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LOCAL SPECTRAL PROPERTIES OF COMMUTATORS

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For a pair of continuous linear operators T and S on complex Banach spaces X and Y, respectively, this paper studies the local spectral properties of the commutator $C(S,T)$ given by $C(S,T)(A):=SA-AT$ for all $A\in L(X,Y)$. Under suitable conditions on T and S, the main results provide the single valued extension property, a description of the local spectrum, and a characterization of the spectral subspaces of $C(S,T)$, which encompasses the closedness of these subspaces. The strongest results are obtained for quotients and restrictions of decomposable operators. The theory is based on the recent characterization of such operators by Albrecht and Eschmeier and extends the classical results for decomposable operators due to Colojoară, Foiaş, and Vasilescu to considerably larger classes of operators. Counterexamples from the theory of semishifts are included to illustrate that the assumptions are appropriate. Finally, it is shown that the commutator of two super-decomposable operators is decomposable.

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1. Introduction

One basic issue of local spectral theory is to relate the spectral properties of two given continuous linear operators $T\in L(X)$ and $S\in L(Y)$ on complex Banach spaces X and Y, respectively, when T and S are linked by some continuous linear mapping $A\in L(X,Y)$. The classical monographs on this topic, those of Colojoară–Foiaş [**8**] and Vasilescu [**21**], contain several results of this sort, most of them related to the notion of local spectrum. Recall that the *local spectrum* $\sigma_T(x)$ of the operator T at the vector $x\in X$ is defined as the complement in $\mathbf{C}$ of the set $\rho_T(x)$ of all $\lambda\in\mathbf{C}$, for which there exists an analytic function $f:U\to X$ on some open neighborhood U of λ in $\mathbf{C}$ such that $(T-\mu)f(\mu)=x$ for all $\mu\in U$. A significant example is the following result of Colojoară and Foiaş [**8**, Theorem 2.3.3]: if both T and S are decomposable in the sense of Foiaş, then every $A\in L(X,Y)$ satisfies

$$\sigma_S(Ax)\subseteq\sigma_T(x)\ \ \forall x\in X \Leftrightarrow \|C(S,T)^n(A)\|^{1/n}\to 0 \text{ as } n\to\infty, \tag{1}$$

where $C(S,T):L(X,Y)\to L(X,Y)$ denotes the commutator of T and S given by $C(S,T)(A):=SA-AT$ for all $A\in L(X,Y)$. This result is of basic importance, for instance, in the theory of quasinilpotent equivalence of decomposable operators [**8**]. More generally, for an arbitrary compact set K of $\mathbf{C}$, Foiaş and Vasilescu [**10**, Theorem 2.5] have shown that

$$\sigma_S(Ax) \subseteq \sigma_T(x) + K \ \ \forall x \in X \Leftrightarrow \sigma_{C(S,T)}(A) \subseteq K, \tag{2}$$

provided that both T and S are decomposable. In terms of the classical *local spectral subspaces* $X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$ for all closed $F \subseteq \mathbf{C}$, this result may be reformulated as a description of the local spectral subspaces $L_C(K)$ of the commutator $C := C(S, T)$ on the Banach space $L := L(X, Y)$ of all continuous linear operators from X into Y. Indeed, it is easily seen that (2) is equivalent to the identity

$$L_C(K) = \{A \in L(X, Y) : AX_T(F) \subseteq Y_S(\overline{F+K}) \text{ for all closed } F \subseteq \mathbf{C}\} \tag{3}$$

for any closed subset K of $\mathbf{C}$.

It is the purpose of this paper to establish results like (1), (2), and (3) for more general classes of operators, by extending them, in particular, to the much larger classes of restrictions and quotients of decomposable operators. Thanks to recent work of Albrecht and Eschmeier [**2**], we now have intrinsic characterizations of these classes, phrased in terms that have been significant in local spectral theory for some time. The relevant parts of this theory will be briefly reviewed in Section 2 below.

It should be noted immediately that results of type (1), (2), and (3) do require some assumption on the operators involved. In fact, we shall see in Section 2 that these results do *not* hold, if T is the unilateral right shift on the Hilbert space $\ell^2(\mathbf{N})$ or, more generally, any semishift on a Banach space X and S is simply the zero operator. Since semishifts are always restrictions of decomposable operators, these examples indicate that one may expect positive results only in a different direction.

Another obstacle is related to the problem of uniqueness of the analytic functions which occur in the definition of the local spectra. Recall from [**8**] that the operator $T \in L(X)$ is said to have the *single valued extension property* (SVEP) if, for any open set $U \subseteq \mathbf{C}$, the only analytic solution of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant function $f \equiv 0$. If T does not have SVEP, then, by [**21**, Proposition 4.3.6], there exists some non-zero $x \in X$ for which $\sigma_T(x)$ is empty. This leads to an obvious obstruction to (2) and hence also to (3), whenever S has SVEP, A is injective, and K contains $\sigma_{C(S,T)}(A)$. To counteract the effects of the absence of SVEP, we need another class of spectral subspaces, the so-called *glocal spectral subspaces* $\mathscr{X}_T(F)$, the definition of which simply bypasses this problem: for given $T \in L(X)$ and any closed $F \subseteq \mathbf{C}$, let

$$\mathscr{X}_T(F) := \{x \in X : \exists \text{ analytic } f : \mathbf{C} \backslash F \to X \text{ so that } (T - \lambda)f(\lambda) = x \ \ \forall \lambda \in \mathbf{C} \backslash F\}.$$

Note how the analytic functions in this definition are required to be defined globally outside the set F. These spectral subspaces date back to early work of Errett Bishop [**5**] and have been fundamental in the recent progress of local spectral theory, for instance in connection with functional models and invariant subspaces [**2**], [**7**] and also in the theory of spectral inclusions for operators on Banach spaces [**14**], [**15**], [**18**]. It is obvious that $\mathscr{X}_T(F) \subseteq X_T(F)$ for all closed $F \subseteq \mathbf{C}$ and that equality holds whenever T has SVEP. Further properties of the glocal spectral subspaces will be mentioned in Section 2.

Probably the most basic problem in the local spectral theory of commutators is to establish SVEP for the commutator $C(S,T)$ under suitable conditions on T and S. This problem will be addressed in Section 3. In particular, we shall see that $C(S,T)$ has SVEP if both T and S are semishifts, although (1), (2), and (3) are bound to fail in this situation.

We then proceed to the description of the spectral subspaces of the commutator in the spirit of (3), but with glocal spectral subspaces replacing the local ones. The main result of Section 4 provides the desired description of the *glocal* subspaces $\mathcal{L}_C(K)$ of the commutator $C:=C(S,T)$ for all *convex* closed sets $K\subseteq\mathbf{C}$, assuming only that T is the quotient of a decomposable operator, whereas S is completely arbitrary. This theorem implies, in particular, a remarkable generalization of the classical result due to Colojoară and Foiaş [**8**]: the equivalence (1), as it stands, remains valid, whenever T is a quotient of a decomposable operator and S has SVEP.

In Section 5, we strengthen the assumption on S to that of Dunford's property (C), thus requiring all of its spectral subspaces to be closed. With the same assumption on T as before, we shall prove that the local and glocal spectral subspaces of the commutator $C(S,T)$ coincide and that they allow the desired description of type (3) for arbitrary closed sets $K\subseteq\mathbf{C}$, not just for the convex ones. This implies, in particular, that $C(S,T)$ inherits Dunford's property (C) from S and extends the result of Foiaş and Vasilescu [**10**] to the considerably more general case of quotients and restrictions of decomposable operators. An essential tool for this extension will be the theory of analytical functional models developed by Albrecht and Eschmeier [**2**]. As an application of this theorem, we finally prove, in Section 6, that the commutator of two super-decomposable operators is decomposable.

