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EIGENFUNCTIONS OF OPERATOR-VALUED ANALYTIC FUNCTIONS

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This paper is a sequel to [**2**], whose primary purposes are to clarify and generalize the concept introduced there of an eigenfunction of an inner function, and to answer questions raised there concerning the equivalence of several possible forms of the definition. A new definition, proposed here, leads to a complete characterization of the eigenfunctions of Potapov inner functions of normal operators, and the result is more satisfactory than [**2**, Theorem 3.4], although the latter is used strongly in the proof.

Let $\mathscr{V}$ be an inner function in the sense of Lax; i.e., $\mathscr{V}(e^{i\theta})$ is almost everywhere (a.e.) a unitary operator on a separable Hilbert space $\mathscr{H}$, and $\mathscr{V}$ belongs weakly to the Hardy class H^2. An analytic function q (which will have to be a scalar inner function) was defined to be an eigenfunction of $\mathscr{V}$ if the set of z in the disk $\{z: |z| \leqq 1\}$ for which $\mathscr{V}(z) - q(z)I$ is invertible is a set of linear measure 0 on the circle $\{z: |z| = 1\}$. We begin with two examples which show that the boundary condition, $q(e^{i\theta}) \in \sigma(\mathscr{V}(e^{i\theta}))$ a.e., where σ denotes spectrum, and the interior condition, $q(z) \in \sigma(\mathscr{V}(z))$, are independent of each other, a question left open in [**2**].

LEMMA 1. *There exists an inner function $\mathscr{V}$ and an analytic function f such that $f(z) \in \sigma(\mathscr{V}(z))$ if $|z| < 1$, but $f(z) \notin \sigma(\mathscr{V}(z))$ if $|z| = 1$.*

Proof. Let $\{z_n\}$ be dense in the disk $\{z: |z| \leqq 1\}$, where each $|z_n| < 1$. Let $\{e_n\}$ be an orthonormal basis for $\mathscr{H}$ and define $\mathscr{V}$ by

$$\mathscr{V}(z)e_n = q_n(z)e_n,$$

where

$$q_n(z) = (z - z_n)(1 - \bar{z}_n z)^{-1}.$$

Then $\mathscr{V}$ is inner and $0 \in \sigma(\mathscr{V}(z_n))$ for all n. Elementary estimates show that

$$|q_n(z) - q_n(w)| \leqq \frac{|z - w|}{(1 - |z|)(1 - |w|)}$$

if $|z|, |w| < 1$, which implies that $\mathscr{V}(z) \to \mathscr{V}(w)$ in operator norm as $z \to w$, if $|w| < 1$. It is easy to see that if $T_n \to T$ in operator norm, $\lambda_n \in \sigma(T_n)$ and $\lambda_n \to \lambda$, then $\lambda \in \sigma(T)$. Thus $0 \in \sigma(\mathscr{V}(z))$ for all $|z| < 1$. But $0 \notin \sigma(\mathscr{V}(e^{i\theta}))$ since $\mathscr{V}(e^{i\theta})$ is unitary, and it suffices to take $f(z) \equiv 0$.

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LEMMA 2. *There exists an inner function $\mathscr{V}$ and a scalar inner function q such that $q(e^{i\theta}) \in \sigma(\mathscr{V}(e^{i\theta}))$ a.e., but $q(z) \notin \sigma(\mathscr{V}(z))$ if $|z| < 1$.*

Proof. Let $\{e_n\}$, $-\infty < n < \infty$, be an orthonormal basis for $\mathscr{H}$ and define $\mathscr{V}$ by $\mathscr{V}(z)e_n = ze_{n+1}$. Then by known properties [**1**, pp. 42, 46] of shifts,

$$\sigma(\mathscr{V}(re^{i\theta})) = \{z: |z| = r\}.$$

Thus we can take $q(z) \equiv 1$.

The concept of an eigenfunction of an inner function should generalize to arbitrary analytic operator-valued functions. The most direct approach would be the following: Let $A(z)$ be an analytic operator-valued function defined on an open set Ω. Then an analytic function f is an eigenfunction of A if $f(z) \in \sigma(A(z))$ for all $z \in \Omega$. The problem, as Lemma 1 shows, is that this leads to an unnatural situation in which the 0 function can be an eigenfunction of an inner function. The resulting spectrum (i.e., the set of eigenfunctions) will also depend on the domain of A. For example, let T be a bounded operator whose spectrum is $\{z: |z| \leqq \frac{1}{2}\}$ and let $A(z) = T$ for $|z| < 1$. The resulting spectrum includes all analytic functions f such that $|f(z)| < \frac{1}{2}$ for $|z| < 1$ and is much larger than the natural spectrum which ought to consist only of the constant functions whose range is in $\sigma(T)$. We can get around this difficulty by requiring that f extend analytically wherever A extends and that $f(z) \in \sigma(A(z))$ for all analytic continuations of A and f.

