Canad. Math. Bull Vol. 62 (1), 2019 pp. 119–130
http://dx.doi.org/10.4153/CMB-2017-077-0

Finsler Warped Product Metrics of Douglas Type

Huaifu Liu and Xiaohuan Mo

Abstract. In this paper, we study the warped structures of Finsler metrics. We obtain the differential equation that characterizes Finsler warped product metrics with vanishing Douglas curvature. By solving this equation, we obtain all Finsler warped product Douglas metrics. Some new Douglas Finsler metrics of this type are produced by using known spherically symmetric Douglas metrics.

1 Introduction

A Finsler metric on a manifold M is a *Douglas metric* if its Douglas curvature vanishes. The Douglas curvature was introduced by J. Douglas in 1927 [6]. Its importance in Finsler geometry is due to the fact that it is a projective invariant. Namely, if two Finsler metrics F and $\overline{F}$ are projectively equivalent, then F and $\overline{F}$ have the same Douglas curvature.

The warped product metric was first introduced by Bishop and O'Neil to study Riemannian manifolds of negative curvature as a generalization of Riemannian product metric [3]. They have mainly been used in the efforts to construct new examples of Riemannian manifolds with prescribed conditions on the curvatures. The warped product metric was later extended to the case of Finsler manifolds by the work of Chen–Shen–Zhao and Kozma–Peter–Varga [4, 8]. These metrics are called *Finsler warped product metrics*.

It is worth mentioning the recent observation by Chen, Shen, and Zhao that spherically symmetric Finsler metrics have warped product structure [4]. Recall that a Finsler metric F is said to be *spherically symmetric* if the orthogonal group acts as isometries on F [9–11].

In [10], the authors obtain the differential equation that characterizes the spherically symmetric Finsler metrics with vanishing Douglas curvature. Furthermore they obtain all the spherically symmetric Douglas metrics by solving this equation.

We know that there are a lot of Finsler warped product metrics that are not spherically symmetric (see (5.3)). Therefore, it is a natural problem to study Finsler warped product metrics with vanishing Douglas curvature.

In this paper, we first characterize such metrics in terms of a differential equation (Theorem 1.1), which can be reduced to a quasi-linear partial differential equation.

Received by the editors September 27, 2017.
Published electronically January 24, 2018.
This work is supported by BNSF(1164009), Beijing Postdoctoral Research Foundation and the National Natural Science Foundation of China 11371032 and 11771020. The second author (Xiaohuan Mo) is the corresponding author.
AMS subject classification: **22E46**, 53C30.
Keywords: Finsler metric, warped product, Douglas metric, spherically symmetric metric.

Then, by solving this equation, we obtain all the warped product Douglas metrics (Theorem 1.2). By using the characteristics of this equation and known spherically symmetric Douglas metrics, we construct explicitly some new warped product Douglas metrics (see Section 5), including the revised Funk's metric [4].

Consider the product manifold $M := I \times \check{M}$ where I is an interval of $\mathbb{R}$ and $\check{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric $\check{\alpha}$.

Finsler metrics on M, given in the form

$$F(u,v) := \check{\alpha}(\check{u},\check{v})\phi\Big(u^1, \frac{v^1}{\check{\alpha}(\check{u},\check{v})}\Big),$$

where $u = (u^1, \check{u})$, $v = v^1 \frac{\partial}{\partial u^1} + \check{v}$ and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$, are called *Finsler warped product metrics.*

In Section 3, we prove the following theorem.

Theorem 1.1 *The warped product metric $F = \check{\alpha}\phi(r,s)$, $r = u^1$, $s = \frac{v^1}{\check{\alpha}}$ is of Douglas type if and only if ϕ satisfies*

$$(\phi - s\phi_s)_r = [f(r)s^2 + g(r)]\phi_{ss}, \tag{1.1}$$

where $f = f(r)$ and $g = g(r)$ are two differentiable functions.

One can show that under generic conditions, the differential equation (1.1) is equivalent to a quasi-linear partial differential equation. By using the characteristic equation, our next result provides the general solution of (1.1).

Theorem 1.2 *Let $f(r)$ and $g(r)$ be differential functions of $r \in I$ such that (4.1) holds. Then for $s > 0$, the general solution of (1.1) is*

$$\phi(r,s) = sh(r) - s\int_{s_0}^{s} \sigma^{-2}\zeta(\varphi(r,\sigma))d\sigma, \tag{1.2}$$

where $s_0 \in (0, s]$,

$$\varphi(r,\sigma) = e^{-\int 2f(r)dr}\sigma^2 + \int 2g(r)e^{-\int 2f(r)dr}dr, \tag{1.3}$$

and h and ζ are arbitrary differentiable real functions of r and φ, respectively. Moreover, any warped product Douglas metric on $I \times \check{M}$ is given by

$$F(u,v) = \check{\alpha}(\check{u},\check{v})\phi\left(u^1, \frac{v^1}{\check{\alpha}(\check{u},\check{v})}\right),$$

where ϕ is of the form (1.2) and ζ satisfies

$$\zeta > 0, \quad \zeta' < 0. \tag{1.4}$$

Theorem 1.1 tells us the following interesting fact: the disappearance of Douglas curvature for a Finsler warped product metric is independent of the Riemannian metric $\check{\alpha}$ on $\check{M}$. It follows that we can construct explicitly some new warped product Douglas metrics by using known spherically symmetric Douglas metrics in Section 5.

