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SUBDIRECT DECOMPOSITIONS OF THE LATTICE OF VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

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It is shown that if $\mathbf{V}$ is an element of the lattice $\mathcal{L}(\mathbf{CR})$ of the title then the map given by $\mathbf{U} \to (\mathbf{V} \wedge \mathbf{U}, \mathbf{V} \vee \mathbf{U})$ is a complete lattice embedding of $\mathcal{L}(\mathbf{CR})$ into $(\mathbf{V}] \times [\mathbf{V})$ if and only if $\mathbf{V}$ is a join-infinitely distributive element. In this case the image of the map is a subdirect product of the principal ideal $(\mathbf{V}]$ by the principal filter $[\mathbf{V})$ generated by $\mathbf{V}$. Some important varieties in $\mathcal{L}(\mathbf{CR})$ are shown to be join-infinitely distributive.

0. INTRODUCTION

Completely regular semigroups are semigroups that are unions of their subgroups. They form the class $\mathbf{CR}$ of universal algebras that have an associative binary operation (multiplication) and a unary operation (inversion) and satisfy the identities

$$xx^{-1}x = x,\ xx^{-1} = x^{-1}x,\ \left(x^{-1}\right)^{-1} = x.$$

Hence $\mathbf{CR}$ is a variety. Denote by $\mathcal{L}(\mathbf{CR})$ the lattice of subvarieties of $\mathbf{CR}$ ordered by inclusion.

Although many papers have appeared in which $\mathcal{L}(\mathbf{CR})$ or some of its sublattices have been studied, we still have little knowledge of the structure of the lattice. As might be expected the best known portion of $\mathcal{L}(\mathbf{CR})$ is at the bottom of the lattice. The sublattice consisting of the varieties of bands has been fully described in [**1**, **2**, **3**] and [**5**]. The sublattice consisting of the varieties of completely simple semigroups is relatively well known (for example see [**8**, **11**, **16**] and [**21**]). Other sublattices have been studied in [**4**, **7**, **14**, **20**] and [**22**]. Recently some general results on $\mathcal{L}(\mathbf{CR})$ have appeared in [**9**, **10**, **12**, **13**, **15**, **17**, **18**, **19**] and [**23**].

In [**12**], $\mathcal{L}(\mathbf{CR})$ is shown to be modular. It is well known (see [**6**]) that $\mathbf{V}$ is a neutral element in a lattice $\mathcal{L}$ (or $\mathbf{V}$ is a distributive element in a modular lattice) if and only if there is a lattice embedding of $\mathcal{L}$ into the direct product of $(\mathbf{V}]$ by $[\mathbf{V})$ given by $\mathbf{U} \to (\mathbf{V} \wedge \mathbf{U}, \mathbf{V} \vee \mathbf{U})$, where $(\mathbf{V}]$ and $[\mathbf{V})$ are respectively the principal ideal and principal dual ideal of $\mathcal{L}$ generated by $\mathbf{V}$. A stronger result applies if $\mathcal{L}$ is complete; then $\mathbf{V}$ is both meet- and join-infinitely distributive and separates $\mathcal{L}$ if and only if the embedding is a complete lattice embedding. This and other relevant results are stated in

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Section 1. In Section 2, it is seen that join-infinitely distributive elements are also meet-infinitely distributive and separate $\mathcal{L}(\mathbf{CR})$. Some join-infinitely distributive elements are found; these include all of the varieties of bands, and the varieties respectively of all groups, all completely simple semigroups, all orthodox members of $\mathbf{CR}$, and all locally orthodox members of $\mathbf{CR}$.

1. Relevant lattice theory

The reference used for this section is [6]. The results are either straightforward refinements to complete lattices of well-known results, or are easily proven; hence the proofs are omitted.

Let $\mathcal{L}$ be a lattice. An element $\mathbf{V} \in \mathcal{L}$ is *distributive* if $\mathbf{V} \vee (\mathbf{U} \wedge \mathbf{W}) = (\mathbf{V} \vee \mathbf{U}) \wedge (\mathbf{V} \vee \mathbf{W})$ for all $\mathbf{U}, \mathbf{W} \in \mathcal{L}$. An element $\mathbf{V}$ is *neutral* if $\mathbf{V}$ is distributive and dually distributive and *separates* $\mathcal{L}$ in the sense that $\mathbf{V} \vee \mathbf{U} = \mathbf{V} \vee \mathbf{W}$ and $\mathbf{V} \wedge \mathbf{U} = \mathbf{V} \wedge \mathbf{W}$ together imply $\mathbf{U} = \mathbf{W}$.