Analytic continuation, however, is not sufficient to eliminate the pathology exhibited in Lemma 1, since the inner function given there has the unit circle as its natural boundary. The following considerations will suffice to complete the definition. If an operator T is a direct sum $T_1 \oplus T_2$, then

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2).$$

This is false for operator-valued functions since if $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, where $A(z)\mathscr{H}_1 \subset \mathscr{H}_1$ and $A(z)\mathscr{H}_2 \subset \mathscr{H}_2$, then $A|_{\mathscr{H}_1}$ and $A|_{\mathscr{H}_2}$ might be continuable to different regions and therefore $\sigma(A_1) \cup \sigma(A_2)$ could be smaller than $\sigma(A)$. An explicit example can be given as follows: let $\{T_n\}$ be a sequence of bounded normal operators with spectrum

$$\left\{z: \frac{n-1}{n} \leqq |z| \leqq \frac{n}{n+1}\right\}.$$

Let $\mathscr{H}$ be the infinite direct sum $\bigoplus \mathscr{H}_n$, where each $\mathscr{H}_n$ is infinite-dimensional, and define an inner function $\mathscr{V}$ to be $(z - T_n)(I - zT_n^*)^{-1}$ on $\mathscr{H}_n$. The spectrum of $\mathscr{V}$ will include any analytic function f such that $|f(z)| < 1$ for $|z| < 1$. If we write $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_1^{\perp}$ and let $\mathscr{V}_1 = \mathscr{V}|_{\mathscr{H}_1}$, $\mathscr{V}_2 = \mathscr{V}|_{\mathscr{H}_1^{\perp}}$, then $\sigma(\mathscr{V}_1)$ consists entirely of functions of the form $\alpha(z - \lambda)(1 - \bar{\lambda}z)^{-1}$, where $|\lambda| \leqq \frac{1}{2}$ and $|\alpha| = 1$ by [**2**, Theorem 3.4]. Also, every function in $\sigma(\mathscr{V}_2)$ satisfies $\frac{1}{2} < |f(0)| < 1$ (although not all such functions are in $\sigma(\mathscr{V}_2)$). Thus $\sigma(\mathscr{V})$ is larger than $\sigma(\mathscr{V}_1) \cup \sigma(\mathscr{V}_2)$. We therefore add the require-

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ment that if $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ (where $\oplus$ denotes an orthogonal direct sum) and $A(z)\mathscr{H}_1 \subset \mathscr{H}_1$, $A(z)\mathscr{H}_2 \subset \mathscr{H}_2$, then in order for an analytic function f to be in the spectrum of A, it must be in the spectrum of $A|_{\mathscr{H}_1}$ or $A|_{\mathscr{H}_2}$ in the above sense. We note that if $A(z)\mathscr{H}_1 \subset \mathscr{H}_1$ for z in an open set U, then the same is true for any analytic continuation of A. This last requirement does suffice to eliminate the pathology inherent in Lemma 1. We state the completed definition for ease of reference.

Definition 3. Let A be an analytic operator-valued function defined on an open subset U of the complex plane. Then an analytic function f defined on U is an eigenfunction of A if

(i) $f(z) \in \sigma(A(z))$ for all $z \in U$,

(ii) f continues analytically wherever A does and $f(z) \in \sigma(A(z))$ for all continuations,

(iii) If a closed subspace $\mathscr{H}_1$ is invariant under $A(z)$ and $A(z)^*$ for all $z \in U$, and $A_1 = A|_{\mathscr{H}_1}$, $A_2 = A|_{\mathscr{H}_1^\perp}$, then f satisfies both (i) and (ii) for either A_1 or A_2.

Important evidence for the essential appropriateness of the definition of eigenfunctions given in [**2**] was Theorem 3.4 asserting that the spectrum of the Potapov inner function $\mathscr{V}_T(z) = (z - T)(I - zT^*)^{-1}$, where T is a normal operator, $||T|| < 1$, consists entirely of functions of the form $\alpha(z - \lambda)(1 - \bar{\lambda}z)^{-1}$, where $\lambda \in \sigma(T)$ and $|\alpha| = 1$. The fact that αs different from one can occur is a mild pathology, and we observe now that the above definition eliminates this possibility.

THEOREM 4. *Let T be normal, $||T|| < 1$, and let $\mathscr{V}_T(z) = (z - T)(I - zT^*)^{-1}$. Then the eigenfunctions of $\mathscr{V}_T$ in the sense of Definition 3 are precisely the functions $q_\lambda(z) = (z - \lambda)(1 - \bar{\lambda}z)^{-1}$, where $\lambda \in \sigma(T)$.*

Proof. Clearly $q_\lambda \in \sigma(\mathscr{V}_T)$ for $\lambda \in \sigma(T)$ as in [**2**, the proof of Theorem 3.4]. Now fix $\lambda \in \sigma(T)$ and let $q_\lambda = (z - \lambda)(1 - \bar{\lambda}z)^{-1}$. Suppose that αq_λ is an eigenfunction of $\mathscr{V}_T$ in the sense of Definition 3. We must show that $\alpha = 1$.