For related results of Finsler warped product metrics, see [1, 2, 4, 8].

2 Preliminaries

Let M be a manifold and let $TM = \bigcup_{x\in M} T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. We set $TM_o := TM \setminus \{0\}$ where $\{0\}$ stands for $\{(x,0) | x \in M, 0 \in T_xM\}$. A *Finsler metric* on M is a function $F\colon TM \to [0,\infty)$ with the following properties:

(a) F is C^∞ on TM_o;
(b) at each point $x \in M$, the restriction $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM.

A Finsler metric F on $\mathbb{B}^n(\nu)$ is said to be *spherically symmetric* if it satisfies

$$F(Ax, Ay) = F(x, y)$$

for all $x \in \mathbb{B}^n(\nu)$, $y \in T_x\mathbb{B}^n(\nu)$, and $A \in O(n)$. In [14], Zhou (see also [7]) showed the following lemma.

Lemma 2.1 *A Finsler metric F on $\mathbb{B}^n(\nu)$ is spherically symmetric if and only if there is a function $\phi\colon [0,\nu) \times \mathbb{R} \to \mathbb{R}$ such that*

$$F(x, y) = |y|\phi\Big(|x|, \frac{\langle x, y\rangle}{|y|}\Big),$$

where $(x, y) \in T\mathbb{B}^n(\nu)\backslash\{0\}$.

Lemma 2.2 *A spherically symmetric metric is a Finsler warped product metric.*

Proof In fact,

$$F(x, y) = |y|\phi\Big(|x|, \frac{\langle x, y\rangle}{|y|}\Big) = \breve{\alpha}_+\tilde{\phi}(r, s), \tag{2.1}$$

where $\breve{\alpha}_+$ is the standard Riemannian metric on the unit sphere S^{n-1},

$$r := |x|, \quad s := \frac{y^1}{\breve{\alpha}_+}, \quad \tilde{\phi}(r, s) = \sqrt{r^2 + s^2}\,\phi\Big(r, \frac{rs}{\sqrt{r^2 + s^2}}\Big), \tag{2.2}$$

where $y^1 = dr(y)$. ∎

There is a set of local functions ${B_C}^A{}_{DE}$ on TM_o defined by

$${B_C}^A{}_{DE} := \frac{\partial^3 G^A}{\partial y^C \partial y^D \partial y^E}$$

where G^A are the geodesic coefficients of a Finsler metric F [13]. Because this quantity was first introduced by L. Berwald, we call it the *Berwald curvature* [12]. Recall that F is said to be *Berwald* if ${B_C}^A{}_{DE} = 0$.

Throughout this paper, our index conventions are as follows:

$$1 \le A \le B \le \cdots \le n, \quad 2 \le i \le j \le \cdots \le n.$$

https://doi.org/10.4153/CMB-2017-077-0 Published online by Cambridge University Press

3 Douglas Metrics

In this section, we are going to find necessary and sufficient conditions for the Finsler warped product metric to be of vanishing Douglas curvature. We need the following lemma.

Lemma 3.1 *Let $P = P(r,s)$ and $Q = Q(r,s)$ be two functions on a domain $\mathcal{U} \subset \mathbb{R}^2$. Then*

$$\frac{\partial^3}{\partial v^A \partial v^B \partial v^C}(P\breve{\alpha}^2) = 0, \quad \frac{\partial^3}{\partial v^A \partial v^B \partial v^C}(Q\breve{\alpha}^2 \breve{l}^j) = 0 \tag{3.1}$$

if and only if

$$P = a(r)s^2 + b(r), \quad Q = c(r)s, \tag{3.2}$$

where $\breve{l}^i = \frac{v^i}{\breve{\alpha}}$ and $a = a(r)$, $b = b(r)$, and $c = c(r)$ are differentiable functions.

