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DIMENSION THEORY VIA REDUCED BISECTOR CHAINS

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ABSTRACT. Let (X, d) be a metric space and Y and Z subsets of X. We say that Z is a bisector in Y and write $Y \rhd Z$ iff $Y \supset Z$ and there are two distinct points $y_1, y_2 \in Y$ such that $Z = \{z : d(z, y_1) = d(z, y_2)$ and $z \in Y\}$. By a reduced bisector chain in (X, d) of length n we understand a chain $X = X_0 \rhd X_1 \rhd \cdots \rhd X_{n-1} \rhd X_n$ such that $\dim X_n \leq 0$ and $\dim X_{n-1} > 0$. By $r(X, d)$ we denote the maximum length of reduced bisector chains in (X, d). For a metrizable topological space X we introduce the topological invariant $r(X)$ as the minimum of $r(X, d)$ taken over the set of all metrizations d of X. We prove that the function $r(X)$ coincides with the dimension of X on the class of compact metric spaces.

1. **Introduction and notation.** If $x_1, x_2 \in X$ are two distinct points in a metric space (X, d) we denote by $B(x_1, x_2)$ *the bisector of* x_1, x_2, i.e., the set $\{x : d(x, x_1) = d(x, x_2)\}$. If Y is a subset of (X, d) we say that Y is a *bisector* in (X, d) iff there are two distinct points x_1, x_2 in X such that $Y = B(x_1, x_2)$. The relevancy of this concept to topological dimension, denoted in the sequel by $\dim X$, has been brought to light in our recent paper [3] where the following result is obtained.

THEOREM 1.1. *If in a compact metric space (X, d) every bisector has dimension $\leq n-1$ then* $\dim X \leq n$. $(n = 0, 1, \ldots)$

This result depends heavily upon a theorem of J. Nagata (see [4] Theorem 11.2. page 18) and our observation that the family of open half-spaces of a compact metric space (X, d) forms a subbasis for the topology of X. (see [3] Lemma 2.1.)

The inductive character of Theorem 1.1. calls naturally for the consideration of consecutive formation of bisectors. If Y and Z are subsets of a metric space

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(X, d) we say that Z *is a bisector of* Y and write $Y \rhd Z$ iff $Y \supset Z$ and Z is a bisector in (Y, d) where (Y, d) is the metric space induced by the metric d on the subset Y. Thus we have defined the binary relation $\rhd$ between subsets of X which permits us to introduce a *bisector chain* (bc) as a sequence $\{X_i\}_o^n$ of subsets of X satisfying $X_i \rhd X_{i+1}$ for $i = 0, 1, \ldots, n-1$, and shall write it in the form:

$$X_0 \rhd X_i \rhd \cdots \rhd X_{n-1} \rhd X_n \tag{*}$$

In [3] we considered chains starting with X, i.e., $X = X_0$ and proceeding as far as possible, i.e., the terminal member X_n was either a singleton or bisector-empty which means $X_n \rhd \Theta$, where Θ denotes the empty set. At that time we were not aware of certain results due to J. H. Roberts [5] indicating the importance of bc with at most zero-dimensional terminals.

DEFINITION 1.1. A bisector chain (*) in a metric space (X, d) is said to be a *reduced bisector chain* (rbc) iff $X = X_0$, $\dim X_n \leq 0$ and $\dim X_{n-1} > 0$. The integer n is called the *length of the rbc*. The reduced bisector chain has length zero iff it is of the form $X = X_0$ where the metric space (X, d) has dimension zero.