2. Preliminaries from local spectral theory

Throughout this section, we fix a continuous linear operator $T\in L(X)$ on a non-trivial complex Banach space X. Recall that T is said to be *decomposable*, if any open cover $\mathbf{C}=U\cup V$ of the complex plane $\mathbf{C}$ by two open sets U and V yields a splitting of the spectrum $\sigma(T)$ and of the space X in the sense that there exist closed T-invariant linear subspaces Y and Z of X for which $\sigma(T\mid Y)\subseteq U$, $\sigma(T\mid Z)\subseteq V$, and $X=Y+Z$; see [**8**] and [**21**] for an account of the classical theory of decomposable operators.

As mentioned above, Albrecht and Eschmeier [**2**] have recently given intrinsic characterizations of restrictions and quotients of decomposable operators. They show that the operator $T\in L(X)$ is similar to the restriction of a decomposable operator to one of its closed invariant subspaces if and only if T has property (β), a condition first considered by Bishop [**5**]: *property* (β) may be expressed by the requirement that, for every open set $U\subseteq\mathbf{C}$ and every sequence of analytic functions $f_n:U\to X$ for which $(T-\lambda)f_n(\lambda)\to 0$ as $n\to\infty$, uniformly on compact subsets of U, it follows that $f_n(\lambda)\to 0$ as $n\to\infty$, again uniformly on compact subsets of U.

It is immediate that an operator with property (β) will have SVEP. Slightly less trivial, but still straightforward, is the observation that an operator $T\in L(X)$ with (β) will satisfy *Dunford's property* (C). The latter means that, for every closed $F\subseteq\mathbf{C}$, the corresponding local spectral subspace $X_T(F)$ is closed. Conceivably, (β) and (C) are equivalent, but, intriguingly, this is still an open problem. Incidentally, property (C) also entails SVEP. This is contained in Proposition 1 below, which also shows that property (C) is equivalent to the corresponding version for closedness of the glocal spectral subspaces.

The Albrecht–Eschmeier description of quotients of decomposable operators is in terms of the glocal subspaces: the operator $T \in L(X)$ is similar to a quotient of a decomposable operator with respect to a closed invariant subspaces if and only if T has the *decomposition property* (δ), which means that the splitting $X = \mathscr{X}_T(\bar{U}) + \mathscr{X}_T(\bar{V})$ holds for every open cover $\{U, V\}$ of $\mathbf{C}$.

It is not difficult to conclude from [**8**, Proposition 1.3.8] that T is decomposable if and only if T has both (δ) and (C). In particular, it follows that T is decomposable if and only if T has both (δ) and (β). Moreover, since restrictions and quotients of operators are dual concepts, it is no surprise that the Albrecht–Eschmeier descriptions from [**2**] encompass a duality, which completes the aspirations of the original Bishop inquiry: the operator $T \in L(X)$ has (δ) if and only if its adjoint $T^* \in L(X^*)$ on the dual space X^* has (β), and the corresponding statement remains valid if both properties are interchanged.

It will be useful to collect a few basic facts about the glocal subspaces in one place. Some notation will be needed. The closed disc in the complex plane with center $\lambda \in \mathbf{C}$ and radius $r \geqq 0$ will be denoted by $\nabla(\lambda, r)$, the corresponding open disc by $V(\lambda, r)$. Given $x \in X$, the quantity $r_T(x) := \limsup_{n \to \infty} \|T^n(x)\|^{1/n}$ is called the *local spectral radius* of T at x; the choice of language is justified by part (g) of the following result.

Proposition 1. *For any $T \in L(X)$, we have:*

(a) $\mathscr{X}_T(\emptyset) = \{0\}$ *and* $\mathscr{X}_T(F) = \mathscr{X}_T(F \cap \sigma(T))$ *for every closed* $F \subseteq \mathbf{C}$.

(b) $\mathscr{X}_{T-\lambda}(F) = \mathscr{X}_T(F + \lambda)$ *for every closed* $F \subseteq \mathbf{C}$ *and every* $\lambda \in \mathbf{C}$.

(c) $\mathscr{X}_T(\bigcap_\alpha F_\alpha) = \bigcap_\alpha \mathscr{X}_T(F_\alpha)$ *for any family* $(F_\alpha\}$ *of closed convex subsets of* $\mathbf{C}$.

(d) $\mathscr{X}_T(F) = X_T(F)$ *for all closed* $F \subseteq \mathbf{C}$ *if and only if T has SVEP.*

(e) $\mathscr{X}_T(F)$ *is closed for every closed* $F \subseteq \mathbf{C}$ *if and only if T has property* (C).

(f) $\mathscr{X}_T(\nabla(0, r)) = \{x \in X : r_T(x) \leqq r\}$ *for any* $r \geqq 0$.

(g) *If T has SVEP, then* $r_T(x) = \max\{|\lambda| : \lambda \in \sigma_T(x)\}$ *for all non-zero* $x \in X$.

(h) *If T has property* (C), *then T has SVEP.*

Proof. The assertions (a) and (b) are straightforward, (c) has been obtained in [**15**, Proposition 1.3], and (d) is part of [**14**, Proposition 1.1]. To show (e) and (h), we note that property (C) implies SVEP by [**14**, Proposition 1.2] and that the corresponding closedness condition on the glocal spectral subspaces implies SVEP by [**19**, Theorem 2.13]. The stated equivalence in (e) is then clear from (d). Finally, the identity in part (f) has been shown in [**18**, Proposition 2.1], and the assertion (g) follows immediately from (d) and (f).

We close this section with some remarks on a special class of operators, which will be useful to illustrate our general results on commutators. Recall from [**11**] that an isometry $T \in L(X)$ is said to be a *semishift*, if $\bigcap_{n=1}^{\infty} T^n(X) = \{0\}$. It is not hard to see that on Hilbert spaces this class of operators coincides with the pure isometries. Further natural examples include, for any $1 \leqq p \leqq \infty$, the unilateral right shift operators of arbitrary multiplicity on the sequence spaces $\ell^p(\mathbf{N})$ as well as the right translation operators on the Lebesgue spaces $L^p(\mathbf{R}_+)$ on the half line $\mathbf{R}_+$. Some of the local spectral theory of semishifts has been developed in [**17**], but mainly in the context of automatic continuity theory. Here, we shall need only the following observation.

Proposition 2. *If $T \in L(X)$ is a semishift, then T is not decomposable, but has property (β) and hence is similar to a restriction of a decomposable operator. Moreover, for any non-zero $x \in X$, we have $\sigma_T(x) = \nabla(0,1) = \sigma(T)$. Finally, for any closed $F \subseteq \mathbf{C}$, we have $\mathscr{X}_T(F) = X_T(F) = X$ if $\nabla(0,1) \subseteq F$, and $\mathscr{X}_T(F) = X_T(F) = \{0\}$ if $\nabla(0,1) \nsubseteq F$.*

Proof. First observe that, by [**12**, Proposition 1.19], any isometry has property (β) and therefore (C) and SVEP. Hence, given any non-zero $x \in X$, it follows from [**8**, Proposition 1.3.8] that $\sigma(T \mid X_T(\sigma_T(x))) = \sigma_T(x)$. Now suppose that $0 \in \rho_T(x)$ and choose an analytic function $f: U \to X$ on an open neighborhood U of 0 such that $(T - \lambda) f(\lambda) = x$ for all $\lambda \in U$. By [**8**, Proposition 1.1.2], we have $\sigma_T(x) = \sigma_T(f(0))$ and therefore $x \in T(X_T(\sigma_T(x)))$. Iteration of this argument yields that $x \in \bigcap_{n=1}^{\infty} T^n(X)$ and hence that $x = 0$ by the semishift property. This contradiction shows that $0 \in \sigma_T(x) = \sigma(T \mid X_T(\sigma_T(x)))$, which makes $T \mid X_T(\sigma_T(x))$ a non-invertible isometry. Since the spectrum of such isometries is known to be the closed unit disc, we conclude that $\sigma_T(x) = \nabla(0,1) = \sigma(T)$. This shows that a semishift cannot be decomposable and implies also the stated description of its spectral subspaces.