Proof By using Lemma A.2, we have

$$\frac{\partial^3}{\partial v^1 \partial v^1 \partial v^1}(P\breve{\alpha}^2) = \frac{1}{\breve{\alpha}} P_{sss}, \tag{3.3}$$

$$\frac{\partial^3}{\partial v^1 \partial v^1 \partial v^i}(P\breve{\alpha}^2) = -\frac{1}{\breve{\alpha}} P_{sss} \breve{l}_i, \tag{3.4}$$

$$\frac{\partial^3}{\partial v^1 \partial v^i \partial v^j}(P\breve{\alpha}^2) = \frac{s^2}{\breve{\alpha}} P_{sss} \breve{l}_i \breve{l}_j + \frac{1}{\breve{\alpha}}(P_s - sP_{ss})\breve{h}_{ij}, \tag{3.5}$$

$$\frac{\partial^3}{\partial v^i \partial v^j \partial v^k}(P\breve{\alpha}^2) = -\frac{s^3}{\breve{\alpha}} P_{sss} \breve{l}_i \breve{l}_j \breve{l}_k - \frac{s}{\breve{\alpha}}(P_s - sP_{ss})\breve{h}_{ij}\breve{l}_k (i \to j \to k \to i). \tag{3.6}$$

where $\breve{l}_i := \breve{\alpha}_{v^i}, \breve{h}_{ij} := \breve{\alpha}(\breve{l}_i)_{v^j}$ and $i \to j \to k \to i$ denotes cyclic permutation. By using Lemma A.3, we get

$$\frac{\partial^3}{\partial v^1 \partial v^1 \partial v^1}(Q\breve{\alpha}^2 \breve{l}^i) = \frac{1}{\breve{\alpha}} Q_{sss} \breve{l}^i, \tag{3.7}$$

$$\frac{\partial^3}{\partial v^1 \partial v^1 \partial v^j}(Q\breve{\alpha}^2 \breve{l}^i) = \frac{1}{\breve{\alpha}} Q_{ss} \breve{h}^i_j - \frac{s}{\breve{\alpha}} Q_{sss} \breve{l}^i \breve{l}_j, \tag{3.8}$$

$$\frac{\partial^3}{\partial v^1 \partial v^j \partial v^k}(Q\breve{\alpha}^2 \breve{l}^i) = \frac{s^2}{\breve{\alpha}} Q_{sss} \breve{l}^i \breve{l}_j \breve{l}_k - \frac{s}{\breve{\alpha}} Q_{ss}(\breve{h}^i_j \breve{l}_k + \breve{h}^i_k \breve{l}_j + \breve{h}_{jk} \breve{l}^i), \tag{3.9}$$

$$\begin{aligned} \frac{\partial^3}{\partial v^j \partial v^k \partial v^l}(Q\breve{\alpha}^2 \breve{l}^i) = {} & \frac{1}{\breve{\alpha}}[3(Q - sQ_s) - 6s^2 Q_{ss} - s^2 Q_{sss}]\breve{l}^i \breve{l}_j \breve{l}_k \breve{l}_l \\ & + \frac{1}{\breve{\alpha}}(s^2 Q_{ss} - Q_s + sQ_s)\breve{l}_l(\breve{l}^i \breve{a}_{jk} + \delta^i_j \breve{l}_k)(j \to k \to l \to j) \\ & + \frac{1}{\breve{\alpha}}(Q - sQ_s)\delta^i_j \breve{a}_{kl}(j \to k \to l \to j), \end{aligned} \tag{3.10}$$

https://doi.org/10.4153/CMB-2017-077-0 Published online by Cambridge University Press

where $\breve{h}^i_j := \breve{\alpha}(\breve{l}^i)_{v^j}$. Using (3.5) and (3.10), we obtain

$$\frac{\partial^3(P\breve{\alpha}^2)}{\partial v^1 \partial v^k \partial v^l} + \sum_{i=2}^n \frac{\partial^3(Q\breve{\alpha}^2\breve{l}^i)}{\partial v^i \partial v^k \partial v^l} = \frac{1}{\breve{\alpha}}\big[(n-2)s^2 Q_{ss} - s^3 Q_{sss} + s^2 P_{sss}\big]\breve{l}_k \breve{l}_l + \frac{1}{\breve{\alpha}}\big[s^2 Q_{ss} + n(Q - sQ_s) + P_s - sP_{ss}\big]\breve{h}_{kl}. \tag{3.11}$$

First suppose that (3.1) holds. Combining (3.1) with (3.3), we get

$$P_{sss} = 0. \tag{3.12}$$

Plugging this into (3.5), we have

$$\frac{1}{\breve{\alpha}}(P_s - sP_{ss})\breve{h}_{ij} = 0.$$

Note that $\operatorname{rank}(\breve{h}_{ij}) = n-2$. It follows that when $n \geq 3$,

$$P_s - sP_{ss} = 0. \tag{3.13}$$

Hence, we get the first equation of (3.2). From (3.7) and (3.8), we get

$$Q_{sss} = 0, \quad Q_{ss} = 0. \tag{3.14}$$

Plugging (3.12), (3.13), and (3.14) into (3.11) yields $(Q - sQ_s)\breve{h}_{kl} = 0$. It follows that $Q - sQ_s = 0$. Thus, we see that the solution of Q is $Q = c(r)s$.

Conversely, suppose that (3.2) holds. From (3.3)–(3.10), one immediately obtain (3.1). ∎

By considering $P = \Phi$ and $Q = \Psi$ in Lemma 3.1, we obtain the following corollary.