The question arises as to whether a metric space (X, d) possesses a *rbc*. If $\dim X = 0$, then, by the definition, the bc $X = X_0$ is the only *rbc* in (X, d) and its length is zero. Assume now $\dim X > 0$. This implies that X is an infinite set which in turn implies the existence of bisectors $B(x_1, x_2)$ in X. If for some $x_1, x_2 \in X$ the bisector $B(x_1, x_2)$ is empty, i.e., $\dim B(x_1, x_2) = -1$, then the chain $X \rhd \Theta$ is a *rbc* of length 1. If $B(x_1, x_2) \neq \Theta$ the dimension of $B(x_1, x_2)$ is either 0, in which case the chain $X \rhd B(x_1, x_2)$ is again a *rbc* of length 1, or the dimension of $B(x_1, x_2)$ is >0 and the process continues applying the above reasoning to $B(x_1, x_2)$. This means that if $\dim X > 0$ three cases may be considered:

(1) there exists in (X, d) an infinite chain $X = X_0 \rhd X_1 \rhd \cdots \rhd X_n \rhd \cdots$ with $\dim X_n > 0$ for $n = 0, 1, \ldots$.

(2) There exists in (X, d) a *rbc* of arbitrary large length.

(3) The length n of *rbc*'s in (X, d) is bounded.

We now assign to every nonempty metric space (X, d) a non-negative integer (or ∞) which we call the *maximal length of rbc in* (X, d) and denote it by $r(X, d)$ as follows:

(a) We set $r(X, d) = 0$ iff $\dim X = 0$.

(b) We set $r(X, d) = \max\{n : \text{there exists a } rbc \text{ in } (X, d) \text{ of length } n\}$ iff $\dim X > 0$ and case (3) takes place.

(c) We set $r(X, d) = \infty$ iff $\dim X > 0$ and either case (1) or case (2) takes place.

For a metrizable topological space X we introduce the topological invariant

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$r(X)$ as the minimum of $\{r(X, d): d \in M(X)\}$ where $M(X)$ denotes the set of all metrics on X inducing the topology of X, or expressed in equivalent terms: $r(X)$ is the minimum of $r(Y, d)$, where (Y, d) ranges through the class of metric spaces homeomorphic to X.

The purpose of this paper is to prove the following two statements.

THEOREM 1.2. *The function $r(X)$ coincides with* $\dim X$ *on the class of compact metric spaces.*

THEOREM 1.3. *For the n-dimensional Euclidean space E^n we have $r(E^n)=n$ for $n=1,2,\ldots$*

2. **Relation between bisectors and the geometric theory of J. H. Roberts.**

LEMMA 2.1. *Let $Y_0 \rhd Y_1 \rhd \cdots \rhd Y_n$ be a bc in a metric space (X, d). Then there exists a bc $X=X_0 \rhd X_1 \rhd \cdots \rhd X_n$ in (X, d) such that $Y_i = X_i \cap Y_0$ for $i=0, 1, \ldots n$.*

Proof. For $i=1,2,\ldots n$, Y_i is a bisector in Y_{i-1}, hence there are two distinct points y'_{i-1} and y''_{i-1} in Y_{i-1} such that $Y_i = B(y'_{i-1}, y''_{i-1}) \cap Y_{i-1}$.

Defining recursively $X_1 = B(y'_0, y''_0)$

$$X_2 = B(y'_1, y''_1) \cap X_1$$

and

$$\vdots$$

$$X_n = B(y'_{n-1}, y''_{n-1}) \cap X_{n-1}$$

we obtain the chain of required properties.

COROLLARY 2.2. *The function $r(X)$ is monotonic, i.e., if Y is a nonempty subset of a metrizable topological space X then we have $r(Y) \le r(X)$.*

Proof. Let $d \in M(X)$ be a metric on X for which $r(X)=r(X, d)$. Assume now that the statement is false, i.e., $r(X) < r(Y)$. Since $r(Y) \le r(Y, d)$ we obtain $r(X, d) < r(Y, d)$. The assumption $r(X) < r(Y)$ implies that $r(X)$ is finite, say $n \ge 0$. Thus, there exists in (Y, d) a *bc* $Y = Y_0 \rhd Y_1 \rhd \cdots \rhd Y_n \rhd Y_{n+1}$ for which $\dim Y_n > 0$. Lemma 2.1. implies the existence of a *bc* $X = X_0 \rhd X_1 \rhd \cdots \rhd X_n \rhd X_{n+1}$ with $Y_n = X_n \cap Y_0$. Since $Y_n \subset X_n$ and the dimension function $\dim X$ is monotonic we have that $\dim X_n > 0$ implying that $r(X, d)$ is at least $n+1$ contrary to our assumption.