An element $\mathbf{V}$ of a complete lattice $\mathcal{L}$ is called *meet-infinitely distributive* (MID) if $\mathbf{V} \vee (\bigwedge \mathcal{A}) = \bigwedge\{\mathbf{V} \vee \mathbf{A};\ \mathbf{A} \in \mathcal{A}\}$ for each subset $\mathcal{A}$ of $\mathcal{L}$. An element satisfying the dual property is called *join-infinitely distributive* (JID).

By [6], $\mathbf{V} \in \mathcal{L}$ is neutral if and only if there is a lattice embedding of $\mathcal{L}$ into the direct product of $(\mathbf{V}]$ by $[\mathbf{V})$ given by $\mathbf{U} \to (\mathbf{V} \wedge \mathbf{U}, \mathbf{V} \vee \mathbf{U})$. Furthermore, if $\mathcal{L}$ is modular then distributivity, dual distributivity, and neutrality are equivalent properties for an element of $\mathcal{L}$. Clearly a (MID) element is distributive.

Suppose $\mathbf{V}, \mathbf{X} \in \mathcal{L}$ and $\mathbf{V} \geqslant \mathbf{X}$. Define $\mathbf{V}(\mathbf{X})$, if it exists, to be the greatest element of $\mathcal{L}$ such that $\mathbf{V}(X) \wedge \mathbf{V} = \mathbf{X}$. In the terminology of [6], $\mathbf{V}(\mathbf{X})$ is the pseudocomplement of $\mathbf{V}$ relative to $\mathbf{X}$.

Theorem 1.1. *Let $\mathcal{L}$ be a complete lattice and $V \in \mathcal{L}$. Then the map*

$$\alpha\colon \mathcal{L} \to (\mathbf{V}] \times [\mathbf{V});\mathbf{U}\alpha = (\mathbf{V} \wedge \mathbf{U},\ \mathbf{V} \vee \mathbf{U})$$

is a complete injective lattice homomorphism if and only if $\mathbf{V}$ is a (MID) and (JID) element that separates $\mathcal{L}$. In this case $\mathbf{V}(\mathbf{X})$ exists for each $\mathbf{X} \leqslant \mathbf{V}$ and the range of α is the subdirect product

$$\textit{range}\ \alpha = \{(\mathbf{X},\ \mathbf{Y}) \in (\mathbf{V}] \times [\mathbf{V});\ \mathbf{Y} \leqslant \mathbf{V} \vee \mathbf{V}(\mathbf{X})\}.$$

The lattice isomorphism $\beta\colon \mathcal{L} \to$ range α given by $\mathbf{U}\beta = \mathbf{U}\alpha$ has its inverse given by $(\mathbf{X},\ \mathbf{Y})\beta^{-1} = \mathbf{V}(\mathbf{X}) \wedge \mathbf{Y}$.

An element $\mathbf{V}$ of a complete lattice $\mathcal{L}$ is *compact* if for each subset $\mathcal{A}$ of $\mathcal{L}$ then $\mathbf{V} \leqslant \bigvee \mathcal{A}$ implies $\mathbf{V} \leqslant \bigvee \mathcal{A}_0$ for some finite subset $\mathcal{A}_0$ of $\mathcal{A}$. A complete lattice $\mathcal{L}$ is *algebraic* if each element of $\mathcal{L}$ is a join of compact elements.

PROPOSITION 1.2. *Let* $\mathbf{V}$ *be a dually distributive element of an algebraic lattice* $\mathcal{L}$. *Then* $\mathbf{V}$ *is a (JID) element of* $\mathcal{L}$.

It is easy to see that the lattice of congruences of a universal algebra is algebraic, as is the sublattice of fully invariant congruences. Consequently $\mathcal{L}(\mathbf{CR})$ is dually algebraic (see Section 2). Since a (JID) element of $\mathcal{L}(\mathbf{CR})$ is dually distributive and $\mathcal{L}(\mathbf{CR})$ is modular then the element is also neutral; so it is distributive and therefore is also an (MID) element. In light of this and Theorem 1.1 our attention will be concentrated on (JID) elements. It should be noted that the results of the paper (other than Theorem 2.11) have analogues for dually distributive elements.

PROPOSITION 1.3. *Let* $\mathbf{V}$ *be a (JID) element of a complete lattice* $\mathcal{L}$. *If* $\mathbf{U}$ *is a (JID) element of* $(\mathbf{V}]$ *then* $\mathbf{U}$ *is a (JID) element of* $\mathcal{L}$.

PROPOSITION 1.4. *Let* $\mathbf{V}$ *be a neutral (JID) element of a complete lattice* $\mathcal{L}$. *If* $\mathbf{U}$ *is a (JID) element of* $[\mathbf{V})$ *then* $\mathbf{U}$ *is a (JID) element of* $\mathcal{L}$.