Note that it follows, in particular, that semishifts do not have property (δ). It is now easy to see that the results mentioned in the introduction do not hold in general.

Example. The equivalence (1) does not extend to the case that $T \in L(X)$ is a semishift. Indeed, in this case, it is immediate from Proposition 2 that the inclusion $\sigma_S(Ax) \subseteq \sigma_T(x)$ holds for all $x \in X$ and all $A \in L(X, Y)$, provided only that the operator $S \in L(Y)$ satisfies $\sigma(S) \subseteq \nabla(0,1)$. However, if $S = 0$ and A is bounded below, then clearly, by the spectral radius formula, $\|C(S,T)^n(A)\|^{1/n} = \|AT^n\|^{1/n} \to 1$ as $n \to \infty$, which shows that, in general, the left hand side of (1) does not imply the right hand side. We note in passing that the reverse implication in (1) does hold without any restriction on the operators involved; this is clear from the proof of the classical result [**8**, Theorem 2.3.3] and follows also from [**14**, Proposition 2.2].

Similarly, neither (2) not (3) is valid, when $T \in L(X)$ is a semishift and $S \in L(Y)$ is the zero operator. More precisely, in this situation, the space on the right hand side of (3) coincides with the whole space $L(X, Y)$, whenever the closed set $K \subseteq \mathbf{C}$ satisfies $K \cap \nabla(0,1) \neq \emptyset$. However, the identity $L_C(K) = L(X, Y)$ can only hold for the commutator $C := C(0, T)$, when K contains the entire unit circle, since, by elementary local spectral theory, the surjectivity spectrum $\sigma_{su}(C) := \{\lambda \in \mathbf{C} : C - \lambda \text{ is not surjective}\}$ of the commutator C contains the boundary of $\sigma(C) = -\sigma(T) = \nabla(0,1)$ and since $\sigma_{su}(C) = \sigma_C(A)$ for at least one $A \in L(X, Y)$, see [**16**, Lemmas 1 and 2]. Consequently, in this case, (2) and (3) do not hold for any closed set $K \subseteq \mathbf{C}$ which touches the unit disc, but does not contain the unit circle.

In the following section, we shall modify these ideas to obtain counterexamples of this type, even when the commutator $C(S, T)$ has SVEP.

3. The single valued extension property for $C(S, T)$

Our standard situation in this section will be that $T \in L(X)$ and $S \in L(Y)$ are given operators on complex Banach spaces X and Y, respectively, and that $C := C(C, T)$ is their commutator on the Banach space $L(X, Y)$. It is routine to verify that

$$C^n(A) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A T^k \quad \text{for all } n \in \mathbf{N} \text{ and } A \in L(X, Y). \tag{4}$$

If $A \neq 0$ and C has SVEP, then the local spectral radius formula of Propostion 1 shows that $r_C(A) = 0$ if and only if $\sigma_C(A) = \{0\}$. Since [**10**] proves that C has SVEP, when both T and S are decomposable, this means that the equivalence (1) is indeed a special case of (2), as implied earlier.

We shall now uncover more general conditions under which SVEP for the commutator can be obtained. The following result is implicit in the proof of [**14**, Theorem 2.4], but for completeness we include a short argument.

Lemma 3. *Suppose that the operator $S \in L(Y)$ has property (C) and that, for a given $\varepsilon > 0$, the operator $T \in L(X)$ satisfies the condition that the linear span of the set of all $x \in X$ with $\operatorname{diam} \sigma_T(x) \leqq \varepsilon$ is dense in X. If $H : U \to L(X, Y)$ is analytic on an open disc U of radius ε and if $(C - \lambda)H(\lambda) = 0$ for all $\lambda \in U$, then $H \equiv 0$ on U.*

Proof. Given any $\lambda \in U$, we have $(S - \lambda)H(\lambda) = H(\lambda)T$. For all $x \in X$, it follows that $\sigma_{S-\lambda}(H(\lambda)x) \subseteq \sigma_T(x)$ and hence $H(\lambda)x \in Y_{S-\lambda}(\sigma_T(x)) = Y_S(\sigma_T(x) + \lambda)$. Now choose two closed discs $D_1, D_2 \subseteq U$ with non-empty interiors and distance strictly greater than ε. For $k = 1, 2$ and arbitrary $x \in X$, we obtain that $H(\lambda)x \in Y_S(\sigma_T(x) + D_k)$ for all $\lambda \in D_k$ and consequently, by property (C) and the identity theorem for analytic functions, even for all $\lambda \in U$. Thus $H(\lambda)x \in Y_S((\sigma_T(x) + D_1) \cap (\sigma_T(x) + D_2))$ for all $\lambda \in U$. Since $\sigma_T(x) + D_1$ and $\sigma_T(x) + D_2$ are disjoint whenever $\operatorname{diam} \sigma_T(x) \leqq \varepsilon$ and since $Y_S(\emptyset) = \{0\}$ by SVEP, we conclude that $H(\lambda)x = 0$ for all $\lambda \in U$ and all $x \in X$ with $\operatorname{diam} \sigma_T(x) \leqq \varepsilon$. The assumption on T now implies that $H \equiv 0$ on U.

Proposition 4. *If $S \in L(Y)$ has property (C) and $T \in L(X)$ has property (δ), then $C(S, T)$ has SVEP.*

Proof. Because of (δ), we know that T is the quotient of some decomposable operator R on a suitable Banach space Z. Given an arbitrary $\varepsilon>0$, choose finitely many open discs of diameter ε so that $\sigma(R)\subseteq U_1\cup\cdots\cup U_n$. By [**21**, Theorem 4.4.28], it follows that $Z=Z_R((\overline{U_1})+\cdots+Z_R(\overline{U_n})$ and hence that $X=X_T(\overline{U_1})+\cdots+X_T(\overline{U_n})$, which shows that T satisfies the condition of Lemma 3. The assertion follows.

Property (δ) is not the only spectral decomposition condition on T which, in conjunction with property (C) for S, will ensure the applicability of Lemma 3. For instance, it follows easily from this lemma that $C(S,T)$ will have SVEP, whenever S has property (C) and T has the *weak spectral decomposition property*. This property means that, for every open cover $\{U_1,\ldots,U_n\}$ of $\mathbf{C}$, there exist closed T-invariant linear subspaces X_k of X, for which $\sigma(T|X_k)\subseteq U_k$ for $k=1,\ldots,n$ and for which the sum $X_1+\cdots+X_n$ is dense in X. Note that the example of Albrecht [**1**] shows that operators with the weak spectral decomposition property need not have property (δ), even if one insists on property (C) as an additional assumption. Conversely, it has been observed in [**14**, Example 1.5] that the unilateral left shift on the Hilbert space $\ell^2(\mathbf{N})$ has property (δ), but not the weak spectral decomposition property.

