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DADE'S CONJECTURE FOR CHEVALLEY GROUPS $G_2(q)$ IN NON-DEFINING CHARACTERISTICS

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ABSTRACT. This paper is part of a program to study the conjecture of E. C. Dade on counting characters in blocks for several finite groups of Lie type. The local structures of certain radical chains of Chevalley groups of type G_2 are given and the ordinary conjecture is confirmed for the groups when the characteristic of the modular representation is distinct from the defining characteristic of the groups.

0. **Introduction.** Let G be a finite group, r a prime and B an r-block of G. In his paper [10], Dade conjectured that the number of ordinary irreducible characters of B with a fixed height can be expressed as an alternating sum of the numbers of ordinary irreducible characters of related heights in related blocks B' of certain local r-subgroups of G. It is mentioned on page 187 of [10] that the final form of the conjecture can be confirmed by verifying it for all non-abelian finite simple groups. In this paper, we prove the ordinary conjecture for the Chevalley groups of type G_2 when r is distinct from the characteristic of the group.

In Section 1 we fix some notation and state the ordinary conjecture. In Section 2 we first simplify the family of radical r-chains $\mathcal{R}(G)$ to get a G-invariant subfamily $\mathcal{R}^0(G)$, and then determine the local structures of radical r-chains in $\mathcal{R}^0(G)$, where G is a Chevalley group of type G_2. In Section 3 and 4 we prove Dade's ordinary conjecture for G when r is odd and even, respectively. It turns out that the Alperin-McKay and the Brauer height conjectures imply the ordinary conjecture of Dade when an r-block of G has a non-cyclic abelian defect group.

1. **The ordinary conjecture of Dade.** Throughout this paper we shall follow the notation of Dade [10]. Let $\mathrm{Irr}(G)$ be the set of all irreducible ordinary characters of a finite group G and $\mathrm{Blk}(G)$ the set of all r-blocks of G. Let C be an r-subgroup chain of G,

$$C: P_0 < P_1 < \cdots < P_n. \tag{1.1}$$

Then $n = |C|$ is called the *length* of C, $C(C) = C_G(C) = C_G(P_n)$ and

$$N(C) = N_G(C) = N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n) \tag{1.2}$$

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are called *centralizer* and *normalizer* of C in G,

(1.3) $C_k\colon P_0 < P_1 < \cdots < P_k$ and $C^k\colon 1 < P_{k+1} < \cdots < P_n, \quad 0 \le k \le n-1$

are called *k-th initial* and *final* r-subchains of C, respectively. Note that the definition of C^k does not agree with that of Dade [10, p. 191]. Thus $N_G(C) \le N_G(C_{n-1}) \le \cdots \le N_G(C_0) = N(P_0)$ and $N_G(C_{k+1}) = N_{N_G(C_k)}(P_{k+1})$ for all $k \ge 0$. In addition, C is called a *radical* r-chain if it satisfies the following two conditions:

(a) $P_0 = O_r(G)$ and (b) $P_k = O_r\big(N_G(C_k)\big)$

for all $1 \le k \le n$. Thus P_{k+1} is a radical subgroup of $N(C_k)$, where $0 \le k \le n-1$ and an r-subgroup R of G is *radical* if $R = O_r\big(N_G(R)\big)$. Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical r-chains of G.

Given $C \in \mathcal{R}(G)$, $B \in \operatorname{Blk}(G)$ and d a non-negative integer, let $\operatorname{Blk}\big(N(C) \mid B\big) = \big\{b \in \operatorname{Blk}\big(N(C)\big) : b^G = B\big\}$ (in Brauer sense) and let $\mathrm{k}\big(N(C), B, d\big)$ be the number of characters of the set

(1.4) $$\operatorname{Irr}\big(N(C), B, d\big) = \big\{\psi \in \operatorname{Irr}\big(N(C)\big) : B(\psi)^G = B, \text{ and } \mathrm{d}(\psi) = d\big\},$$

where $B(\psi)$ is the block of $N(C)$ containing ψ and $\mathrm{d}(\psi)$ is the r-defect of ψ (see [10], (5.5) for the definition). Then the following is Dade's ordinary conjecture, [10, Conjecture 6.3].

DADE'S CONJECTURE. *If $O_r(G) = 1$ and B is an r-block of a finite group G with defect* $\mathrm{d}(B) > 0$, *and if d is a non-negative integer, then*

(1.5) $$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}\big(N(C), B, d\big) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits in $\mathcal{R}$.

Given $h \in \mathbb{Z}$ and $B \in \operatorname{Blk}(G)$, let $\operatorname{Irr}(B)$ be the set of irreducible ordinary characters of B, $\mathrm{k}(B) = |\operatorname{Irr}(B)|$ and let $\mathrm{k}(B, h)$ be the characters of the set

(1.6) $$\operatorname{Irr}(B, h) = \{\chi \in \operatorname{Irr}(B) : \mathrm{h}(\chi) = h\},$$

where the *height* $\mathrm{h}(\chi)$ of $\chi \in \operatorname{Irr}(B)$ is defined on page 151 of [12].

2. **Radical chains of $G_2(q)$.** The notation and terminology of Section 1 are continued in this section. Let $\mathbb{F}_q$ be the field of q elements of characteristic p distinct from r, and G the Chevalley group $G_2(q)$. Throughout this paper we shall also follow the notation of [2] and [18]. In particular, if $\eta = \pm$, then $2_\eta^{1+2\gamma}$ denotes the extraspecial 2-group of order $2^{1+2\gamma}$ and type η, and if r is odd, then $r_\eta^{1+2\gamma}$ denotes the extraspecial r-group of order $r^{1+2\gamma}$ with exponent r or r^2 according as $\eta = +$ or $-$. Given an integer $n \ge 0$, denote by D_{2n} a dihedral group of order $2n$, by E_{r^n} an elementary abelian group of order r^n, and by $\mathbb{Z}_n$ a cyclic group of order n.

For a sign $\delta = \pm$, let T_δ, T_2^δ, and T_3^δ be maximal tori of G such that $T_\delta \simeq \mathbb{Z}_{q-\delta} \times \mathbb{Z}_{q-\delta}$, $T_2^\delta \simeq \mathbb{Z}_{q^2+\delta q+1}$, and $T_3^\delta \simeq \mathbb{Z}_{q^2-1}$. Here, for simplicity, we always identify $q-\delta$ with $q-\delta 1$.

Denote by K_δ a maximal subgroup of G which contains a subgroup $L_\delta \simeq \mathrm{SL}(3,\delta q)$ of index 2, where $\mathrm{SL}(3,-q) = \mathrm{SU}(3,q)$. Thus K_δ is L_δ extended by an involutory outer automorphism (see [5, p. 254]). Moreover, T_δ and T_i^δ for $i = 2, 3$ can be embedded as maximal tori of K_δ (see (15.2) of [5]) and $N_G(T_\delta)/T_\delta \simeq D_{12}$, $N_G(T_2^\delta)/T_2^\delta \simeq \mathbb{Z}_6$, $N_G(T_3^\delta)/T_3^\delta \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (see Table I of [18]). In addition, if q is odd, then by Theorem A of [18] and Theorem 3 (2) of [5], G has only one conjugacy class of involutions z, and $C_G(z) \simeq \mathrm{SO}^+(4,q)$. By Lemma 2.4 of [18], G has only one class of subgroups E_8, and $C_G(E_8) = E_8$, $N_G(E_8)/E_8 \simeq \mathrm{GL}(3,2)$.

We may always suppose a block $B \in \mathrm{Blk}(G)$ has a non-cyclic group, since otherwise Dade's conjecture for B follows by Theorem 9.1 of [10]. So $r|(q^2-1)$ (see p. 25 of [2]). Let r^a or r^{a+1} be the exact power of r dividing q^2-1 according as r is odd or even, and let the sign ϵ be chosen such that $r^a|q-\epsilon$.

Let $\Phi(G,r)$ be a set of representatives for conjugacy G-classes of radical r-subgroups of G and denote by H^* a non-conjugate subgroup of G which is isomorphic to a subgroup H of G. Suppose $r \geq 3$, then by pages 358–159 of [15] and Table II of [18], G has two classes of subgroups isomorphic to $\mathrm{GL}(2,\epsilon q)$ with representatives L and L^*. We may suppose $L \leq K_\epsilon$, so that $Z(L_\epsilon) = \Omega_1\Big(O_3\big(Z(L)\big)\Big)$ when $r = 3$. Let $\mathbb{Z}_{r^a} = O_r\big(Z(L)\big)$ and $\mathbb{Z}^*_{r^a} = O_r\big(Z(L^*)\big)$. The elements of $\Phi(G,r)$ can be obtained from (1D), (1E) and (1G) of [2] and their proofs. Suppose $r \geq 5$. Then we may take

$$\Phi(G,r) = \{1, \mathbb{Z}_{r^a}, \mathbb{Z}^*_{r^a}, O_r(T_\epsilon)\}.$$

Suppose $r = 3$. Then we may take

$$\Phi(G,3) = \begin{cases} \{1, Z(L_\epsilon) = \mathbb{Z}_3, \mathbb{Z}_3^*, O_3(T_\epsilon), E = S\} & \text{if } a = 1 \text{ and } 3 \neq q-\epsilon, \\ \{1, Z(L_\epsilon) = \mathbb{Z}_3, \mathbb{Z}_3^*, E = S\} & \text{if } a = 1 \text{ and } 3 = q-\epsilon, \\ \{1, Z(L_\epsilon), \mathbb{Z}_{3^a}, \mathbb{Z}^*_{3^a}, O_3(T_\epsilon), E, E^*, S\} & \text{if } a \geq 2 \text{ and } 3^a \neq q-\epsilon, \\ \{1, Z(L_\epsilon), \mathbb{Z}_{3^a}, \mathbb{Z}^*_{3^a}, E, E^*, S\} & \text{if } a \geq 2 \text{ and } 3^a = q-\epsilon, \end{cases}$$

where $E \simeq E^* \simeq 3_+^{1+2}$ and $S \in \mathrm{Syl}_3(G)$ is a Sylow 3-subgroup.