In order to formulate the geometrical result of J. H. Roberts we need to make some trivial observations concerning the *bc* in Euclidean spaces E_n.

LEMMA 2.3. *Every bisector Y in the n-dimensional Euclidean space (E^n, e) $(n=1,2,\ldots)$ equipped with the Euclidean metric e is a hyperplane, i.e., an affine subspace of E^n of dimension $n-1$, and conversely every affine subspace of dimension $n-1$ is a bisector in (E^n, e).*

Proof. If $x_1, x_2 \in E^n$ and $x_1 \neq x_2$, the bisector $B(x_1, x_2)$ can be defined as a hyperplane passing through the point $1/2(x_1+x_2)$ and orthogonal to the vector x_2-x_1; it is clear that every hyperplane can be obtained this way.

COROLLARY 2.4. *If $E^n = Y_0 \rhd Y_1 \rhd \cdots \rhd Y_k$ is a bc in the Euclidean space (E^n, e) $(n=1,2,\ldots)$ then each member Y_i is an affine subspace of dimension $n-i$, $i=1,2,\ldots,k$, and conversely, if Y is an affine subset of E^n of dimension $m (0 \leq m \leq n)$ then there exists a bc $E^n = Y_0 \rhd Y_1 \rhd \cdots \rhd Y_k = Y$ of length $k = n-m$ connecting E^n and Y.*

Proof. Straight-forward by induction on K.

THEOREM 2.5. *Let X be a nonempty subset of (E^{2n+1}, e) such that* $\dim (X \cap Y) \leq 0$ *for every affine subset Y of E^{2n+1} of dimension $n+1$. Then $r(X, e) \leq n$ where (X, e) is the metric space induced on X by the euclidean metric e.*

Proof. Assume that $X = X_0 \rhd X_1 \rhd \cdots \rhd X_k$ is an arbitrary *bc* in (X, e) such that $\dim X_k > 0$. Lemma 2.1. implies that there is a *bc* in (E^{2n+1}, e) $E^{2n+1} = Y_0 \rhd Y_1 \rhd \cdots \rhd Y_k$ with $X_k = Y_k \cap X$. Since $\dim X_k > 0$ and $\dim (Y \cap X) \leq 0$ for every affine subset of dimension $n+1$, this implies that $\dim Y_k > n+1$. On the other hand we know that the dimension of Y_k is precisely $2n+1-k$, so that $k < n$ showing that no *rbc* in (X, e) can be longer than n.

We now confront this result with the theorem of J. H. Roberts ([5] Theorem 12).

THEOREM 2.6. *If a separable metric space X has dimension n then there exists a topological embedding $f: X \to E^{2n+1}$ such that* $\dim (f(X) \cap Y) \leq 0$ *for every affine subset Y of E^{2n+1} of dimension $n+1$.*

COROLLARY 2.7. *If X is a separable metrizable space then $r(X) \leq \dim X$.*

Proof. If $\dim X = \infty$ there is nothing to prove, therefore assume $\dim X$ finite, say $n \geq 0$. Theorem 2.6. implies that a homeomorphic image of X, namely $f(X)$ satisfies the hypothesis of Theorem 2.5. furnishing $r(f(X), e) \leq n$ from which our assertion follows.

3. **Proofs of Theorems 1.2. and 1.3.** To prove Theorem 1.2. means to show that for every non-negative integer $k \geq 0$ we have

$$r(X) = k \quad \text{if and only if} \quad \dim X = k \qquad (**)$$

for every compact metrizable space X. We shall proceed by induction on k. For $k=0$ the statement (**) is true by the very definition of $r(X)$. In order to carry out the induction step we need

LEMMA 3.1. *Let X be a metrizable topological space with $r(X) < \infty$, and assume that $Y \subset X$ is a bisector in (X, d) where $d \in M(X)$ is such that $r(X) = r(X, d)$. Then $r(Y) < r(X)$.*

Proof. The length of an *rbc* in (Y, d) is not greater than $r(X)-1$. Thus $r(Y, d)<r(X)$ and therefore $r(Y)<r(X)$.

