - \xi}{z_{n-1} - \xi}$$

$$= 1 - O((z_{n-1} - \xi)^{k-1}), \quad n \to \infty \quad \text{by (1).}$$

Thus

$$\frac{|z_{n-1} - \xi|}{|z_{n-1} - z_n|} = 1 + O(|z_{n-1} - \xi|^{k-1}), \quad n \to \infty .$$

This, together with (2), gives for $n \to \infty$:

$$\begin{aligned} |z_n - \xi| &= A|z_{n-1} - \xi|^k + O(|z_{n-1} - \xi|^{k+1}) \\ (4) \qquad &= A|z_{n-1} - z_n|^k + O(|z_{n-1} - \xi|^{k^2-k}) \\ &\quad + O(|z_{n-1} - \xi|^{k+1}) \end{aligned}$$

https://doi.org/10.4153/CMB-1964-041-1 Published online by Cambridge University Press

$$= A|z_{n-1} - z_n|^k + O(|z_{n-1} - \xi|^{k+1}), \; k > 2 ,$$

i.e.,

$$z_n - \xi = A|z_{n-1} - z_n|^k \operatorname{sgn}(z_n - \xi) + O(|z_{n-1} - \xi|^{k+1}) .$$

Therefore, if we choose Z_n^* as in (3), we have

$$|Z_n^* - \xi| = O(|z_{n-1} - \xi|^{k+1}) = O(\delta^{k^2+k}), \quad n \to \infty$$

where

$$\delta = |z_{n-2} - \xi| .$$

Note: In the case of $k=2$, nothing is gained by replacing z_n with Z_n^*. It is obvious from (4) that for $k=2$, $|Z_n^* - \xi| = O(\delta^4)$. On the other hand, $|z_n - \xi| = O(\delta^4)$.

The trouble with this acceleration method is, of course, the determination of the value A. If, however, approximations of the values of the first, second and third derivatives in the vicinity of the desired zero ξ of the function $f(z)$ are known, this will not pose a serious problem in case of any of the third order algorithms in use. Simple asymptotic expansions from ξ will show that e.g., (i) in case of Frame's algorithm:

$$(5) \qquad F(z) = z - \frac{2f(z)f'(z)}{2f'(z)^2 - f(z)f''(z)} \qquad (k=3) ;$$

an approximation of $\left| \frac{1}{12\, f'(\xi)^2} [3f''(\xi)^2 - 2f'(\xi)f'''(\xi)] \right|$

can be taken as A.

(ii) in case of the algorithm:

https://doi.org/10.4153/CMB-1964-041-1 Published online by Cambridge University Press

$$(6) \qquad F(z) = z - \frac{f(z)}{f'(z)} - \frac{f(z)^2 f''(z)}{2f'(z)^3} \qquad (k=3) \; ;$$

an approximation of $\left|\frac{1}{2f'(\xi)^3}\,[f'(\xi)f''(\xi)^2 + \frac{5}{3}f'(\xi)^2 f'''(\xi)]\right|$ can be taken as value A.

This issue is entirely avoided in the use of the following acceleration method, and is therefore perhaps better suited for practical application.

THEOREM 2. Given an algorithm of order $k > 1$ which yields after n applications

$$\frac{|z_n - \xi|}{|z_{n-1} - \xi|^k} = A + O(|z_{n-1} - \xi|) \, , \quad A \neq 0, \quad n \to \infty \, .$$

Then the approximation to ξ will be improved if z_n is replaced by

$$(7) \qquad Z_n = z_n - \frac{|z_{n-1} - z_n|^{k+1}}{|z_{n-2} - z_{n-1}|^k} \cdot \operatorname{sgn}(z_n - \xi) \, .$$

<u>Proof</u>. We can obviously write:

$$(8) \qquad |z_n - \xi| = A|z_{n-1} - \xi|^k + O(|z_{n-1} - \xi|^{k+1}), \quad n \to \infty$$

$$(9) \qquad |z_{n-1} - \xi| = A|z_{n-2} - \xi|^k + O(|z_{n-2} - \xi|^{k+1}), \quad n \to \infty$$

$$|z_{n-1} - z_n| = |z_{n-1} - \xi| \; \left|1 - \frac{z_n - \xi}{z_{n-1} - \xi}\right|$$

$$= A|z_{n-2} - \xi|^k [1 + O(|z_{n-1} - \xi|^{k-1})$$

https://doi.org/10.4153/CMB-1964-041-1 Published online by Cambridge University Press

$$+ O(|z_{n-2} - \xi|^{k+1})$$

$$+ O(|z_{n-1} - \xi|^{k-1} |z_{n-2} - \xi|^{k+1})]$$

$$= A|z_{n-2} - \xi|^{k} + O(|z_{n-2} - \xi|^{k+1}), \quad n \to \infty$$

$$= A\delta^{k} + O(\delta^{k+1}), \quad n \to \infty .$$

From (9),

$$\frac{|z_{n-2} - z_{n-1}|}{|z_{n-2} - \xi|} = 1 + O(\delta^{k-1}), \quad n \to \infty .$$

Hence:

$$\frac{|z_{n-1} - z_{n}|^{k+1}}{|z_{n-2} - z_{n-1}|^{k}} = A^{k+1} \delta^{k^2} + O(\delta^{k^2+1}), \quad n \to \infty .$$

From (8) and (9):

$$z_n - \xi = |z_n - \xi| \operatorname{sgn}(z_n - \xi) = [A^{k+1} \delta^{k^2} + O(\delta^{k^2+1})] \operatorname{sgn}(z_n - \xi), \ n \to \infty.$$

Therefore $Z_n - \xi = z_n - \xi - \dfrac{|z_{n-1} - z_n|^{k+1}}{|z_{n-2} - z_{n-1}|^{k}} \operatorname{sgn}(z_n - \xi)$

$$= O(\delta^{k^2+1}) \operatorname{sgn}(z_n - \xi), \quad n \to \infty ,$$

i.e., $\quad |Z_n - \xi| = O(\delta^{k^2+1}), \quad n \to \infty$.

https://doi.org/10.4153/CMB-1964-041-1 Published online by Cambridge University Press

Obviously Z_n is a better approximation to ξ than z_n for all $k > 1$. Indeed for $k=2$, the use of (7) gives an order improvement of 25 per cent and for $k=3$, an order improvement of 11.1 per cent. We observe that, theoretically, Z_n^* will give a better approximation to ξ than Z_n, since $k^2 + k > k^2 + 1$ for $k > 1$.

REFERENCES

1. W. Kotzé, On infinitely many algorithms for the solution of an analytic equation, Thesis for the degree M. Sc. (McGill University, 1961). An abstract of this thesis appeared in Canad. Math. Bull. vol. 5, no. 1 (1962), pp. 108-109.

2. J. S. Frame, A variation of Newton's method, Amer. Math. Monthly 51, (1944), pp. 36-38.

3. E. Schröder, Über unendlich viele Algorithmen zur Auflösung der Gleichungen, Math. Ann. 2, (1870), pp. 317-365.

4. H. Schwerdtfeger, Notes on Numerical Analysis I. Polynomial Iteration, Canad. Math. Bull., vol. 2, no. 2 (1959), pp. 97-109.

McGill University