PROPOSITION 1.5. *The set of neutral (JID) elements of a complete lattice* $\mathcal{L}$ *is a sublattice of* $\mathcal{L}$.

2. VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

A remarkable new representation of $\mathcal{L}(\mathbf{CR})$, due to Polák [**18**], will be used. Some preliminary information is needed for its description.

Let X denote a denumerable set and F be the free object in $\mathbf{CR}$ on X. For any variety $\mathbf{V} \in \mathcal{L}(\mathbf{CR})$ there is a fully invariant congruence $\rho_{\mathbf{V}}$ of F such that $F/\rho_{\mathbf{V}}$ is the free object in $\mathbf{V}$ on X. There is a lattice anti-isomorphism of $\mathcal{L}(\mathbf{CR})$ onto the lattice C of fully invariant congruences of F given by $\mathbf{V} \to \rho_{\mathbf{V}}$. Hence, as noted after Proposition 1.2, $\mathcal{L}(\mathbf{CR})$ is dually algebraic.

Let E be the set of idempotents of F and $\mathcal{L}$ and $\mathcal{R}$ be the Green's relations on F. Consider the following equivalence relations on $\mathcal{L}(\mathbf{CR})$, where $\mathbf{U}, \mathbf{V} \in \mathcal{L}(\mathbf{CR})$:

$$\mathbf{U}T_l\mathbf{V} \Leftrightarrow \rho_{\mathbf{U}} \vee \mathcal{L} = \rho_{\mathbf{V}} \vee \mathcal{L};$$

$$\mathbf{U}T_r\mathbf{V} \Leftrightarrow \rho_{\mathbf{U}} \vee \mathcal{R} = \rho_{\mathbf{V}} \vee \mathcal{R};$$

$$\mathbf{U}T\mathbf{V} \Leftrightarrow \rho_{\mathbf{U}} \cap (E \times E) = \rho_{\mathbf{V}} \cap (E \times E);$$

$$\mathbf{U}K\mathbf{V} \Leftrightarrow \cup\{e\rho_{\mathbf{U}}; e \in E\} = \cup\{e\rho_{\mathbf{V}}; e \in E\}.$$

The corresponding equivalence relations on congruences are well-known and significant in the theory of regular semigroups.

The following properties of T_l, T_r, T and K are from [**12**] and [**13**], or [**17**] and [**18**]. For $H \in \{T_l, T_r, T, K\}$, H is a complete lattice congruence on $\mathcal{L}(\mathbf{CR})$. For $\mathbf{V} \in \mathcal{L}(\mathbf{CR})$ the H-class of $\mathbf{V}$ is a closed interval $[\mathbf{V}_H, \mathbf{V}^H]$ of $\mathcal{L}(\mathbf{CR})$; that is;

$\mathbf{U} \in \mathbf{V}H$ if and only if $\mathbf{V}_H \leqslant \mathbf{U} \leqslant \mathbf{V}^H$. The variety $\mathbf{V}$ is uniquely determined by its T- and K-classes; $\{\mathbf{V}\} = \mathbf{V}T \cap \mathbf{V}K$. Furthermore $T = T_l \cap T_r$ while $\mathbf{V}_T = \mathbf{V}_{T_l} \vee \mathbf{V}_{T_r}$ and $\mathbf{V}^T = \mathbf{V}^{T_l} \cap \mathbf{V}^{T_r}$. There are complete lattice endomorphisms of $\mathcal{L}(\mathbf{CR})$ given by $\mathbf{V} \to \mathbf{V}_{T_l}$, $\mathbf{V} \to \mathbf{V}_{T_r}$ and $\mathbf{V} \to \mathbf{V}^K$.

The subvarieties of **CR** that will be mentioned are listed below, along with their members and equivalent formulations.

T	trivial semigroups	
LZ	left zero semigroups	the atoms of $\mathcal{L}(\mathbf{CR})$
RZ	right zero semigroups	the atoms of $\mathcal{L}(\mathbf{CR})$
SL	semilattices	the atoms of $\mathcal{L}(\mathbf{CR})$
ReB	rectangular bands	$\mathbf{LZ} \vee \mathbf{RZ}$, $\mathbf{ReB}_T$
LNB	left normal bands	$\mathbf{LZ} \vee \mathbf{SL}$, $\mathbf{LNB}_T$, $\mathbf{LNB}_{T_r}$
RNB	right normal bands	$\mathbf{RZ} \vee \mathbf{SL}$, $\mathbf{RNB}_T$, $\mathbf{RNB}_{T_l}$
NB	normal bands	$\mathbf{LNB} \vee \mathbf{RNB}$
B	bands	$\mathbf{T}^K$
G	groups	$\mathbf{T}^T$, $\mathbf{G}_K$
CS	Completely simple semigroups	$\mathbf{ReB}^T$, $\mathbf{CS}_K$
O	orthodox semigroups in **CR**	$\mathbf{G}^K$
LO	locally orthodox semigroups in **CR** .	$\mathbf{CS}^K$

The first eight varieties from a principal ideal $(\mathbf{NB}]$ of $\mathcal{L}(\mathbf{CR})$ that is contained in the principal ideal $(\mathbf{B}]$. Also $\mathbf{SL} = \mathbf{SL}_{T_l} = \mathbf{SL}_{T_r}$, $\mathbf{CS}_{T_l} = \mathbf{RZ}$ and $\mathbf{CS}_{T_r} = \mathbf{LZ}$.