We may suppose $\mathbb{Z}_{r^a}, \mathbb{Z}^*_{r^a} \leq O_r(T_\epsilon)$. Define radical r-chains $C(1)$ and $C(2)$ as follows:

$$\text{(2.1)} \qquad \begin{aligned} &C(1)\colon \begin{cases} 1 < \mathbb{Z}_{r^a} < O_r(T_\epsilon) & \text{if } r \geq 5, \\ 1 < Z(L_\epsilon) & \text{if } r = 3, \end{cases} \\ &C(2)\colon 1 < \mathbb{Z}^*_{r^a} < O_r(T_\epsilon) \quad \text{if } r \geq 3. \end{aligned}$$

If $C \in \mathcal{R}(G)$ is given by (1.1) with $|C| \geq 2$, then we may suppose $P_1 \in \Phi(G,r)$ and $P_2 \in \Phi\big(N_G(P_1), r\big)$. Suppose $C_G(P_1) \simeq \mathrm{GL}(2,\epsilon q)$, so that by (1C) of [2], P_2 is radical in $C_G(P_1)$ as $r \geq 3$. Thus we may take $P_2 = O_r(T_\epsilon)$. It follows that if $r \geq 5$, then we may take

$$\mathcal{R}(G)/G = \{C(1)_0, C(1)_1, C(2)_1, C(1)^1, C(1), C(2)\}$$

and set $\mathcal{R}^0(G) = \mathcal{R}(G)$. If $r = 3$, then let $\mathcal{R}^0(G)$ be a G-invariant subfamily of $\mathcal{R}(G)$ such that

$$\text{(2.2)} \qquad \mathcal{R}^0(G)/G = \{C(1)_0, C(1), C(2)_1, C(2)\}.$$

(2A). *Let $G = G_2(q)$. In the notation above, suppose $r \geq 3$. If $B \in \mathrm{Blk}(G)$ with $\mathrm{d}(B) > 0$ and $d \in \mathbb{Z}$ with $d \geq 0$, then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}\big(N(C), B, d\big) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}\big(N(C), B, d\big).$$

PROOF. Suppose $r = 3$ and $C \in \mathcal{R}(G)$ is given by (1.1) with $|C| \geq 1$. We may suppose $P_1 \in \Phi(G,3)$ and $P_1 \leq L_\epsilon$. Let $\Phi^+(G,3)$ be the subset of $\Phi(G,3)$ of non-abelian radical 3-subgroups. If $P_1 \in \Phi^+(G,3)$, then $Z(P_1) = Z(L_\epsilon)$. If $P_1 = \mathbb{Z}_{3^a}$, then $\Omega_1(P_1) = Z(L_\epsilon)$. If $P_1 = O_3(T_\epsilon)$, then $N_G(P_1) = N_G(T_\epsilon) = N_{K_\epsilon}(T_\epsilon)$ (see (1E) (b) of [2]). In all cases $N_G(P_1) \leq N_G\big(Z(L_\epsilon)\big) = K_\epsilon$. Given $R \in \Phi^+(G,3) \cup \{\mathbb{Z}_{3^a}, O_3(T_\epsilon)\}$ with $|R| > 3$, define G-invariant subfamilies $\mathcal{M}^+(G)$ and $\mathcal{M}^0(G)$ of $\mathcal{R}(G)$ such that

$$\begin{aligned} &\mathcal{M}^+(R)/G = \{C' \in \mathcal{R}(G)/G : P_1 = R\}, \quad \text{and} \\ &\mathcal{M}^0(R)/G = \{C' \in \mathcal{R}(G)/C : P_1 = Z(L_\epsilon), P_2 = R\}. \end{aligned} \tag{2.3}$$

If $C' \in \mathcal{M}^+(R)$ is given by (1.1), then

$$g(C') \colon 1 < Z(L_\epsilon) < R = P_1 < P_2 < \ldots < P_n$$

is an element of $\mathcal{M}^0(R)$ and $N(C') = N\big(g(C')\big)$. Thus

$$\mathrm{k}\big(N(C'), B, d\big) = \mathrm{k}\Big(N\big(g(C')\big), B, d\Big) \tag{2.4}$$

and the contributions of C' and $g(C')$ in the sum (1.5) is zero. It is clear that g induces a bijection between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$. We may suppose

$$C \in \mathcal{X}(G) = \mathcal{R}(G) \Big\backslash \bigcup_R \big(\mathcal{M}^0(R) \cup \mathcal{M}^+(R)\big),$$

where R runs over the set $\Phi^+(G,3) \cup \{\mathbb{Z}_{3^a}, O_3(T_\epsilon)\}$ with $|R| > 3$. So $P_1 \in \{Z(L_\epsilon), \mathbb{Z}_{3^a}^*\}$.

Suppose $|C| \geq 2$. If $P_1 = Z(L_\epsilon)$, then by (1C) of [2], we may take $P_2 \in \Phi(L_\epsilon, 3)$, so that $P_2 \in \Phi(G,3) \backslash \{1, Z(L_\epsilon), \mathbb{Z}_{3^a}^*\}$ and $C \in \mathcal{M}^0(P_2)$. Contradiction. If $P_1 = \mathbb{Z}_{3^a}^*$, then as shown before (2A), we may take $P_2 = O_3(T_\epsilon)$ and $C = C(2)$. It follows that $\mathcal{X}(G) = \mathcal{R}^0(G)$, and this proves (2A).

Suppose $r = 2$, so that q is odd and $\Phi(G,2)$ is given in Section 2 of [2]. A 2-group R is called of *symplectic type* if R is a central product EP over $Z(E) = \Omega_1\big(Z(P)\big)$ of an extraspecial 2-group E and a 2-group P such that P is either cyclic or isomorphic to a semidihedral group SD_{2^β}, a dihedral group D_{2^β} or a generalized quaternion group Q_{2^β} of order 2^β, where $\beta \geq 4$. Given a sign δ, suppose T_δ is a torus of $L_\delta = \mathrm{SL}(3, \delta q)$ and of $C_G(z) = \mathrm{SO}^+(4,q)$. As shown in the proof of (3B) of [2], we may suppose

$$N(T_\delta) = N_G(T_\delta) = \langle T_\delta, \rho, \sigma, \tau \rangle, \tag{2.5}$$

$N_{L_\delta}(T_\delta) = \langle T_\delta, \sigma, \tau \rangle$ and $N_{\mathrm{SO}^+(4,q)}(T_\delta) = \langle T_\delta, \rho, \tau \rangle$, where $|\rho| = |\tau| = 2$, $|\sigma| = 3$, $\rho \in Z\big(N(T_\delta)/T_\delta\big)$, and $\tau\sigma\tau = \sigma^{-1}$. In addition, $\langle \rho, \sigma, \tau \rangle = D_{12}$, ρ acts on T_δ by $t^\rho = t^{-1}$ for all $t \in T_\delta$.

If $2^a \neq q-\epsilon$, then $O_2(T_\epsilon) \neq T_\epsilon$ and $O_2(T_{-\epsilon}) \neq T_{-\epsilon}$. From Section 2 of [2], we may take

$$\Phi(G,2) = \{1, \mathbb{Z}_2, \mathbb{Z}_{2^a}, \mathbb{Z}_{2^{a+1}}, E_8, O_2(T_{-\epsilon}), O_2(T_\epsilon), \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, \langle O_2(T_\epsilon), \rho\rangle, EP, S\},$$

where $\mathbb{Z}_2 = O_2\big(C_G(z)\big) = O_2\big(\mathrm{SO}^+(4,q)\big)$, $\mathbb{Z}_{2^a} = O_2\big(\mathrm{GL}(2,\epsilon q)\big)$, $\mathbb{Z}_{2^{a+1}} = O_2(T_3^\delta)$, EP is a non-abelian 2-groups of symplectic type, and $S \in \mathrm{Syl}_2(G)$. In addition, G has two classes of $\mathbb{Z}_{2^a}, \mathbb{Z}_{2^{a+1}}, \mathbb{Z}_{2^a}\wr\mathbb{Z}_2$, and EP except when $EP = 2_+^{1+4}$, $q \equiv \pm 3 \pmod 8$, and $EP = 2_+^{1+2}\mathbb{Z}_{2^a}$, in which cases G has one class of EP. If $2^a = q-\epsilon$ and $q \neq 3$, then $O_2(T_\epsilon) = T_\epsilon$ and $O_2(T_{-\epsilon}) \neq T_{-\epsilon}$, $O_2(T_\epsilon)$ is not a radical 2-subgroup of G since $O_2\big(N_G(T_\epsilon)/T_\epsilon\big) \simeq \mathbb{Z}_2$, so that by Section 2 of [2], we may take

$$\Phi(G,2) = \{1, \mathbb{Z}_2, \mathbb{Z}_{2^a}, \mathbb{Z}_{2^{a+1}}, E_8, O_2(T_{-\epsilon}), \langle O_2(T_\epsilon), \rho\rangle, EP, S\},$$

and similarly, if $q=3$, then $\mathbb{Z}_{2^{a+1}}$, $O_2(T_\epsilon)$ and $O_2(T_{-\epsilon})$ are non-radical 2-subgroups in G, and so we may take

$$\Phi(G,2) = \{1, \mathbb{Z}_2, \mathbb{Z}_{2^a}, E_8, \langle O_2(T_\epsilon), \rho\rangle, EP, S\},$$

where the radical 2-subgroups are defined as above.