We shall now describe a rather different situation, in which we also may conclude that the commutator has SVEP. Recently, Sun [**20**] has obtained the remarkable result that the sum and product of two commuting continuous linear operators, both with property (C), will have SVEP. This is based on the following interesting lemma from [**20**], restated here with the minimal assumptions available: property (C) is needed only for S, not for T.

Lemma 5. *Suppose that $S,T\in L(X)$ are commuting operators of which S has property (C) and that Z is a closed linear subspace of X, which is hyperinvariant for T. Suppose also that $f:U\to Z$ is an analytic function on some open disc U of radius $\varepsilon>0$ such that $(S+T-\lambda)f(\lambda)=0$ for all $\lambda\in U$. Then $\operatorname{diam}\sigma(T|Z)<2\varepsilon$ implies that $f\equiv0$ on U, while $\operatorname{diam}\sigma(T|Z)\geqq2\varepsilon$ implies that there exists a closed set $F\subseteq\mathbf{C}$ with $\operatorname{diam}F\leqq\operatorname{diam}\sigma(T|Z)-2\varepsilon$ such that $f(\lambda)\in X_S(F)$ for all $\lambda\in U$.*

We mention this somewhat technical looking result, because it has an interesting consequence for perturbations of semishifts. Note that the only condition on the perturbation T in the following result involves the size of its spectrum.

Proposition 6. *Let $S,T\in L(X)$ be commuting operators such that S is a semishift and that $\operatorname{diam}\sigma(T)\leqq2$. Then $S+T$ has SVEP.*

Proof. Consider an analytic function $f:U\to Y$ on an open disc U such that $(S+T-\lambda)f(\lambda)=0$ for all $\lambda\in U$, let ε denote the radius of U, and apply the preceding lemma with $Z=Y$. The case $\operatorname{diam}\sigma(T)<2\varepsilon$ brings us directly to the conclusion desired for SVEP, so we may suppose that $\operatorname{diam}\sigma(T)\geqq2\varepsilon$. Thus we obtain a closed set $F\subseteq\mathbf{C}$ with $\operatorname{diam}F\leqq\operatorname{diam}\sigma(T)-2\varepsilon<2$ for which $f(\lambda)\in X_S(F)$ for all $\lambda\in U$. But $\operatorname{diam}F<2$

implies that $\nabla(0,1)\not\subseteq F$ and therefore $X_S(F)=\{0\}$ by Proposition 2. We conclude that $f\equiv 0$ on U.

In Proposition 6, the condition $\operatorname{diam}\sigma(T)\leqq 2$ is sharp in the following sense: as soon as the commutant of S contains at least one operator V without SVEP, we have an example to show that this condition cannot be weakened. Indeed, we may suppose that $\|V\|=1$ and then take $T:=\varepsilon V-S$ for any chosen $\varepsilon>0$. This operator T will have $\operatorname{diam}\sigma(T)\leqq 2+\varepsilon$, yet $S+T$ will not have SVEP.

The following easy consequence of Proposition 6 supplements our list of results on SVEP for commutators.

Corollary 7. *If $S\in L(Y)$ is a semishift and $T\in L(X)$ satisfies* $\operatorname{diam}\sigma(T)\leqq 2$, *then $C(S,T)$ has SVEP.*

Proof. Consider the multiplication operators L_S and R_T on $L(X,Y)$ given by $L_S(A):=SA$ and $R_T(A):=AT$ for all $A\in L(X,Y)$. Then L_S is a semishift and $\operatorname{diam}\sigma(R_T)\leqq 2$. Since L_S and R_T commute and satisfy $C(S,T)=L_S-R_T$, it follows from Proposition 6 that $C(S,T)$ has SVEP.

Proposition 8. *If $T\in L(X)$ and $S\in L(Y)$ are arbitrary semishifts, then T and S are restrictions of decomposable operators and the commutator $C:=C(S,T)$ has SVEP, but none of the statements* (1), (2), *or* (3) *hold in this situation.*

Proof. The first assertions are clear from Proposition 2 and Corollary 7. Next observe that $\sigma(C)=\sigma_{su}(C)=\nabla(0,2)$. Indeed, elementary spectral theory shows that $\sigma(C)\subseteq\sigma(L_S)-\sigma(R_T)=\sigma(S)-\sigma(T)=\nabla(0,2)$, and conversely it follows from [**9**, Theorem 5] that $\sigma(C)\supseteq\sigma_{su}(C)\supseteq\sigma_{su}(S)-\sigma_{ap}(T)=\nabla(0,2)$, since, for any non-invertible isometry, the surjectivity spectrum is the unit disc and the approximate point spectrum is the unit circle. Now, if $K\subseteq\mathbf{C}$ is any closed set with $0\in K$, then it is clear from Proposition 2 that every $A\in L(X,Y)$ satisfies $AX_T(F)\subseteq Y_S(\overline{F+K})$ for all closed $F\subseteq\mathbf{C}$. On the other hand, it follows from [**16**, Lemma 2] that $L_C(K)=L(X,Y)$ holds if and only if K contains $\sigma(C)=\sigma_{su}(C)=\nabla(0,2)$. We conclude that here the identity (3) ceases to be true in a rather drastic way: it does not hold for any closed set $K\subseteq\mathbf{C}$ which contains the origin, but not the entire spectrum $\sigma(C)=\nabla(0,2)$. In particular, it follows that the equivalences (1) and (2) cannot hold in the present situation.

Example. It is also easy to give a more concrete counterexample in this setting: take $T=S$ to be the right shift and A to be the left shift on the Hilbert space $X:=Y:=\ell^2(\mathbf{N})$. Then it is easily seen that $C(T,T)^n(A)=-T^{n-1}P$ for all $n\in\mathbf{N}$, where P denotes the projection onto the first component. It follows that $\|C(T,T)^n(A)\|=1$ for all $n\in\mathbf{N}$ and therefore $r_{C(T,T)}(A)=1$, although $\sigma_T(Ax)\subseteq\sigma_T(x)$ for all $x\in X$ by Proposition 2. This illustrates how (1), (2), and (3) fail to hold in this situation.

4. Spectral subspaces of $C(S, T)$: the convex case

The first observation of this section is simple enough and seemingly innocuous. However, as we shall see in the proof of Theorem 10, it has quite far-reaching consequences. As before, let X and Y denote complex Banach spaces.

Lemma 9. *If $r \geqq 0$ and if the operators $T_n \in L(X, Y)$ for all $n \in \mathbf{N}$ satisfy the condition that $\limsup_{n\to\infty} \|T_n x\|^{1/n} \leqq r$ for all $x \in X$, then $\limsup_{n\to\infty} \|T_n\|^{1/n} \leqq r$.*

Proof. Let $\varepsilon > 0$ be given. Since $\limsup_{n\to\infty} \|T_n x\|^{1/n} \leqq r$ for all $x \in X$, it follows that, for each $x \in X$, there is a number $M_x \geqq 1$ such that $\|T_n x/(r+\varepsilon)^n\| \leqq M_x$ for all $n \in \mathbf{N}$. The uniform boundedness principle then implies the existence of a number $M \geqq 1$ for which $\|T_n/(r+\varepsilon)^n\| \leqq M$ for all $n \in \mathbf{N}$, and from this we conclude that $\limsup_{n\to\infty} \|T_n\|^{1/n} \leqq r + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the claim of the lemma follows.