Now assume that the validity of the statement (**) has been established for all values $k=0, 1, \ldots, n$ and assume

(a) X compact and $r(X)=n+1$. Consider the metric space (X, d) where d is such that $r(X)=r(X, d)$. Lemma 3.1. implies that every bisector Y in (X, d) is such that $r(Y)\leq n$ which by the induction hypothesis yields that $\dim Y\leq n$ from which we conclude, using Theorem 1.1. that $\dim X\leq n+1$. Confronting this result with Corollary 2.7., we finally have $\dim X=n+1$, which proves one half of the statement. To prove the second half assume

(b) X compact and $\dim X=n+1$. From Corollary 2.7. we know $r(X)\leq n+1$. But if $r(X)<n+1$ then the induction hypothesis would yield $r(X)=\dim X<n+1$ contrary to the assumption. Thus we have $r(X)=n+1$ and the proof of Theorem 1.2. is complete.

We now prove Theorem 1.3. as an easy corollary of Theorem 1.2. and the monotonic property of the function $r(X)$.

From Corollary 2.4. follows that $r(E^n, e)=n(n=1, 2, \ldots)$ implying that $r(E^n)\leq n(n=1, 2, \ldots)$.

Denoting by I_n the unit cube in E_n we obtain from Theorem 1.2. that $r(I^n)=n(n=1, 2, \ldots)$ and since $I^n\subset E^n$ Corollary 2.2. yields finally $r(E^n)=n(n=1, 2, \ldots)$ what had to be shown.

4. **Some logical aspects of our results.** In our paper [3] we introduced the function $b(X, d)$ as the maximum length of *bcs* in a metric space (X, d) and the corresponding topological invariant $b(X)$ as the minimum of $\{b(X, d): d\in M(X)\}$.

Despite similarity between the definitions of $b(X, d)$ and $r(X, d)$ there is an essential difference between these notions from the point of view of formal logic and we need some definitions to bring this distinction to light.

Definition 4.1. For a nonempty metric space (X, d) we introduce the ternary relation $R\subset X\times X\times X$ on X setting $(x_1, x_2, x_3)\in R$ iff $x_1\neq x_2$ and $x_3\in B(x_1, x_2)$.

Remark 4.1. The relation R is defined naturally by the concept of bisector $B(x_1, x_2)$. In the sequel we shall also deal with two quaternary relations on X, $I\subset X\times X\times X\times X$ and $E\subset X\times X\times X\times X$ defined on a metric space (X, d) by $(x_1, x_2, x_3, x_4)\in I$ iff $d(x_1, x_2)\leq d(x_3, x_4)$ and $(x_1, x_2, x_3, x_4)\in E$ iff $d(x_1, x_2)=d(x_3, x_4)$ respectively. It is obvious that E can be expressed in terms of I and R in turn can be expressed in terms of E. We can say that the relations R, E and I introduce on X the bisector-, equational- and inequality-structure, respectively. The corresponding languages which can talk about these structures shall be denoted by L, L_E and L_I respectively.

DEFINITION 4.2. Let L denote the first order language containing besides the logical connectives, $\neg$, V, $\wedge$, $\rightarrow$, $\exists$ and $\forall$ and variables $x_1, x_2, \ldots$ only one ternary predicate symbol R^*. If P is a property of a metric space we say that P *is expressible in the language* L provided there is a sentence S (i.e., a formula without free variables) in L such that a metric space (X, d) has the property P if and only if (X, d) is a model of S, assuming that the predicate symbol R^* is interpreted by R in X. The fact that (X, d) is a model of S will be denoted by $(X, d) \models S$.

REMARK 4.2. Analogously we understand the expressability of P in the language L_E or L_I. Since L can be conceived as a sublanguage of L_E and L_E as a sublanguage of L_I the expressability in L implies that in L_E and consequently in L_I.