We define chains $C(i)$ for $1 \leq i \leq 4$ as follows.

$$(2.6)\qquad \begin{aligned} &C(1)\colon 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon}),\\ &C(2)\colon \begin{cases} 1 < \mathbb{Z}_2 < O_2(T_\epsilon) & \text{if } 2^a \neq q-\epsilon,\\ 1 < \langle O_2(T_\epsilon), \rho\rangle < S' & \text{if } 2^a = q-\epsilon,\end{cases}\\ &C(3)\colon 1 < E_8 < 2_+^{1+4},\\ &C(4)\colon 1 < E_8 < W < S'',\end{aligned}$$

where $2_+^{1+4} \leq N_G(E_8)$, $W = \big\langle \rho, \Omega_2\big(O_2(T_\epsilon)\big)\big\rangle \leq N_G(E_8)$ containing $E_8 = \big\langle \rho, \Omega_1\big(O_2(T_\epsilon)\big)\big\rangle$, $W/E_8 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, $S'' \in \mathrm{Syl}_2\big(N_G(E_8)\big)$ containing W, and $S' \in \mathrm{Syl}_2\big(N_G(T_\epsilon)\big)$. In the following we are going to show that each $C(j) \in \mathcal{R}(G)$ except when $j=1$ and $q=3$, in which case $C(1)^1, C(1) \notin \mathcal{R}(G)$. Let $\mathcal{R}^0(G)$ be a G-invariant subfamily of $\mathcal{R}(G)$ such that if $q \neq 3$, then

$$(2.7)\quad \mathcal{R}^0(G)/G = \{C(1)_0, C(1)_1, C(1)^1, C(1), C^1(2), C(2), C(3)_1, C(3), C(4)_2, C(4)\},$$

where $C^1(2) = C(2)^1$ or $C(2)_1$ according to whether $2^a \neq q-\epsilon$ or $2^a = q-\epsilon$; if $q=3$, then

$$(2.8)\qquad \mathcal{R}^0(G)/G = \{C(1)_0, C(1)_1, C(2)_1, C(2), C(3)_1, C(3), C(4)_2, C(4)\}.$$

We have the following proposition.

(2B). *If B is a 2-block of $G = G_2(q)$ with* $\mathrm{d}(B) > 0$ *and if* $d \in \mathbb{Z}$ *with* $d \geq 0$, *then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}\big(N_G(C), B, d\big) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}\big(N_G(C), B, d\big).$$

PROOF. The proof of (2B) is similar to that of (2A). Suppose $C \in \mathcal{R}(G)$ is given by (1.1) with $|C| \geq 1$ and $P_1 \in \Phi(G,2)$. Let $N(C) = N_G(C)$ and $C(C) = C_G(C)$.

(1) Suppose $P_1 = \mathbb{Z}_2$ and $|C| \geq 2$, so that we may take $P_2 \in \Phi\big(\mathrm{SO}^+(4,q),2\big)$. As shown in the proof of (2B) and (2C) of [2], we may suppose $P_2 \in \Phi(G,2)$ and both E_8 and $\langle O_2(T_\epsilon), \rho\rangle$ are non-radical 2-subgroups of $\mathrm{SO}^+(4,q)$. Let

$$\Omega = \{1, \mathbb{Z}_2, O_2(T_{-\epsilon}), E_8, O_2(T_\epsilon), \langle O_2(T_\epsilon), \rho\rangle\} \subseteq \Phi(G,2)$$

and take $R \in \Phi(G,2)\backslash\Omega$. Then $\mathbb{Z}_2 = \Omega_1\big(Z(R)\big)$. Replace $Z(L_\epsilon)$ by $\mathbb{Z}_2$ in (2.3). Then (2.4) holds and we may suppose

$$C \notin \bigcup_R \big(\mathcal{M}^+(R) \cup \mathcal{M}^0(R)\big),$$

where R runs over the set $\Phi(G,2)\backslash\Omega$. Thus we may take $P_1 \in \Omega$ and if $P_1 = \mathbb{Z}_2$ with $|C| \geq 2$, then we may suppose $P_2 \in \{O_2(T_{-\epsilon}), O_2(T_\epsilon)\}$. In particular, if $q = 3$ and $P_1 = \mathbb{Z}_2$, then $|C| = 1$, and if $2^a = q - \epsilon$ and $q \neq 3$, then $P_2 = O_2(T_{-\epsilon})$.

(2) If $P_1 = O_2(T_\epsilon)$, then $2^a \neq q - \epsilon$, $N(P_1) = N(T_\epsilon) = \langle T_\epsilon, \sigma, \rho, \tau\rangle$ and $\langle \rho, \tau\rangle \in \mathrm{Syl}_2(D_{12})$. Thus we may take

$$\Phi\Big(N\big(O_2(T_\epsilon)\big), 2\Big) = \{O_2(T_\epsilon), \langle O_2(T_\epsilon), \rho\rangle, \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, (\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^*, S'\},$$

where $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2 = \langle O_2(T_\epsilon), \tau\rangle$, $(\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^* = \langle O_2(T_\epsilon), \rho\tau\rangle$ and $S' = \langle O_2(T_\epsilon), \rho, \tau\rangle$. Replace $Z(L_\epsilon)$ by $O_2(T_\epsilon)$ and R by $\langle O_2(T_\epsilon), \rho\rangle$ in (2.3). Then (2.4) holds. Thus we may suppose $P_2 \in \{\mathbb{Z}_{2^a} \wr \mathbb{Z}_2, (\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^*, S'\}$. Moreover, we may suppose P_1 and $\langle O_2(T_\epsilon), \rho\rangle$ are not conjugate in G.

Suppose $2^a = q - \epsilon$. If $P_1 = \langle O_2(T_\epsilon), \rho\rangle$, then $N(P_1) = N(T_\epsilon)$ and $P_2 = S' \in \mathrm{Syl}_2\big(N(P_1)\big)$. Thus $C(2)$ and $C(2)_1$ are radical chains.

(3) Suppose $C' \in \mathcal{R}(G)$ such that

$$C' \colon 1 < O_2(T_\epsilon) < P_2 < \cdots < P_n,$$

where $P_2 \in \{\mathbb{Z}_{2^a} \wr \mathbb{Z}_2, (\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^*, S'\}$. Define radical 2-chain

$$g(C') \colon 1 < \mathbb{Z}_2 < O_2(T_\epsilon) < P_2 < \cdots < P_n.$$

Then $N(C') = N\big(g(C')\big)$ since $N(T_\epsilon) \cap N(P_2) \leq \langle T_\epsilon, \rho, \tau\rangle \leq N(\mathbb{Z}_2)$. Thus (2.4) holds for C' and $g(C')$. Since $N_{N(\mathbb{Z}_2)}\big(O_2(T_\epsilon)\big) = \langle T_\epsilon, \rho, \tau\rangle$, it follows that the normalizer of $\langle O_2(T_\epsilon), \rho\rangle$ in $N_{N(\mathbb{Z}_2)}\big(O_2(T_\epsilon)\big)$ is a Sylow 2-subgroup $\langle O_2(T_\epsilon), \rho, \tau\rangle$ of $N(T_\epsilon)$, so that $\langle O_2(T_\epsilon), \rho\rangle$ is a non-radical subgroup of $N_{N(\mathbb{Z}_2)}\big(O_2(T_\epsilon)\big)$. Thus we may take

$$\Phi\Big(N_{N(\mathbb{Z}_2)}\big(O_2(T_\epsilon)\big), 2\Big) = \{O_2(T_\epsilon), \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, (\mathbb{Z}_{2^a} \wr \mathbb{Z}_2)^*, S'\}.$$

It follows that the remaining chains $C \in \mathcal{R}(G)$ such that P_1 is conjugate to $\mathbb{Z}_2$ or $O_2(T_\epsilon)$ and $|P_2| \neq 4$ have representatives, $C(1)_1$, $C(2)$ and $C(2)^1$.

(4) If $P_1 = E_8$, then $N(E_8)/E_8 = \mathrm{GL}(3,2)$ and we may take $P_2/E_8 \in \Phi\big(\mathrm{GL}(3,2), 2\big)$. It follows by the Borel and Tits theorem, [8] that either $P_2/E_8 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $P_2 = S''$. In the formal case, P_2/E_8 is the unipotent radical of a parabolic subgroup of GL(3, 2), and $N_{N(E_8)}(P_2)/P_2 \simeq \mathrm{GL}(2,2)$. So $N(E_8)$ has two non-conjugate subgroups K and W such that $K/E_8 \simeq W/E_8 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. We may suppose $P_2 \in \{K, W, S''\}$.

In the notation of (2B) of [2] $2_+^{1+4} = \langle x_1, x_2, x_3, x_4\rangle$ can be regarded as a subgroup of $C_G(z) = \mathrm{SO}^+(4,q)$, where $\langle x_1, x_2\rangle \simeq \langle x_3, x_4\rangle \simeq D_8$. Thus $\langle z, x_1, x_3\rangle \simeq E_8$. Since G has exactly one class of subgroups E_8, we may suppose $E_8 = \langle z, x_1, x_3\rangle$ and so $2_+^{1+4} \leq N(E_8)$. The group induced by actions of elements of 2_+^{1+4} on E_8 is the unipotent radical of a parabolic subgroup of $N(E_8)/E_8$. So we may suppose $K = 2_+^{1+4}$.