Corollary 3.2 *Let $\Phi(r,s)$ and $\Psi(r,s)$ be two functions defined by*

$$\Phi = \frac{s^2(\omega_r\omega_{ss} - \omega_s\omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega\omega_{ss} - \omega_s^2)}, \quad \Psi = \frac{s(\omega_r\omega_{ss} - \omega_s\omega_{rs}) + \omega_s\omega_r}{2(2\omega\omega_{ss} - \omega_s^2)}, \tag{3.15}$$

where

$$\omega = \phi^2. \tag{3.16}$$

Then the Finsler warped product metric $F = \breve{\alpha}\phi\big(u^1, \frac{v^1}{\breve{\alpha}}\big)$ is Berwald, i.e., ${B_C}^A{}_{DE} = 0$, *if and only if*

$$\Phi = a(r)s^2 + b(r), \quad \Psi = c(r)s,$$

where $a = a(r)$, $b = b(r)$ and $c = c(r)$ are differentiable functions.

Proof Combining Lemma 3.1 and Lemma A.1 proves the corollary. ∎

In [6], J. Douglas introduced the local functions ${D_B}^A{}_{CD}$ on TM defined by

$${D_B}^A{}_{CD} := \frac{\partial^3}{\partial v^B \partial v^C \partial v^D}\Big(G^A - \frac{1}{n+1}\sum_{E=1}^n \frac{\partial G^E}{\partial v^E} v^A\Big) \tag{3.17}$$

https://doi.org/10.4153/CMB-2017-077-0 Published online by Cambridge University Press

in local coordinate $u^1, \dots, u^n$ and $v = v^A \frac{\partial}{\partial v^A}$, where G^A are the geodesic coefficients of F [13]. These functions are called the *Douglas curvatures*. From Lemma A.1, we have

$$\begin{aligned}\sum_{A=1}^n \frac{\partial G^A}{\partial v^A} &= \frac{\partial G^1}{\partial v^1} + \sum_{j=2}^n \frac{\partial G^j}{\partial v^j}\\ &= (\Psi - s\Psi_s)\breve{\alpha}\breve{l}_j\breve{l}^j + \Psi\breve{\alpha}\sum_{j=2}^n \delta_j^j + \Phi_s\breve{\alpha} + \sum_{j=2}^n \frac{\partial \breve{G}^j}{\partial v^j}\\ &= \sum_{j=2}^n \frac{\partial \breve{G}^j}{\partial v^j} + (\Phi_s + n\Psi - s\Psi_s)\breve{\alpha},\end{aligned}$$

where $\breve{G}^j$ are the geodesic coefficients of the Riemannian metric $\breve{\alpha}$, and Φ and Ψ are given in (3.15). It follows that

$$G^1 - \frac{1}{n+1}\sum_{A=1}^n \frac{\partial G^A}{\partial v^A} v^1 = (\Phi + s\Theta)\breve{\alpha}^2 - \frac{1}{n+1}\sum_{j=2}^n \frac{\partial \breve{G}^j}{\partial v^j} v^1, \tag{3.18}$$

$$G^k - \frac{1}{n+1}\sum_{A=1}^n \frac{\partial G^A}{\partial v^A} v^k = (\Psi + \Theta)\breve{\alpha}^2\breve{l}^k + \breve{G}^k - \frac{1}{n+1}\sum_{j=2}^n \frac{\partial \breve{G}^j}{\partial v^j} v^k \tag{3.19}$$

where

$$\Theta := -\frac{1}{n+1}(\Phi_s + n\Psi - s\Psi_s). \tag{3.20}$$

For a Riemannian metric $\breve{\alpha}$, we have

$$\breve{G}^k = \breve{\Gamma}^k_{ij}(\breve{u})v^i v^j.$$

Hence, both

$$-\frac{1}{n+1}\sum_{j=2}^n \frac{\partial \breve{G}^j}{\partial v^j} v^1 \quad \text{and} \quad \breve{G}^k - \frac{1}{n+1}\sum_{j=2}^n \frac{\partial \breve{G}^j}{\partial v^j} v^k$$

are quadratic in $v = v^A \frac{\partial}{\partial v^A}$.

Combining this with (3.17), (3.18), (3.19), and Lemma 3.1, we conclude that F has vanishing Douglas curvature if and only if

$$\Phi + s\Theta = a(r)s^2 + b(r), \quad \Psi + \Theta = c(r)s. \tag{3.21}$$

Suppose that F is of Douglas type, that is, it has vanishing Douglas curvature. Then (3.21) implies that

$$\Phi - s\Psi = \xi(r)s^2 + \eta(r), \tag{3.22}$$

where

$$\xi(r) := a(r) - c(r), \quad \eta(r) := b(r).$$

Conversely, suppose that (3.22) holds. It follows from (3.22) that

$$\Phi = s\Psi + \xi(r)s^2 + \eta(r). \tag{3.23}$$

Differentiating (3.23) with respect to s, we obtain $\Phi_s = \Psi + s\Psi_s + 2s\xi$. Together with (3.20), this yields

$$\Theta = -\Psi - \frac{2}{n+1}\xi s.$$

https://doi.org/10.4153/CMB-2017-077-0 Published online by Cambridge University Press

It follows that

$$\Phi + s\Theta = \frac{n-1}{n+1}\xi(r)s^2 + \eta(r), \quad \Psi + \Theta = -\frac{2}{n+1}\xi(r)s,$$

where we have made use of (3.23). From (3.21), we conclude that F is of Douglas type. Thus, we obtain the following lemma.