An immediate application of Lemma 9 is the following: if we take $T_n := T^n$ for all $n \in \mathbf{N}$ with the arbitrary operator $T \in L(X)$, then we obtain for the spectral radius $r(T)$ that $r(T) = \sup_{x \in X} r_T(x)$. Of course, this relationship between the spectral radius and its local counterpart may also be seen in other ways.

Given two operators $T \in L(X)$ and $S \in L(Y)$, we now proceed to characterize the *glocal* spectral subspaces $\mathscr{L}_C(K)$ of the commutator $C := C(S, T)$ on the space $L := L(X, Y)$ for certain closed sets $K \subseteq \mathbf{C}$. Note that the spaces $\mathscr{L}_C(K)$ coincide with the corresponding local spectral subspaces $L_C(K)$, whenever C has SVEP. In (3) the analogue of one of the inclusions holds in general, if phrased in terms of the glocal subspaces. This version has been obtained in [**15**, Proposition 2.1] by a refinement of the Foiaş–Vasilescu methods [**10**] and asserts that

$$\mathscr{L}_C(K) \subseteq \{A \in L(X, Y) : A\mathscr{X}_T(F) \subseteq \mathscr{Y}_S(\overline{F+K}) \text{ for all closed } F \subseteq \mathbf{C}\}, \tag{5}$$

without any particular assumption on the operators T and S and the closed set $K \subseteq \mathbf{C}$. From Proposition 8, we know that the inclusion (5) will be strict in general, even if T is the restriction of a decomposable operator. However, the dual condition on T, namely that of being the quotient of a decomposable operator, together with a geometric restriction on the set K, will allow us to obtain the following positive result for completely arbitrary $S \in L(Y)$.

Theorem 10. *If $T \in L(X)$ has property (δ) then, for any $S \in L(Y)$ and any closed convex set $K \subseteq \mathbf{C}$, we have that*

$$\mathscr{L}_C(K) = \{A \in L(X, Y) : A\mathscr{X}_T(F) \subseteq \mathscr{Y}_S(\overline{F+K}) \textit{ for all closed } F \subseteq \mathbf{C}\}. \tag{6}$$

Proof. Clearly, this result covers the case where K is a closed disc, and indeed our proof is based on establishing the equality for such sets first. We start with the special case that K is a closed disc of the form $K=\nabla(0,r)$ for some $r\geqq 0$. By (5), it suffices to verify the inclusion $\supseteq$ of (6). Hence let $A\in L(X,Y)$ be any operator such that $A\mathscr{X}_T(F)\subseteq\mathscr{Y}_S(F+\nabla(0,r))$ for all closed $F\subseteq\mathbf{C}$. By part (f) of Proposition 1, we have to prove that $r_C(A)\leqq r$. Since we assume the operator T to have property (δ), we know from [2] that there exist a decomposable operator $R\in L(Z)$ on some Banach space Z and a surjection $Q\in L(Z,X)$ such that $TQ=QR$. Let $\varepsilon>0$ be given and choose, by compactness, finitely many $\lambda_1,\ldots,\lambda_m\in\mathbf{C}$ such that $\sigma(R)\subseteq V(\lambda_1,\varepsilon)\cup\cdots\cup V(\lambda_m,\varepsilon)$. Let $Z_i:=Z_R(\nabla(\lambda_i,\varepsilon))$ and note that $Z_i=Z_{R-\lambda_i}(\nabla(0,\varepsilon))$ for $i=1,\ldots,m$, by part (b) of Proposition 1. Since R is decomposable, we have $Z=Z_1+\cdots+Z_m$, and each of the spaces Z_i for $i=1,\ldots,m$ is closed and invariant under R. Moreover, if we let $R_i=(R-\lambda_i)\,|\,Z_i$, then $\sigma(R_i)\subseteq\nabla(0,\varepsilon)$ for $i=1,\ldots,m$, for instance, by [**8**, Proposition 1.3.8]. Hence, by the formula for the spectral radius, there exists a number $M_\varepsilon\geqq 1$ such that $\|R_i^n\|\leqq M_\varepsilon(2\varepsilon)^n$ for each $i=1,\ldots,m$ and all $n\in\mathbf{N}$. Next, let $Y_i:=\mathscr{Y}_{S-\lambda_i}(\nabla(0,\varepsilon+r))=\mathscr{Y}_S(\nabla(\lambda_i,\varepsilon+r))$ and $S_i:=(S-\lambda_i)\,|\,Y_i$ for $i=1,\ldots,m$. Since the condition $TQ=QR$ obviously yields $QZ_i\subseteq\mathscr{X}_T(\nabla(\lambda_i,\varepsilon))$, it follows from our assumption on A that $AQZ_i\subseteq Y_i$ for $i=1,\ldots,m$. Thus part (f) of Proposition 1 implies that $\limsup_{n\to\infty}\|S_i^nAQz\|^{1/n}\leqq r+\varepsilon$ for all $z\in Z_i$ and $i=1,\ldots,m$. Consequently, from Lemma 9 applied to $T_n:=S_i^nAQ\,|\,Z_i$ for all $n\in\mathbf{N}$, we obtain a number $N_\varepsilon\geqq 1$ such that $\|S_i^nAQ\,|\,Z_i\|\leqq N_\varepsilon(r+2\varepsilon)^n$ for each $i=1,\ldots,m$ and all $n\in\mathbf{N}$. Now, let $x\in X$ be arbitrarily given and write $x=Qz_1+\cdots+Qz_m$, where $z_i\in Z_i$ for $i=1,\ldots,m$. Then, for each $n\in\mathbf{N}$, we obtain, via (4), that

$$\begin{aligned}\|C(S,T)^n(A)x\| &\leqq \sum_{i=1}^m \|C(S_i,R_i)^n(AQ)z_i\|\\ &=\sum_{i=1}^m\left\|\sum_{k=0}^n\binom{n}{k}(-1)^kS_i^{n-k}(AQ)R_i^kz_i\right\|\\ &\leqq\sum_{i=1}^m\sum_{k=0}^n\binom{n}{k}\|S_i^{n-k}AQ\,|\,Z_i\|\,\|R_i^k\|\,\|z_i\|\\ &\leqq\sum_{i=1}^m\sum_{k=0}^n\binom{n}{k}M_\varepsilon N_\varepsilon(r+2\varepsilon)^{n-k}(2\varepsilon)^k\|z_i\|\\ &=(r+4\varepsilon)^nM_\varepsilon N_\varepsilon\sum_{i=1}^m\|z_i\|.\end{aligned}$$

It follows that $\limsup_{n\to\infty}\|C(S,T)^n(A)x\|^{1/n}\leqq r+4\varepsilon$ for each $x\in X$. Another application of Lemma 9 yields that $\limsup_{n\to\infty}\|C(S,T)^n(A)\|^{1/n}\leqq r+4\varepsilon$. Since $\varepsilon>0$ is arbitrary, it follows that $r_C(A)\leqq r$. By part (f) of Proposition 1, this means precisely that $A\in\mathscr{L}_C(\nabla(0,r))$, which completes the proof of (6) for the special case of a disc of the form $K=\nabla(0,r)$.

Next, given an arbitrary closed disc $K=\nabla(\lambda,r)$ centered at $\lambda\in\mathbf{C}$, we obtain, via part (b) of Proposition 1, that

$$\begin{aligned}\mathscr{L}_{C(S,T)}(\nabla(\lambda,r)) &= \mathscr{L}_{C(S,T)-\lambda}(\nabla(0,r)) = \mathscr{L}_{C(S-\lambda,T)}(\nabla(0,r))\\ &= \{A\in L(X,Y): A\mathscr{X}_T(F)\subseteq \mathscr{Y}_{S-\lambda}(F+\nabla(0,r)) \text{ for all closed } F\subseteq \mathbf{C}\}\\ &= \{A\in L(X,Y): A\mathscr{X}_T(F)\subseteq \mathscr{Y}_S(F+\nabla(\lambda,r)) \text{ for all closed } F\subseteq \mathbf{C}\}.\end{aligned}$$

This shows that (6) holds for all closed discs.