Let $N(T_\epsilon) = \langle T_\epsilon, \rho, \sigma, \tau\rangle$, $\Omega_2\big(O_2(T_\epsilon)\big) = \langle u, w\rangle$ and $X = \langle \rho, u, w\rangle$, where ρ, σ, τ are given by (2.5). Then $|u| = |w| = 4$ and $\sigma, \tau \in N\big(\Omega_1(X)\big)$. But $\Omega_1(X) = \langle \rho, u^2, w^2\rangle \simeq E_8$, so we may identify $\Omega_1(X)$ with E_8. Let $N' = \langle \sigma, \tau, X\rangle$. Then $N' \leq N(E_8)$, $N'/X \simeq S_3$ and $X/\Omega_1(X) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. It follows from the actions of elements of N' on $E_8 = \Omega_1(X)$ that N'/E_8 is a parabolic subgroup of $N(E_8)/E_8$, so that $X = O_2(N')$ is a radical 2-subgroup of $N(E_8)$. Since $Z(X) = \langle u^2, w^2\rangle$ has order 4, it follows that $X \not\simeq K$ and we may suppose $W = X$. If $a \neq 2$, then $N' \neq N(W)$ since $O_2(T_\epsilon) \leq N(W)$. If $a = 2$, then $W = \langle \rho, O_2(T_\epsilon)\rangle$ and $N' = N(W)$.

Apply (2.4) to

$$C' \colon 1 < E_8 < 2_+^{1+4} < S'' \text{ and } g(C') \colon 1 < E_8 < S''.$$

Then the remaining chains in $\mathcal{R}(G)$ with $P_1 \simeq E_8$ have representatives $C(3)$, $C(3)_1$, $C(4)$ and $C(4)_2$.

(5) If $P_1 = O_2(T_{-\epsilon})$, then $q \neq 3$ and $N(P_1) = N(T_{-\epsilon}) = \langle T_{-\epsilon}, \sigma, \tau, \rho\rangle$, where σ, τ, ρ are given as before. Thus we may suppose $P_2 \in \{E_8, D_8, D_8^*, Q\}$, where $E_8 = \langle P_1, \rho\rangle$, $D_8 = \langle P_1, \tau\rangle$, $D_8^* = \langle P_1, \rho\tau\rangle$, and $Q = \langle E_8, \tau\rangle \in \mathrm{Syl}_2\big(N(P_1)\big)$. Let

$$C' \colon 1 < O_2(T_{-\epsilon}) < E_8 \text{ and } g(C') \colon 1 < O_2(T_{-\epsilon}) < E_8 < Q.$$

Since $C(E_8) = E_8 \trianglelefteq N(C')$, it follows by Corollary V.3.11 of [12] that $\mathrm{Irr}\big(b(C')\big) = \mathrm{Irr}\big(N(C')\big)$, where $b(C')$ is the principal block $B_0\big(N(C')\big)$ of $N(C')$. Similarly, $\mathrm{Irr}\Big(b\big(g(C')\big)\Big) = \mathrm{Irr}\Big(N\big(g(C')\big)\Big)$ with $b\big(g(C')\big) = B_0\Big(N\big(g(C')\big)\Big)$. Now $N\big(g(C')\big) = Q$ and $N(C') = \langle Q, \sigma\rangle$. By Clifford thaory, Irr(Q) has 8 linear characters and 2 characters of degree 2, and $\mathrm{Irr}(\langle Q, \sigma\rangle)$ has 4 linear characters, 2 characters of degree 2 and 4 of degree 3. Thus $\mathrm{k}\big(b(C'), h\big) = \mathrm{k}\Big(b\big(g(C')\big), h\Big)$ and by (5.7) of [10],

$$\mathrm{k}\big(N(C'), B, d\big) = \mathrm{k}\Big(N\big(g(C')\big), B, d\Big),$$

so that (2.4) holds and we may take $P_2 \in \{D_8, D_8^*, Q\}$. Let

$$C' \colon 1 < O_2(T_{-\epsilon}) < P_2 < \cdots < P_n, \text{ and}$$
$$g(C') \colon 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon}) < P_2 < \cdots < P_n.$$

Then $g(C') \in \mathcal{R}(G)$ and $N(C') = N\big(g(C')\big)$ since $N(T_{-\epsilon}) \cap N(P_2) \leq T_{-\epsilon}Q \leq N(\mathbb{Z}_2)$. So (2.4) holds and a proof similar to that of (3) above shows that we may take

$$\Phi\Big(N_{N(\mathbb{Z}_2)}\big(O_2(T_{-\epsilon})\big), 2\Big) = \{O_2(T_{-\epsilon}), D_8, D_8^*, Q\}.$$

Thus the remaining radical chains $C \in \mathcal{R}(G)$ such that P_1 is conjugate to $\mathbb{Z}_2$ or $O_2(T_{-\epsilon})$ and P_2 is not conjugate to $O_2(T_\epsilon)$ have representatives, $C(1)_1$, $C(1)$ and $C(1)^1$. Thus (2B) follows.

(2C). *Let* $C \in \mathcal{R}^0(G)$, $C(C) = C_G(C)$, *and* $N(C) = N_G(C)$. *Suppose* $r \geq 3$ *and* $|C| \geq 1$. *Then*

C	$C(C)$	$N(C)$	Conditions
$C(1)_1: 1 < \mathbb{Z}_{r^a}$	$\mathrm{GL}(2, \epsilon q)$	$\langle \mathrm{GL}(2, \epsilon q), \rho \rangle$	$r \geq 5$
$C(1)^1: 1 < O_r(T_\epsilon)$	T_ϵ	$N_G(T_\epsilon)$	$r \geq 5$
$C(2)_1: 1 < \mathbb{Z}_{r^a}^*$	$\mathrm{GL}(2, \epsilon q)$	$\langle \mathrm{GL}(2, \epsilon q), \rho \rangle$	$r \geq 3$
$C(1): 1 < \mathbb{Z}_{r^a} < O_r(T_\epsilon)$	T_ϵ	$\langle T_\epsilon, \rho, \tau \rangle$	$r \geq 5$
$C(2): 1 < \mathbb{Z}_{r^a}^* < O_r(T_\epsilon)$	T_ϵ	$\langle T_\epsilon, \rho, \tau \rangle$	$r \geq 3$
$C(1): 1 < Z(L_\epsilon)$	L_ϵ	K_ϵ	$r = 3$

where ρ, σ, τ *are given by* (2.5).

PROOF. The proof is given either by that of (2A) or by Section 1 of [2].

(2D). *Let* $C \in \mathcal{R}^0(G)$, $C(C) = C_G(C)$, *and* $N(C) = N_G(C)$. *Suppose* $r = 2$ *and* $|C| \geq 1$. *If* $2^a \neq q - \epsilon$, *then*

C	$C(C)$	$N(C)/C(C)P_{\lvert C \rvert}$
$C(1)_1: 1 < \mathbb{Z}_2$	$\mathrm{SO}^+(4, q)$	1
$C(1)^1: 1 < O_2(T_{-\epsilon})$	$\langle T_{-\epsilon}, \rho \rangle$	$\mathrm{GL}(2, 2)$
$C(2)^1: 1 < O_2(T_\epsilon)$	T_ϵ	D_{12}
$C(3)_1: 1 < E_8$	E_8	$\mathrm{GL}(3, 2)$
$C(1): 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon})$	$\langle T_{-\epsilon}, \rho \rangle$	$\mathbb{Z}_2$
$C(2): 1 < \mathbb{Z}_2 < O_2(T_\epsilon)$	T_ϵ	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$C(3): 1 < E_8 < 2_+^{1+4}$	$Z(2_+^{1+4})$	$\mathrm{GL}(2, 2)$
$C(4)_2: 1 < E_8 < W$	$Z(W)$	$\mathrm{GL}(2, 2)$
$C(4): 1 < E_8 < W < S''$	$Z(S'')$	1

where ρ, σ, τ *are given by* (2.5), $E_8 = \big\langle \rho, \Omega_1\big(O_2(T_\epsilon)\big)\big\rangle$, $W = \big\langle \rho, \Omega_2\big(O_2(T_\epsilon)\big)\big\rangle \leq N(E_8)$ *with* $N_{N(E_8)}(W) = \langle \sigma, \tau, W \rangle$, *and* $S'' \in \mathrm{Syl}_2\big(N(E_8)\big)$ *containing* W. *If* $2^a = q - \epsilon$, *then* $C(2)_1: 1 < \langle O_2(T_\epsilon), \rho \rangle$ *and* $C(2): 1 < \langle O_2(T_\epsilon), \rho \rangle < S'$, *where* $S' \in \mathrm{Syl}_2\big(N(T_\epsilon)\big)$. *Moreover,* $C\big(C(2)_1\big) = \Omega_1\big(O_2(T_\epsilon)\big)$, $N\big(C(2)_1\big) = N(T_\epsilon)$, $C\big(C(2)\big) = Z(S')$, *and* $N\big(C(2)\big) = S'$.

PROOF. The proof is given either by that of (2B) or by Section 2 of [2].

3. **The conjecture for odd primes.** The notation and terminology of Sections 1 and 2 are continued in this section. We shall identify a dual group of $G = G_2(q)$ with G. Let $\mathcal{E}\big(G, (s)\big)$ be the set of the irreducible constituents of Deligne-Lusztig generalized characters associated with the conjugacy class (s) of a semisimple element $s \in G$, and let

$$\mathcal{E}_r\big(G, (s)\big) = \bigcup_u \mathcal{E}\big(G, (su)\big),$$

where $s \in G_{r'}$ is semisimple and u runs over all the r-elements of $C_G(s)$. Then $\mathcal{E}_r\big(G, (s)\big)$ is a union of r-blocks.