The representation of $[\mathbf{SL}, \mathbf{CR}]$ that follows is an interpretation of the main theorem of [**18**] based on the description in [**12**].

Let $\Gamma = \{(i, n);\ i \in \{0, 1\}, n \in \mathbf{Z}, n \geqslant 0\}$. Let $\mathcal{P} = \{L, R, O\}$ be a three element meet semilattice with $L \wedge R = O$. Let Δ be the ordinal sum of $\mathcal{P}$ with $\mathcal{L}(\mathbf{CR})/K$; so Δ is a complete lattice with $\vee\mathcal{P} = \mathbf{T}K$.

For $\mathbf{V} \in \mathcal{L}(\mathbf{CR})$ define inductively for $(0, n), (1, n) \in \Gamma$,

$$\mathbf{V}(0,0) = \mathbf{V}(1,0) = \mathbf{V},\ \mathbf{V}(0, n+1) = \mathbf{V}(1,n)_{T_r},\ \mathbf{V}(1, n+1) = \mathbf{V}(0,n)_{T_l}.$$

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Now define for $\mathbf{V} \in [\mathbf{SL}, \mathbf{CR}]$ and $(i,j) \in \Gamma$,

$$\mathbf{V}_{ij} = \begin{cases} O & \text{if } j \geqslant 1 \text{ and } \mathbf{V}(i,j) = \mathbf{SL}, \\ L & \text{if } j \geqslant 1,\ i = 0 \text{ and } \mathbf{V}(i,j) = \mathbf{LNB}, \\ R & \text{if } j \geqslant 1,\ i = 1 \text{ and } \mathbf{V}(i,j) = \mathbf{RNB}, \\ \mathbf{V}(i,j)K & \text{otherwise.} \end{cases}$$

With the direct power of Δ by Γ denoted by Δ^{Γ}, define a map

$$\xi\colon [\mathbf{SL}, \mathbf{CR}] \to \Delta^{\Gamma}\ ;\ \mathbf{V}\xi(i,j) = \mathbf{V}_{ij}.$$

By [**18**], ξ is a lattice embedding whose image is a complete sublattice of Δ^{Γ}. Hence

LEMMA 2.1. *ξ is a complete lattice embedding of $[\mathbf{SL}, \mathbf{CR}]$ into Δ^{Γ}.*

Recently Pastijn [**12**] used the representation to prove that $\mathcal{L}(\mathbf{CR})$ is modular. Hence by [**6**]

PROPOSITION 2.2. *For $\mathbf{V} \in \mathcal{L}(\mathbf{CR})$ the following are equivalent:*
(i) $\mathbf{V}$ is distributive; (ii) $\mathbf{V}$ is dually distributive; (iii) $\mathbf{V}$ is neutral; and (iv) there is a lattice embedding of $\mathcal{L}(\mathbf{CR})$ into $(\mathbf{V}] \times [\mathbf{V})$ given by $\mathbf{U} \to (\mathbf{V} \cap \mathbf{U}, \mathbf{V} \vee \mathbf{U})$.

The only neutral elements of $\mathcal{L}(\mathbf{CR})$ that have been listed in the literature are the elements of the principal ideal $(\mathbf{NB}]$ (see [**9**]).

PROPOSITION 2.3. *The set $(\mathbf{NB}] \cup \{\mathbf{G}, \mathbf{CS}, \mathbf{CR}\}$ generates a sublattice of (JID) elements of $\mathcal{L}(\mathbf{CR})$.*

PROOF: In [**9**] it was shown that $\mathbf{G}$ and $\mathbf{CS}$ are dually distributive in $\mathcal{L}(\mathbf{CR})$. The proof in [**9**] generalises directly to show that $\mathbf{G}$ and $\mathbf{CS}$ are also (JID) elements. Obviously $\mathbf{CR}$ is a (JID) element.