THEOREM 4.1. *The property $b(X, d) = n$ for $n = 0, 1, \ldots$ is expressible in L.*

Proof. The sentence $S_0 = \forall x_1 \forall x_2 \forall x_3 \neg R^*(x_1x_2x_3)$ says precisely that every bisector in (X, d) is empty. Applying this result to any bisector $B(x_4, x_5)$ of a metric space (X, d) we can express the fact that $b(X, d) \leq 1$ by the formula $S_1' = \forall x_1 \forall x_2 \forall x_3 \forall x_4 \forall x_5 [R^*(x_4x_5x_1) \wedge R^*(x_4x_5x_2) \wedge R^*(x_4x_5x_3) \rightarrow \neg R^*(x_1x_2x_3)]$. Proceeding inductively we can produce formulas S_n' expressing the property $b(X, d) \models S_n (n = 1, 2, \ldots)$. Thus the statement $b(X, d) = 0$ is equivalent to $(X, d) \models S_0$ and the statement $b(X, d) = n$ for $n = 1, 2, \ldots$ is equivalent to $(X, d) \models S_n$ where $S_n = S_n' \wedge \neg S_{n-1}'$ and where we set $S_0' = S_0$.

It is clear that this statement is no longer true if we pass from the property $b(X, d) = n$ to the property $r(X, d) = n$ since the conditions $\dim X_n \leq 0$ and $\dim X_{n-1} > 0$ involved in the definition of *rbc* are not in any obvious way describable in terms of the relation R. This is the main reason why the results obtained in this paper cannot be considered as definite.

DEFINITION 4.3. To each sentence S in the language L we assign the topological property P_s defined on the class of metrizable spaces as follows. We say that a space X has the topological property P_s iff there is a metric $d \in M(X)$ for which $(X, d) \models S$, and we express this by saying that $P_s(X)$ is true.

DEFINITION 4.4. Let P be a topological property and C a subclass of the class of metrizable spaces. We say that P *is expressible in the language L on the class C* provided there are sentences $S_1, S_2, \ldots, S_m$ in L and a formula $F(p_1, p_2, \ldots, p_m)$ of the sentential logic such that for $X \in C$ the truth value of $P(X)$ coincides with that of $F(P_{s_1}(X), P_{s_1}(X), \ldots P_{s_m}(X)]$.

THEOREM 4.2. *The topological property $b(X) = n$ for $n = 0, 1, \ldots$ is expressible in L on the class of metrizable spaces.*

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Proof. If $n=0$ the fact $b(X)=0$ means that there is $d \in M(X)$ with $(X, d) \models S_0$ showing that the property $b(X)=0$ is expressible. Assume now $n>1$. In this case the fact $b(X)=n$ means that there is $d \in M(X)$ for which $b(X, d)=n$ but it is not true that there is such $d' \in M(X)$ for which $b(X, d')=n-1$. Thus the statement $\{[d \in M(X)(x, d) \models S_n]$ and not $[d' \in M(x)(X, d') \models S_{n-1}]\}$ is the desired expression of the property $b(x)=n$.

COROLLARY 4.3. *The property* $\dim X=0$ *is expressible in L on the class of compact metrizable spaces.*

Proof. This follows readily from the above theorem and the basic result of our paper [3] where we proved that if X is compact then $\dim X=0$ iff $b(X)=0$.

Our main conjecture is that for arbitrary $n \geq 0$ the property $\dim X=n$ is expressible in L on the class of compact metrizable spaces.

Our belief in the truth of this conjecture is supported by a result of J. de Groot (see [1] or [4] page 154, Corollary to Theorem V.5). This result can easily be translated in the language L_I and it reads:

THEOREM 4.4. (De Groot) *The property* $\dim X=n(n=0,1,\dots)$ *is expressible in the language* L_I *on the class of compact metrizable spaces.*

Analogously, our result [2] on the metric rigidity if translated in the equational language L_E reads:

The property $\dim X=0$ is expressible in the language L_E on the class of separable metrizable spaces.

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