Now, let K be any closed, convex and bounded subset of $\mathbf{C}$. In this case, K is the intersection of all the closed discs containing it, i.e. we have $K=\bigcap_\alpha K_\alpha$ for a family of closed discs $K_\alpha\subseteq\mathbf{C}$. To establish the inclusion $\supseteq$ of (6) in this case, consider an arbitrary operator $A\in L(X,Y)$, for which $A\mathscr{X}_T(F)\subseteq\mathscr{Y}_S(F+K)$ for all closed $F\subseteq\mathbf{C}$. Then we have $A\mathscr{X}_T(F)\subseteq\mathscr{Y}_S(F+K_\alpha)$ for every α and every closed set $F\subseteq\mathbf{C}$. Hence, by the preceding part of the proof, we see that $A\in\mathscr{L}_C(K_\alpha)$ for every α and therefore $A\in\bigcap_\alpha\mathscr{L}_C(K_\alpha)$. Since all the sets K_α are convex, it follows from part (c) of Proposition 1 that the latter space equals $\mathscr{L}_C(K)$. Thus $A\in\mathscr{L}_C(K)$, which, together with (5), completes the proof of the identity (6) in the case of bounded sets.

The final step consists in reducing the general case to the bounded one. This may be done as follows: given an arbitrary closed convex set $K\subseteq\mathbf{C}$, we choose a radius $s\geqq 0$ large enough that $\sigma(T)\cup\sigma(S)\subseteq\nabla(0,s)$. Then $K\cap\nabla(0,2s)$ is closed, convex and bounded. If $A\in L(X,Y)$ has the property that $A\mathscr{X}_T(F)\subseteq\mathscr{Y}_S(\overline{F+K})$ for all closed sets $F\subseteq\mathbf{C}$, then it follows from part (a) of Proposition 1 that

$$\begin{aligned}A\mathscr{X}_T(F) &= A\mathscr{X}_T(F\cap\sigma(T)) = A\mathscr{X}_T(F\cap\nabla(0,s))\subseteq\mathscr{Y}_S((F\cap\nabla(0,s))+K)\\ &= \mathscr{Y}_S(((F\cap\nabla(0,s))+K)\cap\nabla(0,s))\subseteq\mathscr{Y}_S(F+(K\cap\nabla(0,2s)))\end{aligned}$$

for all closed $F\subseteq\mathbf{C}$. Consequently, we obtain from the result in the bounded case that $A\in\mathscr{L}_C(K\cap\nabla(0,2s))\subseteq\mathscr{L}_C(K)$. By (5), this completes the proof of the theorem.

The following is a general version of the equivalence (1).

Corollary 11. *If $T\in L(X)$ has property (δ) and $S\in L(Y)$ has SVEP, then the following assertions are equivalent:*

(a) $AX_T(F)\subseteq Y_S(F)$ *for all closed* $F\subseteq\mathbf{C}$.

(b) $A\mathscr{X}_T(F)\subseteq\mathscr{Y}_S(F)$ *for all closed* $F\subseteq\mathbf{C}$.

(c) $r_{C(S,T)}(A)=0$.

Proof. With $K:=\{0\}$, the equivalence of (b) and (c) is immediate from Theorem 10 and Proposition 1, while (c) always implies (a) by [**14**, Proposition 2.2]. Finally, since S is assumed to have SVEP, assertion (a) implies that $A\mathscr{X}_T(F)\subseteq AX_T(F)\subseteq Y_S(F)=\mathscr{Y}_S(F)$ for all closed $F\subseteq\mathbf{C}$ and hence (b).

5. Spectral subspaces of $C(S,T)$: the general case

The purpose of this section is to obtain, for suitable operators T and S, the description (6) of the spectral subspaces $\mathscr{L}_C(K)$ of the commutator $C:=C(S,T)$ without any geometric condition on the underlying closed subset K of the complex plane. As mentioned earlier, Foiaş and Vasilescu have established formula (3) for arbitrary compact sets $K\subseteq\mathbf{C}$, provided that both T and S are decomposable; see [**10**, Theorem 2.5] and also [**21**, Theorem 4.6.7]. Examination of their proof reveals that the condition on S may be relaxed to that of assuming only Dunford's property (C). Moreover, using the argument from the last step in the proof of Theorem 10, it is easily seen that the formula extends from compact sets to arbitrary closed subsets. It follows that the identity (6) holds for arbitrary closed $K\subseteq\mathbf{C}$, whenever T is decomposable and S has property (C).

In Theorem 13, we shall extend this result to the case that T has property (δ) and S has property (C). It seems that the very long and technical argument of [**10**] is hard to mimic in this more general situation, but fortunately the Albrecht–Eschmeier characterization [**2**] of property (δ) suggests another natural approach. In fact, if T has (δ), then T is similar to the quotient of a decomposable operator R on a suitable Banach space Z, hence the classical Foiaş–Vasilescu result will apply to the commutator $C(S,R)$, and it remains to compare the local spectra of $C(S,R)$ and $C(S,T)$. The last step turns out to be a bit harder than one might expect and will require a closer look at the analytic functional models developed by Albrecht and Eschmeier [**2**].

A diagram will illustrate what is going on. We assume that X, Y, and Z are complex Banach spaces and that $T\in L(X)$, $S\in L(Y)$, and $R\in L(Z)$ are given operators. Moreover, we suppose that T is a quotient of R; thus there exists a surjective operator $Q\in L(Z,X)$, which intertwines in the sense that $TQ=QR$. Let the kernel of Q be denoted by Z_0. Then Z_0 is invariant under R, and we may define $R_0:=R\,|\,Z_0\in L(Z_0)$. Finally, let $i\in L(Z_0,Z)$ be the canonical injection and consider an arbitrary operator $A\in L(X,Y)$. Note that the following diagram commutes on the left side and in the middle, but there is no such restriction on the right.

$$\begin{array}{cccc} Z_0 \xrightarrow{i} & Z \xrightarrow{Q} & X \xrightarrow{A} & Y \\ R_0\downarrow & R\downarrow & T\downarrow & S\downarrow \\ Z_0 \xrightarrow{i} & Z \xrightarrow{Q} & X \xrightarrow{A} & Y \end{array} \tag{7}$$

Lemma 12. *If, in diagram* (7), *the commutator* $C(S,R_0)$ *has SVEP, then we have* $\sigma_{C(S,R)}(AQ)=\sigma_{C(S,T)}(A)$. *In particular, this identity holds whenever* R_0 *has property* (δ) *and* S *has property* (C).

Proof. By Proposition 4, the last statement is indeed a consequence of the first. As pointed out to the authors by the referee, the main claim is immediate from the following simple principle, which has to be applied to suitable commutators: if

$$0\longrightarrow X\xrightarrow{j} Y\xrightarrow{q} Z$$

is an exact sequence of continuous linear operators between Banach spaces which

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intertwines the operators $T \in L(X)$, $S \in L(Y)$, and $R \in L(Z)$ and if the operator R has SVEP, then $\sigma_S(j(x)) = \sigma_T(x)$ holds for all $x \in X$.