It is well known (for example see [**12**]) that for any $\mathbf{A} \in \mathcal{L}(\mathbf{CR})$,

$$\mathbf{A} \cap \mathbf{LZ} = \mathbf{T} \Leftrightarrow \mathbf{A} \cap \mathbf{LZ} \neq \mathbf{LZ} \Leftrightarrow \mathbf{A} \leqslant \mathbf{RNB} \vee \mathbf{G}.$$

It follows easily that $\mathbf{LZ}$ is a (JID) element of $\mathcal{L}(\mathbf{CR})$. Likewise the atoms $\mathbf{RZ}$ and $\mathbf{SL}$ are (JID) elements of $\mathcal{L}(\mathbf{CR})$. The result follows by Proposition 1.5 and 2.2. ∎

The aim now is to extend the (JID) property of a variety to related varieties.

PROPOSITION 2.4. *([**18**] and [**19**]). For $\mathbf{V} \in \mathcal{L}(\mathbf{CR})$ then*

$$\mathbf{V}^K = [\mathbf{V}^K]_{T_l} = [\mathbf{V}^K]_{T_r} = [\mathbf{V}^K]_T \text{ and } [\mathbf{V}^K]_{ij} = \mathbf{V}K \text{ for all } (i,j) \in \Gamma.$$

COROLLARY 2.5. *Suppose* $\mathbf{V} \in [\mathbf{SL}, \mathbf{CR}]$. *Then* $\mathbf{V}$ *is a (JID) element in* $\mathcal{L}(\mathbf{CR})$ *if and only if* $\mathbf{V}_{00}$ *is a (JID) element in* $\mathcal{L}(\mathbf{CR})/K$ *and* $\mathbf{V}_{ij}$ *is a (JID) element in* Δ *for each* $(i,j) \in \Gamma$, $j \neq 0$.

PROOF: For any $\mathbf{V} \in \mathcal{L}(\mathbf{CR})$, $\mathbf{V}^K \in [\mathbf{SL}, \mathbf{CR}]$. So the lattice homomorphism $\xi(0,0)$ maps $[\mathbf{SL}, \mathbf{CR}]$ onto $\mathcal{L}(\mathbf{CR})/K$. Furthermore by Proposition 2.4 and the definition of $\xi(i,j)$, the lattice homomorphism $\xi(i,j)$ maps $[\mathbf{SL}, \mathbf{CR}]$ onto Δ. The result follows from Lemma 2.1, and Propositions 1.4 and 2.3. ∎

PROPOSITION 2.6. *If* $\mathbf{V}$ *is a (JID) element of* $\mathcal{L}(\mathbf{CR})$ *then so are* $\mathbf{V}_{T_l}$ *and* $\mathbf{V}_{T_r}$.

PROOF: First suppose $\mathbf{V} \leqslant \mathbf{CS}$. Then $\mathbf{V}_{T_l} \in \{\mathbf{T}, \mathbf{RZ}\}$ and the result is by Proposition 2.3. Alternatively suppose $\mathbf{V} \in [\mathbf{SL}, \mathbf{CR}]$. Then $\mathbf{V}_{T_l} = \mathbf{V}(1,1)$ and $[\mathbf{V}_{T_l}]_{00} = \mathbf{V}(1,1)K$. Hence either $[\mathbf{V}_{T_l}]_{00} = \mathbf{V}_{11}$, or $\mathbf{V}(1,1) \in \{\mathbf{SL}, \mathbf{RNB}\}$ and $[\mathbf{V}_{T_l}]_{00} = \mathbf{T}K$. In the first case $\mathbf{V}_{11} \in \mathcal{L}(\mathbf{CR})/K$ and since $\mathbf{V}_{11}$ is a (JID) element of Δ, it is a (JID) element of the complete sublattice $\mathcal{L}(\mathbf{CR})/K$. Furthermore, $\mathbf{T}K$ is a (JID) element of $\mathcal{L}(\mathbf{CR})/K$, so in both cases $[\mathbf{V}_{T_l}]_{00}$ is a (JID) element of $\mathcal{L}(\mathbf{CR})/K$. Since $\mathbf{V}_{T_l} = \mathbf{V}_{T_l T_l}$ then $\mathbf{V}_{T_l}(1,1) = \mathbf{V}_{T_l}(0,0) = \mathbf{V}_{T_l}(1,0) = \mathbf{V}(1,1)$. Hence when h is even

$$\mathbf{V}_{T_l}(1,h) = V_{T_l}(1,h+1) = \mathbf{V}(1,h+1),\ \mathbf{V}_{T_l}(0,h-1) = \mathbf{V}_{T_l}(0,h) = \mathbf{V}(0,h).$$

Therefore $[\mathbf{V}_{T_l}]_{ij}$ is a (JID) element in Δ for each $(i,j) \in \Gamma$, $j > 0$. By Corollary 2.5 $\mathbf{V}_{T_l}$ is a (JID) element in $\mathcal{L}(\mathbf{CR})$. The proof is similar for $\mathbf{V}_{T_r}$. ∎

PROPOSITION 2.7. *If* $\mathbf{V}$ *is a (JID) element of* $\mathcal{L}(\mathbf{CR})$ *then so is* $\mathbf{V}^K$.