Let $\mathrm{Blk}^0(G, r)$ be the set of r-blocks of G with non-cyclic defect groups. In the rest of this section we suppose $r \geq 3$. By [15] and [16],

$$\mathrm{Blk}^0(G, r) = \{B_1, B_2, B_3, B_a, B_b, B_{X_\alpha}\}, \tag{3.1}$$

where $B_1 = B_0(G)$, and $\alpha = 1$ or 2 according as $\epsilon = +$ or $-$. In addition, if q is even, then there is no block B_2. If $\gcd(q, 3) = 3$, then there is no block B_3. Suppose $B \subseteq \mathcal{E}_r\big(G, (s)\big)$ for some semisimple r'-element s of G. Then $C_G(s)$ is given by (3.4) of [2]. Moreover, suppose $r = 3$, then by Tables I and II of [16],

$$\mathrm{k}(B_1, h) = \begin{cases} 9 & \text{if } h = 0, \\ (3^a - 1) + \frac{1}{12}(3^a - 3)(3^a + 9) & \text{if } a \neq 1 \text{ and } h = 1, \\ 3 & \text{if } a \neq 1 \text{ and } h = 2a - 1, \\ 5 & \text{if } a = 1 \text{ and } h = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

(3A). *Let $R = \mathbb{Z}^*_{r^a}$ or $R \in \{\mathbb{Z}_{r^a}, \mathbb{Z}^*_{r^a}\}$ according as $r = 3$ or $r \geq 5$. Set $H = N_G(R)$. Then Alperin-McKay conjecture holds for each block $B \in \mathrm{Blk}^0(H, r)$ and $\mathrm{k}(B) = \mathrm{k}(B, 0)$.*

PROOF. By (2C), $H = \langle \mathrm{GL}(2, \epsilon q), \rho\rangle$ for some $\rho \in G$. Let D be a defect group of B, $K = N_H(D)$ and $b \in \mathrm{Blk}(K \mid B)$. Then $K = \langle T_\epsilon, \rho, \tau\rangle$ for some $\tau \in \mathrm{GL}(2, \epsilon q)$ and $D \in \mathrm{Syl}_r(H)$. It follows by (8B) of [13] and Clifford theory that $\mathrm{k}(B) = \mathrm{k}(B, 0)$ and $\mathrm{k}(b, 0) = \mathrm{k}(b)$. The equation $\mathrm{k}(B) = \mathrm{k}(b)$ is essentially a consequence of Brauer's permutation lemma and (2H) of [13] (*cf.* the proof of (3G) of [1]). Thus (3A) holds.

(3B). *Let $G = G_2(q)$ and let r be a prime such that $\gcd(r, q) = 1$. If $r \geq 3$, then Dade's conjecture holds for G.*

PROOF. We may suppose $B \in \mathrm{Blk}^0(G, r)$. Let $C = C(2)_1$ and $C' = C(2)$. In the notation of (3A) $N(C) = H$ and $N(C') = K$. By Brauer's First and Third Main Theorems, $\mathrm{Blk}\big(N(C) \mid B\big) = \{b(C)\}$ and $b(C)$ has a defect group $D = O_r(T_\epsilon)$. Again by Brauer's First Main Theorem, $\mathrm{Blk}\big(N(C') \mid B\big) = \{b(C')\}$ and $b(C')^{N(C)} = b(C)$. By (3A), $\mathrm{k}\big(b(C), h\big) = \mathrm{k}\big(b(C'), h\big)$ for all $h \in \mathbb{Z}$ and by (5.7) of [10],

$$\mathrm{k}\big(N(C), B, d\big) = \mathrm{k}\big(N(C'), B, d\big) \tag{3.3}$$

for non-negative integers d. If $r \geq 5$, then let $C = C(1)_1$ and $C' = C(1)$. The same proof as above shows that (3.3) holds.

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Let $C = C(1)$ or $C(1)^1$ according as $r = 3$ or $r \geq 5$. In the former case $N(C) = K_\epsilon$, and in the latter case $N(C) = N(T_\epsilon)$. It suffices to show that

$$\text{(3.4)} \qquad \mathrm{k}(G,B,d) = \mathrm{k}\big(N(C),B,d\big),$$

for $d \in \mathbb{Z}$ with $d \geq 0$. A proof similar to above shows that $\mathrm{Blk}\big(N(C) \mid B\big) = \{b(C)\}$. If $r \geq 5$, then (3.4) follows by (1A) of [3]. Suppose $r = 3$ and B is a non-principal block. Then B has a defect group $D = O_r(T_\epsilon)$. By (1A) of [3], $\mathrm{k}(B) = \mathrm{k}(B,0) = \mathrm{k}(b_N) = \mathrm{k}(b_N,0)$, where $b_N \in \mathrm{Blk}\big(N(D) \mid B\big)$. Since $N(D) = N_{K_\epsilon}(D)$, it follows that $b_N \in \mathrm{Blk}\big(N(D) \mid b(C)\big)$. It suffices to show that

$$\text{(3.5)} \qquad \mathrm{k}(b_N) = \mathrm{k}\big(b(C)\big) = \mathrm{k}\big(b(C),0\big).$$

If q is an odd power of 2, then K_ϵ can be regarded as a maximal subgroup of the Ree group ${}^2F_4(q)$ by Main Theorem (3) of [20]. In this case and moreover, in the case $r|q+1$, (3.5) follows by the proof of (3A) (2) of [4]. If $3|q-1$, then replace $\mathrm{SU}(3,q)$ by $\mathrm{SL}(3,q)$, T_8 by T_ϵ, and some obvious modifications in the proof of (3A) (2) of [4], so that (3.5) still holds.

Finally, suppose $r = 3$ and $B = B_1$. Then $b(C) = B_0(K_\epsilon)$. If $3|q+1$, then (3.4) follows by (3.2) and [4], (3.6). If $3|q-1$, then (3.4) follows by a proof similar to that of (3.6) of [4] with $\mathrm{U}(3,q^2)$ replaced by $\mathrm{GL}(3,q)$, $\mathrm{SU}(3,q^2)$ by $\mathrm{SL}(3,q)$, and some obvious modifications.

4. **The conjecture for the even prime.**

The notation and terminology of Sections 1 and 2 are continued in this section. Suppose $r = 2$. By [17],

$$\mathrm{Blk}^0(G,2) = \{B_1, B_3, B_{1a}, B_{1b}, B_{2a}, B_{2b}, B_{X_1}, B_{X_2}\}$$

where $B_1 = B_0(G)$ and if $\gcd(q,3) = 3$, there is no block B_3. Suppose $B \in \mathrm{Blk}^0(G,2)$ such that $B \subseteq \mathcal{E}_2\big(G,(s)\big)$ for some semisimple $2'$-element s of G, so that $C_G(s)$ is given by (3.9) of [2]. Let $\eta = \pm$ such that $q \equiv \eta \pmod 3$.

Given $B \in \mathrm{Blk}(G)$, a radical chain C is called a *B-chain* if $\mathrm{Blk}\big(N(C) \mid B\big) \neq \emptyset$. If $B \in \mathrm{Blk}^0(G,2)$ has a defect group D, then by (3D) of [2], D is either abelian or $D \in \{SD_{2^{a+2}}, \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, S\}$, where $S \in \mathrm{Syl}_2(G)$.

(4A). *Suppose $B \in \mathrm{Blk}^0(G,2)$ with a defect group D. Then we may take*

$$\mathcal{R}^0(B)/G = \begin{cases} \{1, 1 < \mathbb{Z}_2, 1 < O_2(T_\delta), 1 < \mathbb{Z}_2 < O_2(T_\delta)\} & \text{if } D = O_2(T_\delta), \\ \{1, 1 < \mathbb{Z}_2, 1 < O_2(T_{-\epsilon}), 1 < \mathbb{Z}_2 < O_2(T_{-\epsilon})\} & \text{if } D \simeq SD_{2^{a+2}}, \\ \{1, 1 < \mathbb{Z}_2, 1 < O_2(T_\epsilon), 1 < \mathbb{Z}_2 < O_2(T_\epsilon)\} & \text{if } D \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2, \\ \mathcal{R}^0(G)/G & \text{if } B = B_1. \end{cases}$$

PROOF. The proof follows by that of (3E), (3F) and (3G) of [2] and Lemma 6.9 of [10].

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(4B). *Let $G = G_2(q)$ and $B \in \mathrm{Blk}^0(G,2)\backslash\{B_{X_1}, B_{X_2}\}$.*
(a) *Suppose $B = B_1$. Then*

$$\mathrm{k}(B,h) = \begin{cases} 8 & \text{if } h = 0, \\ 2^{a+1} - 2 & \text{if } h = 1, \\ \frac{1}{12}(2^a - 4)(2^a - 2) & \text{if } h = 2, \\ 2 & \text{if } h = 2a - 1, \\ 2^{a-1} & \text{if } h = a + 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *Suppose $B = B_3$. Then*

$$\mathrm{k}(B,h) = \begin{cases} 2^{a+1} & \text{if } \eta = \epsilon \text{ and } h = 0, \\ 1 + \frac{1}{6}(2^a - 2)(2^a - 1) & \text{if } \eta = \epsilon \text{ and } h = 1, \\ 2^{a-1} & \text{if } \eta = \epsilon \text{ and } h = a, \\ 4 & \text{if } \eta = -\epsilon \text{ and } h = 0, \\ 2^a - 1 & \text{if } \eta = -\epsilon \text{ and } h = 1, \\ 1 & \text{if } \eta = -\epsilon \text{ and } h = a, \\ 0 & \text{otherwise.} \end{cases}$$

(c) *Suppose $B \in \{B_{1a}, B_{2a}, B_{1b}, B_{2b}\}$. Then*

$$\mathrm{k}(B,h) = \begin{cases} 2^{a+1} & \text{if } B \in \{B_{\alpha a}, B_{\alpha b}\} \text{ and } h = 0, \\ 2^{a-1}(2^a - 1) & \text{if } B \in \{B_{\alpha a}, B_{\alpha b}\} \text{ and } h = 1, \\ 2^{a-1} & \text{if } B \in \{B_{\alpha a}, B_{\alpha b}\} \text{ and } h = a, \\ 4 & \text{if } B \in \{B_{\beta a}, B_{\beta b}\} \text{ and } h = 0, \\ 2^a - 1 & \text{if } B \in \{B_{\beta a}, B_{\beta b}\} \text{ and } h = 1, \\ 1 & \text{if } B \in \{B_{\beta a}, B_{\beta b}\} \text{ and } h = a, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha = 1$ or 2 according as $\epsilon = +$ or $-$, and $\beta + \alpha = 3$.