It is not hard to see that $\sigma_{C(S,R)}(AQ)$ is always contained in $\sigma_{C(S,T)}(A)$ and that these two sets have the same polynomially convex hull whenever $C(S,R)$ has SVEP. The following example will show, however, that, in (7), we may well have $\sigma_{C(S,R)}(AQ) \neq \sigma_{C(S,T)}(A)$, even if both R and S are decomposable. Note that, in this case, $C(S,R)$ will have SVEP by Proposition 4, while $C(S,R_0)$ cannot have SVEP by Lemma 12.

Example. Consider the Hilbert spaces $X := Y := \ell^2(\mathbf{N})$ and $Z := \ell^2(\mathbf{Z})$, let $T \in L(X)$ and $R \in L(Z)$ denote the left shifts on X and Z, respectively, and let $Q \in L(Z,X)$ be the canonical surjection such that $TQ = QR$. Then R is a unitary operator and therefore decomposable, and it is well-known and easily seen that $\sigma(R) = \mathbf{T}$, the unit circle. If we take S to be the zero operator on $\ell^2(\mathbf{N})$, then $C(S,R)$ becomes right multiplication by $-R$ on the space $L(Z,X)$, which implies that $\sigma(C(S,R)) = -\sigma(R) = \mathbf{T}$. In particular, it follows that $\sigma_{C(S,R)}(AQ) \subseteq \sigma(C(S,R)) = \mathbf{T}$ for all $A \in L(X)$. On the other hand, note that T is a quotient of the decomposable operator R and therefore has property (δ). Moreover, $C(S,T)$ is the operator of right multiplication by $-T$ on $L(X)$, and from this it is easily seen that $C(S,T)$ is a semishift. By Proposition 2, we conclude that $\sigma_{C(S,T)}(A) = \nabla(0,1) = \sigma(C(S,T))$ for all non-zero $A \in L(X)$. Combining these observations, we obtain that $\sigma_{C(S,R)}(AQ)$ is strictly contained in $\sigma_{C(S,T)}(A)$, whenever $A \in L(X)$ is non-zero.

We now state and prove the main result of this section.

Theorem 13. *Suppose that $T \in L(X)$ has property (δ) and that $S \in L(Y)$ has property (C). Then the commutator $C := C(S,T)$ on the space $L := L(X,Y)$ has property (C), and, for each closed $K \subseteq \mathbf{C}$, we have that*

$$L_C(K) = \mathscr{L}_C(K) = \{A \in L(X,Y) : A\mathscr{X}_T(F) \subseteq \mathscr{Y}_S(\overline{F+K}) \text{ for all closed } F \subseteq \mathbf{C}\}.$$

Proof. We first employ the Albrecht–Eschmeier functional model given in [2, Theorem 2.9]: for any operator $T \in L(X)$, there exist Banach spaces Z_1 and Z, a decomposable operator $R_1 \in L(Z_1)$, an operator $R \in L(Z)$ with property (β), and operators $J \in L(Z_1, Z)$ and $Q \in L(Z,X)$ so that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc} 0 \to & Z_1 & \xrightarrow{J} & Z & \xrightarrow{Q} & X & \to 0 \\ & R_1\downarrow & & R\downarrow & & T\downarrow & \\ 0 \to & Z_1 & \xrightarrow{J} & Z & \xrightarrow{Q} & X & \to 0 \end{array}$$

Remarkably, the Albrecht–Eschmeier model holds for any T, and one may even choose R_1 to be generalized scalar and R to be a restriction of a generalized scalar operator, but this additional information will not be needed here. Since we assume that our given operator T has property (δ), we conclude from [2, Lemma 1.2] that R has (δ) as well. But R already has property (β) and hence becomes decomposable. Moreover, by exactness, the operator $R_0 := R|\ker Q$ is similar to R_1 and hence also decomposable. This means that Lemma 12 applies to the present situation. Now, to prove the theorem, we first note that $C(S,T)$ has SVEP by Proposition 4 and hence that $L_{C(S,T)}(K) = \mathscr{L}_{C(S,T)}(K)$ for each closed $K \subseteq \mathbf{C}$. It remains to show the converse inclusion in (5). Let $K \subseteq \mathbf{C}$ be a given closed set, and suppose that $A \in L(X,Y)$ satisfies $A\mathscr{X}_T(F) \subseteq \mathscr{Y}_S(\overline{F+K})$ for every closed set $F \subseteq \mathbf{C}$. Since $TQ = QR$, we have that $QZ_R(F) = Q\mathscr{X}_R(F) \subseteq \mathscr{X}_T(F)$ and hence $AQZ_R(F) \subseteq \mathscr{Y}_S(\overline{F+K}) = Y_S(\overline{F+K})$ for all closed $F \subseteq \mathbf{C}$. By the Foiaş–Vasilescu result [**10**, Theorem 2.5], we see that $AQ \in L_{C(S,R)}(K)$ and thus $\sigma_{C(S,R)}(AQ) \subseteq K$. But from Lemma 12 we know that $\sigma_{C(S,R)}(AQ) = \sigma_{C(S,T)}(A)$ and therefore $A \in L_{C(S,T)}(K)$. Using (5), we thus obtain the desired description of the spectral subspaces of $C(S,T)$. This characterization shows, in particular, that $C(S,T)$ inherits property (C) from S. The theorem is proved.

Remark. As an immediate consequence of the preceding result, we obtain descriptions of the spectral subspaces for the right and left multiplication operators R_T and L_S on the space $L := L(X,Y)$, where, as before, $R_T(A) := AT$ and $L_S(A) := SA$ for all $A \in L(X,Y)$. Indeed, if $T \in L(X)$ has property (δ), then Theorem 13 shows that R_T has property (C) and that the identity

$$L_{R_T}(K) = \{A \in L(X,Y) : A\mathscr{X}_T(F) = \{0\} \text{ for all closed } F \subseteq \mathbf{C}\backslash K\} \tag{8}$$

holds for each closed $K \subseteq \mathbf{C}$. Similarly, if $S \in L(Y)$ has property (C), then the left multiplication operator L_S has property (C), and, for each closed $K \subseteq \mathbf{C}$, we have

$$L_{L_S}(K) = \{A \in L(X,Y) : AX \subseteq Y_S(K)\}. \tag{9}$$

It is interesting to note that the last formula remains valid under the weaker assumption that S has SVEP. In fact, in this case, it is straightforward to verify the L_S has SVEP and that the inclusion $\subseteq$ holds in (9). To see the converse, let $K \subseteq \mathbf{C}$ be closed and $A \in L(X,Y)$ be such that $AX \subseteq Y_S(K)$. If H_K denotes the Fréchet space of all analytic functions $f : \mathbf{C}\backslash K \to Y$, we obtain a mapping $F : X \to H_K$ such that $(S-\lambda)(F(x)(\lambda)) = Ax$ for all $x \in X$ and all $\lambda \in \mathbf{C}\backslash K$. Since S has SVEP, it follows that F is linear. Moreover, if $x_n \in X$ and $f \in H_K$ are given such that $x_n \to 0$ in X and $F(x_n) \to f$ in the topology of H_K, then, for each fixed $\lambda \in \mathbf{C}\backslash K$, we have that $(S-\lambda)f(\lambda) = \lim_{n\to\infty}(S-\lambda)(F(x_n)(\lambda)) = \lim_{n\to\infty} Ax_n = 0$ and therefore $f \equiv 0$ on $\mathbf{C}\backslash K$, again by SVEP. The closed graph theorem now implies that F is continuous. In particular, it follows that, for each $\lambda \in \mathbf{C}\backslash K$, the operator $F_\lambda : X \to Y$, given by $F_\lambda(x) := F(x)(\lambda)$ for all $x \in X$, belongs to $L(X,Y)$. Since $(L_S - \lambda)F_\lambda = A$ for all $\lambda \in \mathbf{C}\backslash K$, it remains to be seen that F_λ depends analytically on $\lambda \in \mathbf{C}\backslash K$. By Morera's theorem, it suffices to show that this operator function is norm continuous. Hence, let $\lambda \in \mathbf{C}\backslash K$ and choose $r > 0$ such that $\nabla(\lambda, 2r) \subseteq \mathbf{C}\backslash K$. Then, for each fixed $x \in X$, a standard application of Cauchy's integral formula leads to the estimate $\|(F_\lambda(x) - F_\mu(x))/(\lambda-\mu)\| \leq M_x/r$ for all $\mu \in \nabla(\lambda, r)$ with $\mu \neq \lambda$, where $M_x := \sup\{\|F_\zeta(x)\| : \zeta \in \nabla(\lambda, 2r)\} < \infty$. By the uniform boundedness principle, we obtain a constant $M > 0$ such that $\|(F_\lambda - F_\mu)/(\lambda-\mu)\| \leq M$ and therefore $\|F_\lambda - F_\mu\| \leq M|\lambda-\mu|$ for all $\mu \in \nabla(\lambda, r)$