Lemma 3.3 *The Finsler warped product metric $F = \breve{\alpha}\phi(u^1, \frac{v^1}{\breve{\alpha}})$ has vanishing Douglas curvature if and only if (3.22) holds where Φ and Ψ are given in (3.15), and $\xi = \xi(r)$ and $\eta = \eta(r)$ are differentiable functions.*

Proof of Theorem 1.1 By a straightforward computation, one obtains

$$2(\Phi - s\Psi) = \frac{2s\omega\omega_{rs} - 2\omega\omega_r - s\omega_r\omega_s}{2(2\omega\omega_{ss} - \omega_s^2)} = \frac{4\phi^3(s\phi_{rs} - \phi_r)}{4\phi^3\phi_{ss}} = -\frac{(\phi - s\phi_s)_r}{\phi_{ss}}, \tag{3.24}$$

where we used (3.15) and (3.16). Combining (3.24) with Lemma 3.3, we conclude the proof. ∎

Remark Note that a spherically symmetric metric $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ is a Finsler warped product metric. In fact,

$$F = |y|\phi(r, \tilde{s}) = \breve{\alpha}_+\tilde{\phi}(r, s),$$

where

$$\tilde{s} = \frac{rs}{\sqrt{r^2+s^2}} = \frac{\langle x, y\rangle}{|y|}, \quad \tilde{\phi}(r,s) = \sqrt{r^2+s^2}\phi(r,\tilde{s}),$$

where $\breve{\alpha}_+$ is the standard Riemannian metric on the unit sphere S^{n-1}, and we have made use of (2.1) and (2.2). It is easy to verify (*cf.* [10, Theorem 1.1]) that ϕ satisfies the Douglas equations

$$\left[(r^2 - \tilde{s}^2)(2\xi + \eta\tilde{s}^2) - 1\right] r\phi_{\tilde{s}\tilde{s}} - \tilde{s}\phi_{r\tilde{s}} + \phi_r + r(2\xi + \eta\tilde{s}^2)(\phi - \tilde{s}\phi_{\tilde{s}}) = 0 \tag{3.25}$$

if and only if ϕ satisfies (1.1), where either

$$g := r(1 - 2r^2\xi) \quad \text{and} \quad f := \frac{1}{r}\Big(1 + \frac{g}{r} - r^4\eta\Big),$$

or

$$\eta := 1 - fr + \frac{g}{r} \quad \text{and} \quad \xi := \frac{1}{2r^2}\Big(1 - \frac{g}{r}\Big).$$

It follows that (1.1) is equivalent to (3.25) for a spherically symmetric metric.

4 Solutions of (1.1)

Let $f(r)$ and $g(r)$ be functions such that the integrals

$$\int 2f(r)dr, \quad \int 2g(r)e^{-\int 2f(r)dr}dr \tag{4.1}$$

are well defined for $r \in I \subset \mathbb{R}$.

https://doi.org/10.4153/CMB-2017-077-0 Published online by Cambridge University Press

Proof of Theorem 1.2 Equation (1.1) is equivalent to

$$[f(r)s^2 + g(r)]\lambda_s + s\lambda_r = 0, \tag{4.2}$$

where

$$\lambda := \phi - s\phi_s. \tag{4.3}$$

The characteristic equation of the quasi-linear PDE (4.2) is

$$\frac{dr}{s} = \frac{ds}{f(r)s^2 + g(r)} = \frac{d\lambda}{0}. \tag{4.4}$$

It follows that $\varphi(r,s) = c_1$ and $\lambda = c_2$ are independent integrals of (4.4), where φ is given in (1.3). Hence, the solution of (4.2) is $\lambda = \zeta(\varphi(r,s))$, where ζ is any continuously differentiable function. Hence,

$$\phi - s\phi_s = \zeta\big(\varphi(r,s)\big), \tag{4.5}$$

It follows that every solution of (1.1) satisfies (4.5).

Conversely, suppose that (4.5) holds. Then we obtain (4.2) and (4.3). Thus, ϕ satisfies (1.1). We conclude that (4.5) and (1.1) are equivalent.