Choose $\epsilon > 0$ and let $S_\epsilon = \{z: |z - \lambda| \leqq \epsilon\}$. Let $\mathscr{H}_\epsilon = E(S_\epsilon)$, where E is the spectral measure of T. Then $\mathscr{H}_\epsilon$ and $\mathscr{H}_\epsilon^\perp$ are invariant under $\mathscr{V}_T$, and hence αq_λ is in either $\sigma(\mathscr{V}_T(z)|_{\mathscr{H}_\epsilon})$ or $\sigma(\mathscr{V}_T(z)|_{\mathscr{H}_\epsilon^\perp})$ by Definition 3 (iii). Clearly αq_λ is not in $\sigma(\mathscr{V}_T(z)|_{\mathscr{H}_\epsilon^\perp})$, since λ is not in $\sigma(T|_{\mathscr{H}_\epsilon^\perp})$. Let

$$\gamma = \inf_{\mu \notin \sigma(T|_{\mathscr{H}_\epsilon})} |\mu - \lambda|.$$

Then, by [**2**, Theorem 3.5 (ii)], $|1 - \alpha|\,|\lambda| < \gamma$ and $|1 - \alpha|\,|1 - \lambda| < 2\gamma$. (*Note.* In [**2**, the statement of Theorem 3.5], "and" should be replaced by "or".) Therefore, since $\gamma \leqq \epsilon$, $|1 - \alpha|\,|\lambda| < \epsilon$ and $|1 - \alpha|\,|1 - \lambda| < 2\epsilon$. Thus $\alpha = 1$, since this holds for all $\epsilon > 0$.

On the other hand, if we interpret the spectrum in the sense of [**2**], we can say something concerning its topological structure.

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THEOREM 5. *Let A be an operator-valued analytic function defined on $|z| < 1$ and suppose that $||A(z)||$ is bounded for $|z| < 1$. Then the eigenfunctions of A in the sense of* [**2**, *Definition* 1.1] *are closed and bounded in H^∞.*

Proof. Boundedness is clear. If $f \notin \sigma(A)$, then $f(z_0) \notin \sigma(A(z_0))$ for some $|z_0| < 1$. Since $\sigma(A(z_0))$ is closed, there is an ϵ for which any $g \in H^\infty$ such that $||g - f||_\infty < \epsilon$ will satisfy $g(z_0) \notin \sigma(A(z_0))$.

We observe that $\sigma(A)$ need not be compact in H^∞ (the diagonal inner function whose (n, n) term is z^n is a counterexample) although the spectrum of Potapov inner functions is compact by [**2**, Theorem 3.4]. Nothing like Theorem 5 can hold for $\sigma(A)$ in the sense in which it has been defined in this paper, since if $f(z_0) \in \sigma(A(z_0))$ only for some z_0s such that $|z_0| > 1$, any neighbourhood of f in H^∞ will include functions which do not extend beyond the disk. Though, again, the spectrum of Potapov inner functions will be compact by Theorem 4 above. Because of Theorem 5 there may very well be situations in which [**2**, Definition 1.1] is more appropriate than Definition 3.

We conclude with some observations on analytic continuations of operator-valued functions. There are situations in which the natural domain of such a function is not connected. For example, if

$$\sigma(T) = \{z: \tfrac{1}{3} \leqq |z| \leqq \tfrac{2}{3}\},$$

then

$$\mathscr{V}_T(z) = (z - T)(I - zT^*)^{-1}$$

makes sense for $|z| < \frac{3}{2}$ and for $|z| > 3$. To see in what sense this function can be analytically continued from $|z| < \frac{3}{2}$ to $|z| > 3$ we specialize T as follows. Let $\{\alpha_n\}$ be dense in $\{z: \frac{1}{3} \leqq |z| \leqq \frac{2}{3}\}$ and define T by $Te_n = \alpha_n e_n$, where $\{e_n\}$ is an orthonormal basis for $\mathscr{H}$. Then $\mathscr{V}_T$ is diagonal with entries

$$(z - \alpha_n)(1 - \bar{\alpha}_n z)^{-1}.$$

Even though $\mathscr{V}_T$ cannot be continued past $\{z: |z| = \frac{3}{2}\}$, each scalar function of the form $(\mathscr{V}(z), e)$, where

$$e = a_1 e_1 + \ldots + a_n e_n$$

is a finite linear combination of the es, can be so continued (each is meromorphic with poles only at $\alpha_1, \alpha_2, \ldots, \alpha_n$). Thus if $(A(z), e)$ can be continued to an open set Ω' for a dense set of es in $\mathscr{H}$, we can require any eigenfunction f to continue to Ω' in the usual sense and to satisfy $f(z) \in \sigma(A(z))$. There are two problems with this proposal. First, this extended form of analytic continuation will not yield Theorem 4 without condition (iii) of Definition 3. Second, and worse, if the continuations of $(A(z), e)$ and $(A(z), f)$ are along different paths for different es and fs, two distinct eigenfunctions may not be distinguishable as point functions on the original domain of A, and the domains must be taken to be a very complicated (and disconnected) Riemann surface. One could require that (A, e) and (A, f) be continuable along a path independent of e, but this leads to other embarrassing questions. In the absence of any example in which the above considerations yield a more natural spectrum than Definition 3, it seems best to avoid these complications.

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