PROOF. It is an easy consequence of [17].

(4C). *Let q be a power of an odd prime p, and let $B = B_1 = B_0(G)$ be the principal 2-block of $G = G_2(q)$. Then B satisfies Dade's conjecture.*

PROOF. Let $C \in \mathcal{R}^0(G)$ be given by (2.7) or (2.8). By Brauer's Third Main Theorem, $\mathrm{Blk}(N(C) \mid B_1) = \{b(C)\}$, where $b(C) = B_0(N(C))$.

(1) Suppose $q \neq 3$. Set $C = C(1)^1$ and $C' = C(1)$. Let $T = T_{-\epsilon}$ and $N(T) = N_G(T) = \langle T, \rho, \sigma, \tau\rangle$, where ρ, σ, τ are given by (2.5). By (2D), $C(C) = C(C') = \langle T, \rho\rangle$, $N(C) = N(T)$, and $N(C') = \langle C(C'), \tau\rangle$. If $b = B_0(C(C))$ then $E_8 = \langle O_2(T), \rho\rangle$ is a defect group of b, $b^{N(C)} = b(C)$ and $b^{N(C')} = b(C')$. Now $b(T) = B_0(T)$ contains 4 linear characters, and each of them is stabilized by ρ, so $\mathrm{Irr}(b)$ has 8 linear characters. Since τ permutes the set $\mathrm{Irr}(b(T))$, τ stabilizes exactly 2 linear characters of $\mathrm{Irr}(b(T))$. Since $\rho \in Z(D_{12})$, it follows that τ stabilizes exactly 4 characters of $\mathrm{Irr}(b)$. Thus $\mathrm{Irr}(b(C'))$ has 8 linear characters and 2 characters of degree 2, and so $\mathrm{k}(b(C'), 0) = 8$, $\mathrm{k}(b(C'), 1) = 2$, and $\mathrm{k}(b(C'), h) = 0$ for $h \neq 0, 1$.

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Similarly, σ permutes $\mathrm{Irr}\big(b(T)\big)$, and so σ stabilizes only the trivial character of $b(T)$. Thus the principal block $b\big(\langle C(C),\sigma\rangle\big)$ of $\langle C(C),\sigma\rangle$ has 6 linear characters and 2 irreducible characters of degree 3. Since $\langle\sigma,\tau\rangle\simeq S_3$ and $\rho\in Z(D_{12})$, it follows that τ stabilizes the two characters of degree 3 in $\mathrm{Irr}\Big(b\big(\langle C(C),\sigma\rangle\big)\Big)$, so that $\mathrm{Irr}\big(b(C)\big)$ has 4 characters of degree 3. We claim that $\mathrm{Irr}\big(b(C)\big)$ has exactly 4 linear characters, so that τ stabilizes exactly 2 linear characters of $\mathrm{Irr}\Big(b\big(\langle C(C),\sigma\rangle\big)\Big)$ and $\mathrm{Irr}\big(b(C)\big)$ has 2 characters of degree 2. Indeed, $N(C)$ is solvable, so the Alperin-McKay conjecture has an affirmative answer for $b(C)$ (see p. 171 of [12]). If Q is a defect group of $b(C)$, then the principal block $b(Q)$ of $N_{N(C)}(Q)$ corresponds to $b(C)$ under the Brauer correspondence. Thus $\mathrm{k}\big(b(Q),0\big)=\mathrm{k}\big(b(C),0\big)$. We may suppose that $Q\leq N(C')$, so that Q is a defect group of $b(C')$. Since $\sigma\notin N_{N(C)}(Q)$, it follows that $N_{N(C)}(Q)\leq N(C')$, and so $b(Q)$ corresponds to $b(C')$ under the Brauer correspondence. Since $N(C')$ is also solvable, it follows that $\mathrm{k}\big(b(Q),0\big)=\mathrm{k}\big(b(C'),0\big)$. As shown above $\mathrm{k}\big(b(C'),0\big)=8$, so $\mathrm{k}\big(b(C),0\big)=8$. But $\mathrm{Irr}\big(b(C)\big)$ has 4 characters of degree 3, so $\mathrm{Irr}\big(b(C)\big)$ has 4 linear characters and the claim follows. Thus $\mathrm{k}\big(b(C),h\big)=\mathrm{k}\big(b(C'),h\big)$ for all h, and so

$$\mathrm{k}\big(N(C),B,d\big)=\mathrm{k}\big(N(C'),B,d\big) \tag{4.1}$$

for integers $d\geq 0$. If $q=3$, then $C(1)^1$ and $C(1)$ are non-radical 2-chains, so that we may suppose (4.1) holds.

(2) Let $C=C(2)^1$ or $C(2)_1$ according to whether $2^a\neq q-\epsilon$ or $2^a=q-\epsilon$, and let $C'=C(2)$. Let $T=T_\epsilon$ and $N(T)=N_G(T)=\langle T,\rho,\sigma,\tau\rangle$. Then $N(C)=N(T)$ and $N(C')=\langle T,\rho,\tau\rangle$. If $b=B_0(T)$, then $O_2(T)$ is a defect group of b, and $\mathrm{Irr}(b)$ has 2^{2a} linear characters. Regard T as a subgroup of $L_\epsilon=\mathrm{SL}(3,\epsilon q)$. Let ϕ be an isomorphism from T to $\mathrm{Irr}(T)$, $\xi\in\mathrm{Irr}(b)$ and $N(\xi)$ the stabilizer of ξ in $N(C)$. Then $\xi=\phi(y)$ for some $y\in O_2(T)$ such that $y=y_1\times y_2\times y_3$ with $y_1y_2y_3=1$. The same proof as that of (3.2) of [4] shows that

$$N(\xi)=\begin{cases}\langle T,\rho,\tau,\sigma\rangle & \text{if } y=1,\\ \langle T,\rho\rangle & \text{if } y\in\Omega_1\big(O_2(T)\big) \text{ and } y_1\neq y_2,\\ \langle T,\tau\rangle & \text{if } y_1=y_2 \text{ and } y^2\neq 1,\\ \langle T,\rho,\tau\rangle & \text{if } y_1=y_2 \text{ and } |y|=2,\\ \langle T,\rho\tau\rangle & \text{if } y_1=y_2^{-1} \text{ and } y^2\neq 1,\\ T & \text{otherwise.}\end{cases} \tag{4.2}$$

Now $\xi=\phi(y)\in\Omega_1\big(\mathrm{Irr}(b)\big)$ if and only if $y\in\Omega_1\big(O_2(T)\big)$, so D_{12} permutes $\Omega_1\big(\mathrm{Irr}(b)\big)$. As shown in the proof of (4.1) $\Omega_1\big(\mathrm{Irr}(b)\big)$ has 10 extensions in $\mathrm{Irr}\big(b(C)\big)$, 4 linear characters, 4 characters of degree 3, and 2 of degree 2. Let $X(\tau)$ be characters of $\mathrm{Irr}(b)\setminus\Omega_1\big(\mathrm{Irr}(b)\big)$ stabilized by τ. By (4.2), $|X(\tau)|=2^a-2$ and ρ permutes $X(\tau)$. Thus $\mathrm{Irr}\big(b(C)\big)$ contains 2^a-2 characters of degree 6 covering characters in $X(\tau)$. Similarly, $\mathrm{Irr}\big(b(C)\big)$ contains 2^a-2 characters of degree 6 covering characters in $X(\rho\tau)$, where $X(\rho\tau)$ is defined as above. Thus $\mathrm{Irr}(b)$ has $2^{2a}-3\times(2^a-2)-3\times(2^a-2)-4$ characters ξ such that $N(\xi) = T$. So $\mathrm{Irr}\big(b(C)\big)$ has $\frac{1}{12}(2^{2a} - 3\times 2^{a+1} + 8)$ characters of degree 12, and hence

$$\text{(4.3)} \qquad \mathrm{k}\big(b(C), h\big) = \begin{cases} 8 & \text{if } h = 0, \\ 2^{a+1} - 2 & \text{if } h = 1, \\ \frac{1}{12}(2^{2a} - 3\times 2^{a+1} + 8) & \text{if } h = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, by Clifford theory and (4.2), $\mathrm{Irr}\big(b(C')\big)$ has 8 linear characters and 2 characters of degree 2 covering characters in $\Omega_1\big(\mathrm{Irr}(b)\big)$, $2^a - 2$ characters of degree 2 covering characters in $X(\tau)$, $2^a - 2$ of degree 2 covering characters in $X(\rho\tau)$, and $\frac{1}{4}\big(2^{2a} - (2^a - 2) - (2^a - 2) - 4\big)$ characters of degree 4. It follows that

$$\mathrm{k}\big(b(C'), h\big) = \begin{cases} 8 & \text{if } h = 0, \\ 2^{a+1} - 2 & \text{if } h = 1, \\ \frac{1}{4}(2^{2a} - 2^{a+1}) & \text{if } h = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{d}\big(b(C)\big) = \mathrm{d}\big(b(C')\big) = 2a + 2$, it follows that

$$\text{(4.4)} \qquad \begin{aligned} &(-1)^{|C|+1}\,\mathrm{k}\big(N(C), B, d\big) + (-1)^{|C'|+1}\,\mathrm{k}\big(N(C'), B, d\big) \\ &\quad = -\begin{cases} \frac{2}{3}(2^{2a-2} - 1) & \text{if } d = 2a, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(3) Let $C = C(3)_1 \colon 1 < E_8$ and $C' = C(3) \colon 1 < E_8 < K$, where $K = 2_+^{1+4}$. Since $C_{N(C)}(E_8) = E_8 \trianglelefteq N(C)$, it follows that $\mathrm{Irr}\big(b(C)\big) = \mathrm{Irr}\big(N(C)\big)$. Similarly, $\mathrm{Irr}\big(b(C')\big) = \mathrm{Irr}\big(N(C')\big)$.