Now we consider $s \in [s_0, \infty)$ where $s_0 > 0$. Put

$$\phi = s\psi. \tag{4.6}$$

It follows that $\phi_s = \psi + s\psi_s$. Together with (4.5), this yields

$$\zeta\big(\varphi(r,s)\big) = \phi - s\phi_s = -s^2\psi_s.$$

Thus,

$$\psi = h(r) - \int_{s_0}^{s} \sigma^{-2}\zeta\big(\varphi(r,\sigma)\big)\,d\sigma.$$

Plugging this into (4.6) yields (1.2).

Similarly, we can obtain the general solution of (1.1) for $s < 0$.

Differentiating (4.5) with respect to s, we obtain $2se^{-\int 2f dr}\zeta' = -s\phi_{ss}$. It follows that $\phi_{ss} = -2e^{-\int 2f dr}\zeta'$. Combining this with (4.5) and Proposition 5.1 (see Section 5), we get that F is a Finsler metric if and only if (1.4) holds. ∎

5 New Families of Douglas Metrics

In this section, we obtain several new families of warped product metrics as corollaries of Theorem 1.1.

Proposition 5.1 *The warped product metric $F = \breve{\alpha}\phi(u^1, \frac{v^1}{\breve{\alpha}})$ is strongly convex if and only if*

$$\phi - s\phi_s > 0, \quad \phi_{ss} > 0. \tag{5.1}$$

Proof Using (3.16), we get

$$2\omega - s\omega_s = 2\phi(\phi - s\phi_s), \quad 2\omega\omega_{ss} - \omega_s^2 = 4\phi^3\phi_{ss}.$$

Taking these together with [4, Proposition 4.1], we obtain that F is strongly convex if and only if the positive function ϕ satisfies (5.1). ∎

https://doi.org/10.4153/CMB-2017-077-0 Published online by Cambridge University Press

Now let us make an observation on the above results. Assume that

$$F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\Big)$$

is a Finsler warped product metric with vanishing Douglas curvature. We replace its $\breve{\alpha}$ by another Riemannian metric $\hat{\alpha}$. Then Proposition 5.1 and Theorem 1.1 imply that

$$\hat{F}(u,v) = \hat{\alpha}(\breve{u},\breve{v})\phi\Big(u^1, \frac{v^1}{\hat{\alpha}(\breve{u},\breve{v})}\Big)$$

is a strongly convex metric with vanishing Douglas curvature.

Corollary 5.2 *Suppose that $F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\big(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\big)$ is a Finsler warped product metric on $I \times \breve{M}$ and F has vanishing Douglas curvature. Given any Riemannian metric $\hat{\alpha}$ on $\breve{M}$, the new warped product Finsler metric $\hat{F}(u,v) = \hat{\alpha}(\breve{u},\breve{v})\phi(u^1, \frac{v^1}{\hat{\alpha}(\breve{u},\breve{v})})$ has vanishing Douglas curvature.*

Example 5.3 For any differentiable function h of $r := |x|$ such that

$$\phi(r,\tilde{s}) := h(r)\tilde{s} + Ae^{-\frac{r^4}{4}} > 0,$$

where $A > 0$, we have the following Douglas spherically symmetric metric [10]

$$F(x,y) = \langle x,y\rangle h(|x|) + A|y|e^{-\frac{|x|^4}{4}}.$$

Applying Lemma 2.2, its Finsler warped product form is

$$F = \breve{\alpha}_+\sqrt{r^2+s^2}\Big[h(r)\frac{rs}{\sqrt{r^2+s^2}} + Ae^{-\frac{r^4}{4}}\Big] = \breve{\alpha}_+\Big[rh(r)s + Ae^{-\frac{r^4}{4}}\sqrt{r^2+s^2}\Big],$$

where $\breve{\alpha}_+$ is the standard Riemannian metric on S^{n-1}. Let $\hat{\alpha}$ be a Riemannian metric on S^{n-1}. Then

$$\hat{F} = \hat{\alpha}\Big[rh(r)\hat{s} + Ae^{-\frac{r^4}{4}}\sqrt{r^2+\hat{s}^2}\Big]$$

is a new warped product Finsler metric with vanishing Douglas curvature, where

$$\hat{s} := \frac{v^1}{\hat{\alpha}(\breve{u},\breve{v})}. \tag{5.2}$$

Example 5.4 Let $\phi(r,\tilde{s})$ be a function defined by

$$\phi(r,\tilde{s}) := \tilde{s}h(r) + \frac{[1+(1+\mu)r^2][1+\mu(r^2-\tilde{s}^2)]+\tilde{s}^2}{\sqrt{1+\mu(r^2-\tilde{s}^2)}(1+\mu r^2)^2},$$

where $\mu \in \mathbb{R}$, and $h(r)$ is any function such that $\phi(r,\tilde{s})$ is positive. Then the Finsler metric on $\mathbb{B}^n(\nu) \subset R^n$ given by

$$F(x,y) := |y|\phi\Big(|x|, \frac{\langle x,y\rangle}{|y|}\Big),$$

where $\nu = \frac{1}{\sqrt{-\mu}}$ if $\mu < 0$, is a spherically symmetric Douglas metric [10]. By using Lemma 2.2, its Finsler warped product form is

$$F = \breve{\alpha}_+\Big\{rh(r)s + \frac{[1+(1+\mu)r^2][r^2+s^2+\mu r^4]+r^2s^2}{\sqrt{r^2+s^2+\mu r^4}(1+\mu r^2)^2}\Big\}.$$