with $\mu \neq \lambda$. This shows that F_λ does indeed depend continuously on $\lambda \in \mathbf{C}\backslash K$ and hence completes the proof of (9). It would be interesting to know if the corresponding formula (8) for the right multiplication operator R_T can be obtained under suitable weaker assumptions on T. For instance, we do not know if (8) holds, when T^* has SVEP.

6. Decomposability of $C(S, T)$

We finally turn to the problem of decomposability for commutators and multiplication operators. In the case of Hilbert spaces X and Y, this problem has been settled in [3]: indeed, in this case, an operator $T \in L(X)$ is decomposable if and only if R_T is decomposable on $L(X, Y)$, and similarly, an operator $S \in L(Y)$ is decomposable if and only if L_S is decomposable on $L(X, Y)$; moreover, if $T, S \in L(X)$ are both decomposable, then the commutator $C(S, T)$ is decomposable not only on $L(X)$, but also on various operator ideals of $L(X)$, for instance, on the compact operators and on the Hilbert–Schmidt operators. However, in the case of Banach spaces, it seems to be an open problem whether the commutator of two decomposable operators has to be decomposable. Here we shall obtain a positive result for a slightly restricted class of operators, which is large enough to contain basically all the prominent examples of decomposable operators.

Recall from [13] that an operator $T \in L(X)$ on an arbitrary complex Banach space X is said to be *super-decomposable* if, for every open cover $\{U, V\}$ of $\mathbf{C}$, there exists an operator $Q \in L(X)$, which commutes with T and satisfies $\sigma(T|\overline{Q(X)}) \subseteq U$ and $\sigma(T|\overline{(I-Q)(X)}) \subseteq V$, where I denotes the identity operator on X. Standard examples include the spectral operators in the sense of Dunford and, more generally, all operators with a non-analytic functional calculus on an algebra of functions with partitions of unity; see [13] for the theory of super-decomposable operators and [3] for an example of a decomposable operator, which is not super-decomposable. If an operator $T \in L(X)$ has property (C), then it is clear from [8, Proposition 1.3.8] and [13, Theorem 1.4] that T is super-decomposable if and only if, for every open cover $\{U, V\}$ of $\mathbf{C}$, there is a $Q \in L(X)$ such that $QT = TQ$, $Q(X) \subseteq X_T(\bar{U})$, and $(I-Q)(X) \subseteq X_T(\bar{V})$. From this characterization and the descriptions (8) and (9) of the spectral subspaces of multiplication operators, we obtain immediately the following permanence property.

Proposition 14. *If $T \in L(X)$ and $S \in L(Y)$ are super-decomposable on the complex Banach spaces X and Y, respectively, then so are the corresponding multiplication operators R_T and L_S on the space $L(X, Y)$.*

Proof. Given arbitrary open sets $U, V \subseteq \mathbf{C}$ with $U \cup V = \mathbf{C}$, we choose $Q \in L(X)$ such that $QT = TQ$, $Q(X) \subseteq X_T(\bar{U})$, and $(I-Q)(X) \subseteq X_T(\bar{V})$. For each closed set $F \subseteq \mathbf{C}$ with $F \cap \bar{U} = \emptyset$, we obtain that $QX_T(F) \subseteq X_T(F) \cap X_T(\bar{U}) = X_T(F \cap \bar{U}) = X_T(\emptyset) = \{0\}$. By formula (8), this implies that $R_Q(A) = AQ \in L_{R_T}(\bar{U})$ for all $A \in L(X, Y)$. The same argument shows that R_{I-Q} maps $L(X, Y)$ into the space $L_{R_T}(\bar{V})$. Since R_T has property (C) and commutes with R_Q, we conclude that R_T is super-decomposable. In the case of

L_S, the proof is even shorter and therefore omitted; see also [**13**, Theorem 3.6] for the special case $X=Y$.

Note that, by [**3**, Corollary 1.2], there is a certain converse to the preceding result: if the operators R_T and L_S are decomposable, so are T and S, provided, of course, that the underlying Banach spaces are non-zero. Proposition 14 leads to our final result.

Theorem 15. *Let $T\in L(X)$ and $S\in L(Y)$ be operators on complex Banach spaces X and Y. If R_T and L_S are decomposable on the spaces $L(X)$ and $L(Y)$, respectively, then the commutator $C(S,T)$ is decomposable on $L(X,Y)$. In particular, $C(S,T)$ is decomposable whenever both T and S are super-decomposable.*

Proof. We may assume that both spaces X and Y are non-zero and consider first the special case $X=Y$, in which $L(X,Y)=L(X)$ becomes a unital Banach algebra. Then clearly $R_T(AB)=AR_T(B)$ and $L_S(AB)=L_S(A)B$ for all $A,B\in L(X)$. Since R_T and L_S have property (δ), we conclude from [**18**, Theorem 3.1] that $C(S,T)$ has (δ). On the other hand, since we know from [**3**, Corollary 1.2] that T and S are decomposable, we infer from Theorem 13 that $C(S,T)$ also has property (C) and hence becomes decomposable. This settles the case $X=Y$. The general case can be reduced to this special case by considering the canonical extensions of the operators T and S to the direct sum $Z:=X\oplus Y$. Indeed, let $\tilde{T},\tilde{S}\in L(Z)$ be given by $\tilde{T}(x,y):=(Tx,0)$ and $\tilde{S}(x,y):=(0,Sy)$ for all $x\in X$ and $y\in Y$. By [**3**, Corollary 1.2] and [**8**, Proposition 2.1.8], the operators $\tilde{T}$ and $\tilde{S}$ are decomposable on Z. Using (8) and (9), it is then easily seen that the corresponding multiplication operators $R_{\tilde{T}}$ and $L_{\tilde{S}}$ are decomposable on the space $L(Z)$. By the first part of the proof, this implies that $C(\tilde{S},\tilde{T})$ is decomposable on $L(Z)$. A simple verification, based on the description of the spectral subspaces given in Theorem 13, shows that this entails the decomposability of $C(S,T)$ on $L(X,Y)$, which completes the proof of the main assertion. The final statement is now clear from Proposition 14. As pointed out by the referee, this last assertion also follows directly from [**4**, Theorem 3.4] or [**6**, Corollary 2.5].

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