As shown in the proof of (2B) (4) $N(C')/E_8$ is a parabolic subgroup of $\mathrm{GL}(3,2)$ and $N(C')/K = \mathrm{GL}(2,2)$ is a Levi subgroup of $N(C')/E_8$. Thus $N(C')/E_8 \simeq S_4$, where S_4 is a symmetric group on four letters. Let $\mathrm{GL}(2,2) = \langle g, v\rangle$ and $E_8 = \langle x, y, z\rangle$, where $\langle z\rangle = C(C') = Z(K)$, $|g| = 3$ and $|v| = 2$ modulo K. We may suppose $x^g = y$, $z^g = z$ and $y^g = xy$, so that x, y are elements of the commutator subgroup $[N(C'), N(C')]$ and $E_8 \leq [N(C'), N(C')]$. Thus $[N(C'), N(C')]/E_8 \simeq [S_4, S_4] = A_4$ and

$$N(C')/[N(C'), N(C')] \simeq \mathbb{Z}_2.$$

So $N(C')$ has exactly 2 linear characters. A linear character $\xi \in \mathrm{Irr}(K)$ stabilized by $\mathrm{GL}(2,2)$ has an extension to $N(C')$. By Clifford theory, ξ has 2 linear extensions and one of degree 2. But $N(C')$ has exactly two linear characters, so $\mathrm{Irr}(K)$ has exactly one linear character, the trivial character stabilized by both g and v. Similarly,

$$\langle K, g\rangle/[\langle K, g\rangle, \langle K, g\rangle] \simeq \mathbb{Z}_3$$

and $\langle K, v\rangle/[\langle K, v\rangle, \langle K, v\rangle] \simeq E_8$, so g and v stabilize exactly one and four linear characters of $\mathrm{Irr}(K)$, respectively.

Since $K \simeq 2_+^{1+4}$, $\mathrm{Irr}(K)$ has exactly one faithful character χ of degree 4 and 16 linear characters. Thus $\chi^g = \chi^v = \chi$ and χ has an extension to $N(\chi) = N(C')$. So $\mathrm{Irr}(\langle K, g\rangle)$

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has 3 linear characters, 3 characters of degree 4 and 5 characters of degree 3. Moreover, v permutes 3 linear characters and 3 characters of degree 4 in $\mathrm{Irr}(\langle K, g\rangle)$, and v stabilizes exactly 3 characters of degree 3. It follows by Clifford theory that $\mathrm{Irr}\big(N(C')\big)$ has 2 linear characters, 1 character of degree 2, 6 of degree 3, 2 of degree 4, 1 of degree 6 and 1 of degree 8. Thus

$$\mathrm{k}\big(b(C'), h\big) = \begin{cases} 8 & \text{if } h = 0, \\ 2 & \text{if } h = 1, \\ 2 & \text{if } h = 2, \\ 1 & \text{if } h = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{Irr}\big(b(C)\big) = \mathrm{Irr}\big(N(C)\big)$, it follows by Table III of [18] that

$$\mathrm{k}\big(b(C), h\big) = \begin{cases} 8 & \text{if } h = 0, \\ 2 & \text{if } h = 1, \\ 1 & \text{if } h = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{d}\big(b(C)\big) = \mathrm{d}\big(b(C')\big) = 6$, it follows that

$$(-1)^{|C|+1}\,\mathrm{k}\big(N(C), B, d\big) + (-1)^{|C'|+1}\,\mathrm{k}\big(N(C'), B, d\big) = \begin{cases} -2 & \text{if } d = 4, \\ 0 & \text{otherwise.} \end{cases}$$

(4) Let $C = C(4)_2$ and $C' = C(4)$. By (2D), $N(C') = S''$ and $C(S'') = Z(S'')$. So $\mathrm{Irr}\big(b(C')\big) = \mathrm{Irr}\big(N(C')\big)$. In the notation above, we may suppose that $S'' = \langle K, v\rangle$ with $K \simeq 2_+^{1+4}$. As shown in the proof (3) above, v stabilizes exactly 4 linear characters and the faithful character of degree 4 in $\mathrm{Irr}(K)$. By Clifford theory, $\mathrm{Irr}\big(N(C')\big)$ has 8 linear characters, 6 characters of degree 2, and 2 of degree 4. Thus $\mathrm{k}\big(b(C'), 0\big) = 8$, $\mathrm{k}\big(b(C'), 1\big) = 6$, $\mathrm{k}\big(b(C'), 2\big) = 2$, and $\mathrm{k}\big(b(C'), h\big) = 0$ for $h \geq 3$.

As shown in the proof of (2B) (4) $N(C) = \big\langle \sigma, \tau, \rho, \Omega_2\big(O_2(T_\epsilon)\big)\big\rangle$ and $C(C) = Z(W) \simeq E_4$, so that $\mathrm{Irr}\big(b(C)\big) = \mathrm{Irr}\big(N(C)\big)$. Suppose $a = 2$ and $2^a = q - \epsilon$. Then $W = \langle \rho, O_2(T_\epsilon)\rangle$ and $N(C) = N(W) = N(T_\epsilon)$. Thus $\mathrm{k}\big(b(C), h\big)$ is given by (4.3) with $a = 2$. In other cases, $\mathrm{k}\big(b(C), h\big)$ is also given by (4.3) with $a = 2$, since $N(C)$ is independent of the choices of the defining field. Thus $\mathrm{k}\big(b(C), 0\big) = 8$, $\mathrm{k}\big(b(C), 1\big) = 6$, and $\mathrm{k}\big(b(C), h\big) = 0$ for $h \geq 2$. So

$$(-1)^{|C|+1}\,\mathrm{k}\big(N(C), B, d\big) + (-1)^{|C'|+1}\,\mathrm{k}\big(N(C'), B, d\big) = \begin{cases} 2 & \text{if } d = 4, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\mathrm{k}\Big(N\big(C(3)_1\big), h\Big) + \mathrm{k}\Big(N\big(C(4)\big), h\Big) = \mathrm{k}\Big(N\big(C(3)\big), h\Big) + \mathrm{k}\Big(N\big(C(4)_2\big), h\Big) \tag{4.5}$$

for all integers $h \geq 0$.

(5) Suppose $C = C(1)_1$. Then $N(C) = \mathrm{SO}^+(4, q)$. By Theorem 13 of [9], $b(C) = \mathcal{E}_2\big(N(C), (1)\big)$. Let $H = N(C)$, and let s be a 2-element of G and $L = C_G(s)$. If $|s| \geq 4$, then we may suppose $Z(H) = \Omega_1(\langle s\rangle)$, since G has only one class of involutions. So $L = C_H(s)$. Thus $L \in \{\mathrm{GL}(2, \epsilon q), T_\epsilon, T_3^\delta\}$ and L is a regular subgroup of H, where $\delta = \pm$.

By Proposition 6.6 of [11], $\pm R_L^H$ (for some sign) induces a bijection between $\mathcal{E}\big(L,(s)\big)$ and $\mathcal{E}\big(H,(s)\big)$.

We first count the number of characters of $\mathcal{E}\big(H,(s)\big)$, where $L \simeq \mathbb{Z}_{q^2-1}$. Suppose s' is another 2-element of H conjugate to s in G. Then $gsg^{-1} = s'$ for some $g \in G$. Since $C_G(s')$ is cyclic and $C_H(s') \leq C_G(s')$, it follows that $C_H(s') \simeq \mathbb{Z}_{q^2-1}$, so that $Z(H) = \Omega_1(\langle s'\rangle)$. But $Z(H) = \Omega_1(\langle s\rangle)$, so g centralizes $Z(H)$ and $g \in H$. Thus $(s)_G \cap H$ is a single class of H, where $(s)_G$ is the conjugacy G-class containing s. Since G has $\frac{1}{2}2^a$ classes of such 2-elements s (*cf.* 2.2.1 and 2.3.1 of [17]) and since $\mathcal{E}\big(L,(s)_L\big)$ has only one character, it follows that $\mathcal{E}_2\big(H,(1)\big)$ contains 2^{a-1} characters $\chi \in \mathcal{E}\big(H,(s)\big)$, where $C_H(s) \simeq \mathbb{Z}_{q^2-1}$. In addition, all of them have height $a+1$ since $\chi(1) = (H\colon L)_{p'}$.

Suppose $L \simeq \mathrm{GL}(2,\epsilon q)$. Then L is a regular subgroup of H. Since s and s^{-1} are conjugate in H, H has $\frac{1}{2}(2^a-2)$ classes of such 2-elements. For each class (s), $\mathcal{E}\big(L,(s)_L\big)$ has two characters with degrees 1 and q, respectively. Since H has two conjugacy classes of regular subgroups isomorphic to $\mathrm{GL}(2,\epsilon q)$, it follows that $\mathcal{E}_2\big(H,(1)\big)$ contains (2^a-2) characters of degree $(H\colon L)_{p'}$ and (2^a-2) characters of degree $q(H\colon L)_{p'}$ in $\mathcal{E}\big(H,(s)\big)$ with $L = C_H(s) \simeq \mathrm{GL}(2,\epsilon q)$. In particular, $\mathcal{E}_2\big(H,(1)\big)$ contains $2(2^a-2)$ such characters of height 1.