Let $\hat{\alpha}$ be a Riemannian metric on S^{n-1}. Then

$$\hat{F} = \hat{\alpha}\Big\{ rh(r)\hat{s} + \frac{[1+(1+\mu)r^2][r^2+\hat{s}^2+\mu r^4]+r^2\hat{s}^2}{\sqrt{r^2+\hat{s}^2+\mu r^4}(1+\mu r^2)^2}\Big\}.$$

is a new warped product Douglas metric, where $\hat{s}$ is given in (5.2).

Example 5.5 Let $\phi(r,\tilde{s})$ be a function defined by

$$\phi(r,\tilde{s}) := \tilde{s}h(r) + \frac{\sqrt{\zeta\varepsilon r^2+\kappa\tilde{s}^2+\varepsilon}}{\zeta r^2+1},$$

where $\zeta, \varepsilon, \kappa$ are any constant real values such that $(\zeta\varepsilon+\kappa^2)+\varepsilon>0$ and $h(r)$ is any function such that $\phi(r,\tilde{s})$ is positive. Then the spherically symmetric Finsler metric on $\mathbb{B}^n(\nu)\subset R^n$ given by

$$F(x,y) = |y|\phi\Big(|x|, \frac{\langle x,y\rangle}{|y|}\Big),$$

where $\nu = \frac{1}{\sqrt{-\zeta}}$ if $\zeta<0$, is of Douglas type [10]. By Lemma 2.2, its Finsler warped product form is

$$F = \check{\alpha}_+\Big\{ rh(r)s + \frac{\sqrt{\varepsilon(\zeta r^2+1)(r^2+s^2)+\kappa^2r^2s^2}}{1+\zeta r^2}\Big\}.$$

Replacing $\check{\alpha}_+$ by any Riemannian metric $\hat{\alpha}$, one can get the new warped product Douglas metric

$$F = \hat{\alpha}\Big\{ rh(r)\hat{s} + \frac{\sqrt{\varepsilon(\zeta r^2+1)(r^2+\hat{s}^2)+\kappa^2r^2\hat{s}^2}}{1+\zeta r^2}\Big\},$$

where $\hat{s}$ is given in (5.2). In particular, when

$$h(r) := \frac{\kappa}{1+\zeta r^2}, \quad \kappa=\pm1, \quad \zeta=-1, \quad \varepsilon=1,$$

we have

$$\hat{F}_\pm = \hat{\alpha}\frac{\sqrt{\hat{s}^2+r^2(1-r^2)}\pm r\hat{s}}{1-r^2}, \tag{5.3}$$

where $\hat{F}_\pm$ is the revised Funk's metric [4].

Example 5.6 Let $\phi(r,\tilde{s})$ be a function defined by

$$\phi(r,\tilde{s}) := \tilde{s}h(r) + \frac{\kappa\nu\sqrt{\nu^2-r^2+\tilde{s}^2}}{e^{\frac{c\nu^2r^2}{2}}(\nu^2-r^2)},$$

where κ and ν are positive constants, c is a nonzero constant, and $h(r)$ is any function such that ϕ is positive. It follows that

$$F(x,y) = |y|\phi\Big(|x|, \frac{\langle x,y\rangle}{|y|}\Big)$$

is a spherically symmetric Douglas metric [10]. Its Finsler warped product form is

$$F = \check{\alpha}_+\Big\{ rh(r)s + \frac{\kappa\nu\sqrt{\nu^2s^2+r^2(\nu^2-r^2)}}{e^{c\nu^2r^2}(\nu^2-r^2)}\Big\}.$$

Replacing $\breve{\alpha}_+$ by any Riemannian metric $\hat{\alpha}$, one can get the following new warped product Douglas metric

$$F = \hat{\alpha}\Big\{ rh(r)\hat{s} + \frac{\kappa\nu\sqrt{\nu^2\hat{s}^2 + r^2(\nu^2 - r^2)}}{e^{c\nu^2 r^2}(\nu^2 - r^2)}\Big\},$$

where $\hat{s}$ is given in (5.2).

A Appendix

We establish the Lemmas required in the proofs of Lemma 3.1 and Corollary 3.2. From now M will always denote a product manifold $I \times \check{M}$, and F will denote a Finsler warped product metric. Let $\check{l}^i = \frac{v^i}{\check{\alpha}}, \check{l}_i := \check{\alpha}_{v^i}, \check{h}_{ij} := \check{\alpha}(\check{l}_i)_{v^j}, \check{h}^i_j := \check{\alpha}(\check{l}^i)_{v^j}$. The following lemmas can be obtained by straightforward calculations.