PROOF: For all $(i,j) \in \Gamma$, $[\mathbf{V}^K]_{ij} = \mathbf{V}K$ by Proposition 2.4. Since K is a complete congruence then $\mathbf{V}K$ is a (JID) element in $\mathcal{L}(\mathbf{CR})/K$. Let $\mathcal{A}$ be a subset of Δ, $\mathcal{P} = \{L, R, O\}$, $\mathcal{B} = \mathcal{A} \cap (\mathcal{L}(\mathbf{CR})/K)$ and $\mathcal{E} = \mathcal{A} \cap \mathcal{P}$; so $\mathcal{A} = \mathcal{B} \cup \mathcal{E}$ while $\mathbf{V}K \geqslant \bigvee \mathcal{P}$. Then

$$\mathbf{V}K \wedge \left(\bigvee \mathcal{A}\right) = \begin{cases} \mathbf{V}K \wedge (\bigvee \mathcal{B}) = \bigvee\{\mathbf{V}K \wedge B;\ B \in \mathcal{B}\} & \text{if } \mathcal{B} \neq \emptyset \\ \bigvee \mathcal{E} = \bigvee\{\mathbf{V}K \wedge E;\ E \in \mathcal{E}\} & \text{if } \mathcal{B} = \emptyset. \end{cases}$$

Thus $\mathbf{V}K$ is a (JID) element in Δ. The result now follows by Corollary 2.5. ∎

THEOREM 2.8. *For* $\mathbf{V} \in \mathcal{L}(\mathbf{CR})$ *the following are equivalent:*

(i) $\mathbf{V}$ *is a (JID) element of* $\mathcal{L}(\mathbf{CR})$;
(ii) $\mathbf{V}_{T_l}$, $\mathbf{V}_{T_r}$ *and* $\mathbf{V}^K$ *are (JID) elements of* $\mathcal{L}(\mathbf{CR})$;
(iii) $\mathbf{V}_T$ *and* $\mathbf{V}^K$ *are (JID) elements of* $\mathcal{L}(\mathbf{CR})$;
(iv) $\mathbf{V}(i,j)K$ *is a (JID) element of* $\mathcal{L}(\mathbf{CR})/K$ *for all* $(i,j) \in \Gamma$.

PROOF: That (i) implies (ii) is by Proposition 2.6 and 2.7. Since $\mathbf{V}_T = \mathbf{V}_{T_l} \vee \mathbf{V}_{T_r}$ then (ii) implies (iii) by Proposition 1.5. Note that $\mathbf{V}(0,0)K = \mathbf{V}(1,0)K = \mathbf{V}K = (\mathbf{V}^K)K$. Also since $\mathbf{V}_{T_l} = \mathbf{V}_{TT_l}$ and $\mathbf{V}_{T_r} = \mathbf{V}_{TT_r}$ then $\mathbf{V}(i,j) = \mathbf{V}_T(i,j)$ for all $(i,j) \in \Gamma$, $j \neq 0$; that is $\mathbf{V}(i,j)K = \mathbf{V}_T(i,j)K$. Because K is a complete lattice congruence, then by Proposition 2.6, (iii) implies (iv).

Now suppose $\mathbf{V} \in [\mathbf{SL}, \mathbf{CR}]$. Observe that $\mathbf{T}^K$ is a (JID) element of $\mathcal{L}(\mathbf{CR})$ by Proposition 2.3 and 2.7. So by Proposition 2.4 and Corollary 2.5, $\mathbf{T}K$ is a (JID) element of Δ, and then by Proposition 1.3 O, L and R are also (JID) elements of Δ. As in the proof of Proposition 2.7 the (JID) property for $\mathbf{V}(i,j)K$ in $\mathcal{L}(\mathbf{CR})/K$ extends to Δ. But $\mathbf{V}_{ij} \in \{\mathbf{V}(i,j)K, O, L, R\}$ so by Corollary 2.5, (iv) implies (i) in this case.