Lemma A.1 *For the geodesic coefficients G^A of F, we have*

$$G^1 = \Phi\check{\alpha}^2, \qquad G^i = \check{G}^i + \Psi\check{\alpha}^2\check{l}^i,$$
$$\frac{\partial G^1}{\partial v^1} = \Phi_s\check{\alpha}, \qquad \frac{\partial G^k}{\partial v^j} = \frac{\partial \check{G}^k}{\partial v^j} + (\Psi - s\Psi_s)\check{\alpha}\check{l}_j\check{l}^k + \Psi\check{\alpha}\delta^k_j,$$

where Φ and Ψ are given in (3.15).

Lemma A.2 *Let $P = P(r,s)$ be a function on a domain $\mathfrak{U} \subset \mathbb{R}^2$. Then*

$$\frac{\partial^2}{\partial v^1\partial v^1}(P\check{\alpha}^2) = P_{ss},$$
$$\frac{\partial^2}{\partial v^i\partial v^j}(P\check{\alpha}^2) = (2P - 2sP_s + s^2P_{ss})\check{l}_i\check{l}_j + (2P - sP_s)\check{h}_{ij}.$$

Lemma A.3 *Let $Q = Q(r,s)$ be a function on a domain $\mathfrak{U} \subset \mathbb{R}^2$. Then*

$$\frac{\partial^2}{\partial v^1\partial v^1}(Q\check{\alpha}^2\check{l}^i) = Q_{ss}\check{l}^i,$$
$$\frac{\partial^2}{\partial v^j\partial v^k}(Q\check{\alpha}^2\check{l}^i) = (2Q - 2sQ_s + s^2Q_{ss})\check{l}_j\check{l}_k\check{l}^i + (Q - sQ_s)(\check{h}_{jk}\check{l}^i + \check{l}_j\check{h}^i_k + \check{l}_k\check{h}^i_j).$$

References

[1] Y. Alipour-Fakhri and M. M. Rezaii, *The warped Sasaki-Matsumoto metric and bundlelike condition*. J. Math. Phys. **51**(2010), no. 12, 122701, 13 pp. http://dx.doi.org/10.1063/1.3520636

[2] G. S. Asanov, *Finslerian metric functions over the product $R \times M$ and their potential applications*. Rep. Math. Phys. **41**(1998), no. 1, 117–132. http://dx.doi.org/10.1016/S0034-4877(98)80184-5

[3] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*. Trans. Amer. Math. Soc. **145**(1969), 1–49. http://dx.doi.org/10.1090/S0002-9947-1969-0251664-4

[4] B. Chen, Z.Shen, and L. Zhao, *Constructions of Einstein Finsler metrics by warped product*. preprint, 2016.

[5] S.-S. Chern and Z. Shen, *Riemann-Finsler geometry*. Nankai Tracts in Mathematics, 6, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. http://dx.doi.org/10.1142/5263

[6] J. Douglas, *The general geometry of paths*. Ann. of Math. **29**(1927-1928), no. 1–4, 143–168. http://dx.doi.org/10.2307/1967989

[7] L. Huang and X. Mo, *Projectively flat Finsler metrics with orthogonal invariance*. Ann. Polon. Math. **107**(2013), no. 3, 259–270. http://dx.doi.org/10.4064/ap107-3-3

[8] L. Kozma, R. Peter, and C. Varga, *Warped product of Finsler manifolds*. Ann. Univ. Sci. Budapest **44**(2001), 157–170.
[9] P. J. McCarthy and S. F. Rutz, *The general four-dimensional spherically symmetric Finsler space*. Gen. Relativity Gravitation **25**(1993), no. 6, 589–602. http://dx.doi.org/10.1007/BF00757070
[10] X. Mo, N. M. Solórzano, and K. Tenenblat, *On spherically symmetric Finsler metrics with vanishing Douglas curvature*. Differential Geom. Appl. **31**(2013), 746–758. http://dx.doi.org/10.1016/j.difgeo.2013.09.002
[11] S. F. Rutz, *Symmetry in Finsler spaces.* In: Finsler geometry (Seattle, WA, 1995), Contemp. Math., 196, American Mathematical Society, Providence, RI, 1996, pp. 289–300. http://dx.doi.org/10.1090/conm/196/02459
[12] Z. Shen, *On R-quadratic Finsler spaces*. Publ. Math. Debrecen. **58**(2001), no. 1–2, 263–274.
[13] Z. Shen, *Differential geometry of spray and Finsler spaces.* Kluwer Academic Publishers, Dordrecht, 2001.
[14] L. Zhou, *Spherically symmetric Finsler metrics in* R^n. Publ. Math. Debrecen **80**(2012), no. 1–2, 67–77. http://dx.doi.org/10.1007/s10711-011-9639-3

College of Applied Science, Beijing University of Technology, Beijing, 100124, China
e-mail: liuhf@bjut.edu.cn

Key Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing, 100871, China
e-mail: moxh@pku.edu.cn