Finally assume $\mathbf{V} \leqslant \mathbf{CS}$. Then $\mathbf{V}_T \leqslant \mathbf{ReB}$ so $\mathbf{V}T = (\mathbf{V}_T)T$ is a (JID) element of $\mathcal{L}(\mathbf{CR})/T$. Recall that any variety of $\mathcal{L}(\mathbf{CR})$ is uniquely determined by the meet of its T- and K-classes, while T and K are complete congruences. So for any subset $\mathcal{A}$ of $\mathcal{L}(\mathbf{CR})$ and with $\mathbf{U} = \bigvee\{\mathbf{V} \cap \mathbf{A};\ \mathbf{A} \in \mathcal{A}\}$ then, assuming (iv),

$$\{\mathbf{V} \cap \left(\bigvee \mathcal{A}\right)\} = \left(\mathbf{V} \cap \left(\bigvee \mathcal{A}\right)\right)T \cap \left(\mathbf{V} \cap \bigvee \mathcal{A}\right)K = \mathbf{U}T \cap \mathbf{U}K = \{\mathbf{U}\}.$$

Thus (iv) also implies (i) in this case. ∎

COROLLARY 2.9. *The set* $(\mathbf{B}] \cup \{\mathbf{G}, \mathbf{CS}, \mathbf{O}, \mathbf{LO}, \mathbf{CR}\}$ *generates a sublattice of (JID) elements of* $\mathcal{L}(\mathbf{CR})$.

PROOF: Since $\mathbf{T}^K = \mathbf{B}$ then $\mathbf{B}$ is a (JID) element in $\mathcal{L}(\mathbf{CR})$. Furthermore, by the well known description of the lattice of bands, $(\mathbf{B}]$ is an algebraic dually distributive lattice; so each of its elements is a (JID) element of $(\mathbf{B}]$. Thus by Proposition 1.3, each element of $(\mathbf{B}]$ is a (JID) element of $\mathcal{L}(\mathbf{CR})$. Also $\mathbf{O} = \mathbf{G}^K$ and $\mathbf{LO} = \mathbf{CS}^K$ so the result now follows from Proposition 2.3 and 1.5. ∎

The converses of Proposition 2.6 and 2.7 are false as can be seen by the following examples.

Suppose $\mathbf{V}$ is not dually distributive in the lattice $(\mathbf{G}]$ of all varieties of goups. We have $\mathbf{V}_{T_l} = \mathbf{V}_{T_r} = \mathbf{V}_T = \mathbf{T}$ which is a dually distributive element of $\mathcal{L}(\mathbf{CR})$; thus the converse of Proposition 2.6 and of its analogue for dually distributive elements are false. Now put $\mathbf{U} = \mathbf{V}^K \vee \mathbf{G}$. We have $\mathbf{V}(0,\,0)K = \mathbf{V}K$ and $\mathbf{V}(i,j)K = \mathbf{T}K$ for all $(i,j) \in \Gamma$, $j \neq 0$, so by the analogue of Theorem 2.8 $\mathbf{V}K$ is not dually distributive in $\mathcal{L}(\mathbf{CR})/K$. Furthermore $\mathbf{U}_{11} = \left(\mathbf{V}^K \vee \mathbf{G}\right)_{T_l} K = \left(\mathbf{V}^K \vee \mathbf{T}\right)K = \mathbf{V}K$ so by the analogue of Corollary 2.5, $\mathbf{U}$ is not dually distributive in $\mathcal{L}(\mathbf{CR})$. However $\mathbf{U}^K = \left(\mathbf{V}^K \vee \mathbf{G}\right)^K = (\mathbf{V} \vee \mathbf{G})^K = \mathbf{G}^K = \mathbf{O}$ which is dually distributive in $\mathcal{L}(\mathbf{CR})$. Thus the converse of Proposition 2.7 and of its analogue for dually distributive elements are false.

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The following was noted before Proposition 1.3.

PROPOSITION 2.10. *If* $\mathbf{V}$ *is a distributive element of* $\mathcal{L}(\mathbf{CR})$ *then* $\mathbf{V}$ *is an (MID) element.*

As a consequence of this, Proposition 2.2 and Theorem 1.1 we get

THEOREM 2.11. *A variety* $\mathbf{V}$ *of* $\mathcal{L}(\mathbf{CR})$ *is a (JID) element if and only if* $\mathbf{V}(\mathbf{X})$ *exists for each* $\mathbf{X} \leqslant \mathbf{V}$, *there is a lattice isomorphism* β *of* $\mathcal{L}(\mathbf{CR})$ *onto the subdirect product* $\{(\mathbf{X}, \mathbf{Y}) \in (\mathbf{V}] \times [\mathbf{V}) ; \mathbf{Y} \leqslant \mathbf{V} \vee \mathbf{V}(\mathbf{X})\}$ *given by* $\mathbf{U}\beta = (\mathbf{V} \cap \mathbf{U}, \mathbf{V} \vee \mathbf{U})$, *and the subdirect product is a complete sublattice of* $(\mathbf{V}] \times [\mathbf{V})$. *The inverse of* β *is given by* $(\mathbf{X}, \mathbf{Y})\beta^{-1} = \mathbf{Y} \cap \mathbf{V}(\mathbf{X})$.

This result applies for any $\mathbf{V}$ in the sublattice of Corollary 2.9. In particular when $\mathbf{V} = \mathbf{B}$ or $\mathbf{V} = \mathbf{G}$ the Theorem generalises the subdirect decompositions of the lattice of varieties of bands of groups obtained respectively in [**7**] and [**9**]; also when $\mathbf{V} = \mathbf{G}$ it generalises the corresponding result for the lattice of varieties of completely simple semigroups in [**16**].

As was previously mentioned, the lattice $(\mathbf{B}]$ of all varieties of bands has been fully described. By the Theorem and Corollary 2.9, the problem of determining the structure of $\mathcal{L}(\mathbf{CR})$ has been reduced to that of determining the structure of $[\mathbf{B})$. Although they are not fully described there is a considerable body of information on the lattices $(\mathbf{G}]$ and $(\mathbf{CS}]$. By [**19**] the lattices $(\mathbf{O}]$ and $(\mathbf{LO}]$ can be described modulo the K-classes of varieties of $(\mathbf{G}]$ and $(\mathbf{CS}]$ respectively. Consequently, by Theorem 2.11 and Corollary 2.9, the study of $\mathcal{L}(\mathbf{CR})$ has essentially been reduced to a study of $[\mathbf{LO})$.

Remark. The results of this paper were announced at the International Conference on Universal Algebra, Lattices and Semigroups in Lisbon (June 20–24, 1988). At the conference, N.R. Reilly also announced that the varieties of Corollary 2.9 are neutral.

REFERENCES

[1] A.P. Birjukov, 'Varieties of idempotent semigroups', *Algebra i Logika* **9** (1970), 255–273. (Russian) .

[2] C.F. Fennemore, 'All varieties of bands', *Math. Nachr* **48** I (1971), 237–252. II, pp. 253–262.

[3] J.A. Gerhard, 'The lattice of equational classes of idempotent semigroups', *J. Algeba* **15** (1970), 195–224.

[4] J.A. Gerhard and M. Petrich, 'All varieties of regular orthogroups', *Semigroup Forum* **31** (1985), 311–351.

[5] J.A. Gerhard and M. Petrich, 'Varieties of bands revisited', (preprint).

[6] G. Gratzer, *General lattice theory* (Birkhaüser, Basel, 1978).

[7] T.E. Hall and P.R. Jones, 'On the lattice of varieties of bands of groups', *Pacific J. Math.* **91** (1980), 327–337.

[8] P.R. Jones, 'Completely simple semigroups: free products, free semigroups and varieties', *Proc. Roy. Soc. Edinburgh* **A88** (1981), 293–313.

[9] P.R. Jones, 'On the lattice of varieties of completely regular semigroups', *J. Austral. Math. Soc. Ser. A* **35** (1983), 227–235.

[10] P.R. Jones, 'Mal'cev products of varieties of completely regular semigroups', *J. Austral. Math. Soc. Ser. A* **42** (1987), 227–246.

[11] G.I. Masevickiĭ, 'On identities in varieties of completely simple semigroups over abelian groups', *Contemporary algebra, Leningrad* (1978), 81–89. (Russian) .

[12] F. Pastijn, 'The lattice of completely regular semigroup varieties', (preprint).

[13] F. Pastijn and P.G. Trotter, 'Lattices of completely regular semigroup varieties', *Pacific J. Math.* **119** (1985), 191–214.

[14] M. Petrich, 'Varieties of orthodox bands of groups', *Pacific J. Math.* **58** (1975), 209–217.

[15] M. Petrich, 'On the varieties of completely regular semigroups', *Semigroup Forum* **25** (1982), 153–169.

[16] M. Petrich and N.R. Reilly, 'Varieties of groups and of completely simple semigroups', *Bull. Austral. Math. Soc.* **23** (1981), 339–359.

[17] L. Polák, 'On varieties of completely regular semigroups I', *Semigroup Forum* **32** (1985), 97–123.

[18] L. Polák, 'On varieties of completely regular semigroups II', *Semigroup Forum* **36** (1987), 253–284.

[19] L. Polák, 'On varieties of completely regular semigroups III', *Semigroup Forum* **37** (1988), 1–30.

[20] V.V. Rasin, 'On the varieties of bands of groups', *Trans XV all union algebraic conference, Krasnojarsk* (1979). Vol.2, 123 (Russian).

[21] V.V. Rasin, 'On the lattice of varieties of completely simple semigroups', *Semigroup Forum* **17** (1979), 113–122.

[22] V.V. Rasin, 'On the varieties of Cliffordian semigroups', *Semigroup Forum* **23** (1981), 201–220.

[23] N.R. Reilly, 'Varieties of completely regular semigroups', *J. Austral. Math. Soc. Ser. A* **38** (1985), 372–393.

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