Suppose $L \simeq T_\epsilon$, and let $K = N_H(\langle s\rangle)$. Then $K/L \simeq E_4$ is a subgroup of $N_G(L)/L \simeq D_{12}$. In this case $(s)_G \cap H$ is not a single class of H. Let $L = \mathbb{Z}_{q-\epsilon} \times \mathbb{Z}_{q-\epsilon}$ and $x \in \mathbb{Z}_{q-\epsilon}$. It is clear from (4.2) that each element w of $\Omega_1\big(O_2(L)\big)$ is stabilized by a non-trivial element of K/L. Thus $L < C_H(w)$ and so $C_H(w) \not\simeq T_\epsilon$. Similarly, each element $w \in \{x \times x, x \times x^{-1}\}$ is stabilized by a non-trivial element of K/L. Thus H contains $\big(2^{2a} - (2^a-2) - (2^a-2) - 4\big)$ 2-elements s such that $C_H(s) = L$. Since $|K/L| = 4$, H contains exactly $\frac{1}{4}(2^{2a} - 2^{a+1})$ classes of such 2-elements s. But $\mathcal{E}\big(L,(s)_L\big)$ has only one character, so $\mathcal{E}_2\big(H,(1)\big)$ contains $\frac{1}{4}(2^{2a} - 2^{a+1})$ characters in $\mathcal{E}\big(H,(s)\big)$ with $L = C_H(s) \simeq T_\epsilon$. In particular, each such character has height 2.

If $s \in Z(H)$, then there exists a bijection between $\mathcal{E}\big(H,(s)\big)$ and $\mathcal{E}\big(H,(1)\big)$ preserving degrees. Now H has four unipotent irreducible characters with degrees $1, q, q, q^2$, and $|Z(H)| = 2$. So $\mathcal{E}_2\big(H,(1)\big)$ has two characters of degree 1, four of degree q and two of degree q^2, and all of them have height 0.

Finally, let $\Omega = \cup_s \mathcal{E}\big(H,(s)\big)$, where s runs over non-central involutions of H. So $K = C_H(s) = \langle L_0, \rho\rangle$, where $L_0 = \mathrm{SO}^\delta(2,q) \times \mathrm{SO}^\delta(2,q)$, $\delta = \pm$ and ρ is an involution. Since $L_0 \simeq \mathrm{GL}(1,\delta q) \times \mathrm{GL}(1,\delta q)$, it follows that $\mathcal{E}\big(L_0,(1)\big)$ contains only the trivial character of L_0 and $K/L_0 \simeq \mathbb{Z}_2$ acts trivially on $\mathcal{E}\big(L_0,(1)\big)$. The stabilizer in K/L_0 of the trivial character is K/L_0 itself. By Proposition 5.1 of [19], $\mathcal{E}\big(H,(s)\big)$ contains exactly 2 characters. Since H has two classes of non-central involutions, it follows that Ω has 4 characters.

Let $H_1 = H_2 = \mathrm{Sp}(2,q) = \mathrm{SL}(2,q)$, and let V_i be the underlying symplectic space of H_i. Then the tensor product $V = V_1 \otimes V_2$ is a 4-dimensional orthogonal space with plus type. Thus we may suppose $H = \mathrm{SO}(V)$. The group $H_1 \otimes H_2$ is a subgroup of index 2 of H and $H_1 \otimes H_2$ is a central product over $Z(H_1) = Z(H_2)$.

Let α be a generator of the multiplicative group $\mathbb{F}_q^*$, and let $g_\alpha = \operatorname{diag}\{1, \alpha\}$ under a symplectic basis of V_i. Then $g = g_\alpha \otimes g_\alpha \in H$ (*cf.* the proof of [2], (2B)). In addition, g permutes two non-conjugate $H_1 \otimes H_2$-classes, so $g \in H \backslash (H_1 \otimes H_2)$ and $H = \langle H_1 \otimes H_2, g \rangle$. As shown in the proof of [14], Section 5.2 $\mathrm{SL}(2,q)$ has two irreducible characters of degree $\frac{1}{2}(q+1)$ and two of degree $\frac{1}{2}(q-1)$ which are permuted by g_α respectively, since $g_\alpha \in \mathrm{GL}(2,q)$. Moreover, these four characters lie in the principal block of $\mathrm{SL}(2,q)$.

Let $\chi' \in \mathrm{Irr}\big(\mathrm{SL}(2,q)\big)$ with $\chi'(1) = \frac{1}{2}(q-\epsilon)$, and let $\chi'' = (\chi')^{g_\alpha}$. Then χ' and χ'' induce the same linear character on $Z\big(\mathrm{SL}(2,q)\big)$. Thus $\chi' \otimes \chi''$ is an irreducible character of $H_1 \otimes H_2$ and $(\chi' \otimes \chi'')^g = \chi'' \otimes \chi'$. Similarly, $\chi' \otimes \chi' \in \mathrm{Irr}(H_1 \otimes H_2)$ and

$$(\chi' \otimes \chi')^g = \chi'' \otimes \chi''.$$

Thus H has two irreducible characters χ_1 and χ_2 of degree $\frac{1}{2}(q-\epsilon)^2$. Let $\varphi' \in \mathrm{Irr}\big(\mathrm{SL}(2,q)\big)$ with $\varphi'(1) = \frac{1}{2}(q+\epsilon)$, and let $\varphi'' = (\varphi')^{g_\alpha}$. A proof similar to above shows that H has two irreducible characters φ_1 and φ_2 of degree $\frac{1}{2}(q+\epsilon)^2$. It follows from the degrees of characters given above that $\chi_i, \varphi_i \in \Omega$ for $i = 1, 2$. Thus $\Omega = \{\chi_1, \chi_2, \varphi_1, \varphi_2\}$, and $\mathrm{h}(\chi_1) = \mathrm{h}(\chi_2) = 2a-1$, $\mathrm{h}(\varphi_1) = \mathrm{h}(\varphi_2) = 1$.

It follows that

$$\mathrm{k}\big(b(C), h\big) = \begin{cases} 8 & \text{if } h = 0 \\ 2(2^a-2)+2 & \text{if } h = 1, \\ \frac{1}{4}(2^{2a} - 2^{a+1}) & \text{if } h = 2, \\ 2 & \text{if } h = 2a-1, \\ 2^{a-1} & \text{if } h = a+1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows by (4B) (a) and (5.7) of [10] that $\mathrm{k}(G,B,d) = \mathrm{k}\big(N(C),B,d\big)$ except when $d = 2a$, in which case

$$\mathrm{k}(G,B,2a) - \mathrm{k}\big(N(C),B,2a\big) = -\frac{2}{3}(2^{2a-2}-1). \tag{4.6}$$

Thus Dade's conjecture for B follows by (4.4), (4.5) and (4.6). This completes the proof.

(4D). *Let $G = G_2(q)$ and $B \in \mathrm{Blk}^0(G,2)$, where q is a power of an odd prime. If $B \neq B_1 = B_0(G)$, then B satisfies Dade's conjecture.*

PROOF. The proof of (4D) is similar to that of (4C) or (3B), so we sketch a proof. Suppose $B \subseteq \mathcal{E}_2\big(G,(s)\big)$.

(1) If a defect group D of B is abelian, then $D = O_2(T_\delta)$ for some $\delta = \pm$. By (1A) of [3], it suffices to show that

$$\mathrm{k}\big(N(1 < \mathbb{Z}_2), B, d\big) = \mathrm{k}\Big(N\big(1 < \mathbb{Z}_2 < O_2(T_\delta)\big), B, d\Big) \tag{4.7}$$

for all integers $d \geq 0$. Let $C\colon 1 < \mathbb{Z}_2$ and $C'\colon 1 < \mathbb{Z}_2 < O_2(T_\delta)$. Then the Brauer correspondence induces a bijection between $\mathrm{Blk}\big(N(C) \mid B\big)$ and $\mathrm{Blk}\big(N(C') \mid B\big)$. For $b(C') \in \mathrm{Blk}\big(N(C') \mid B\big)$, set $b(C) = b(C')^{N(C)}$. By direct calculation (*cf.* the proof of (3A)), $\mathrm{k}\big(b(C)\big) = \mathrm{k}\big(b(C),0\big) = \mathrm{k}\big(b(C')\big) = \mathrm{k}\big(b(C'),0\big)$ and (4.7) follows.

(2) Suppose $B \in \{B_{1a}, B_{2a}, B_{1b}, B_{2b}\}$. Then $D \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ or $SD_{2^{a+2}}$ according as $B \in \{B_{\alpha a}, B_{\alpha b}\}$ or $B \in \{B_{\beta a}, B_{\beta b}\}$, where $\alpha = 1$ or 2 according as $\epsilon = +$ or $-$, and $\beta + \alpha = 3$.

Suppose $D \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$, so $2^a \not= q - \epsilon$. Let $C = C(2)^1$ and $C' = C(2)$. Then $N(C) = N(T_\epsilon) = T_\epsilon \rtimes D_{12}$, and $N(C') = T_\epsilon \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Since $O_2(T_\epsilon)$ is the only maximal normal abelian subgroup of D, it follows that $N(D) \leq N\big(O_2(T_\epsilon)\big) = N(C)$. Similarly, $N(D) \leq N(C')$ since $\Omega_1\big(Z(D)\big) = \mathbb{Z}_2$ and $N(C') = N_{N(\mathbb{Z}_2)}\big(O_2(T_\epsilon)\big)$. By Brauer's First Main Theorem,

$$\text{(4.8)} \qquad \operatorname{Blk}\big(N(C) \mid B\big) = \{b(C)\} \text{ and } \operatorname{Blk}\big(N(C') \mid B\big) = \{b(C')\}.$$

Let $b \in \operatorname{Blk}\big(C(C') \mid b(C')\big)$. By a result of Fong, Theorem V.3.14 of [12], we may suppose $D \leq N(b)$ and $|N(b){:}\,DT_\epsilon|_2 = 1$, where $N(b)$ is the stabilizer of b in $N(C)$. Thus $N(b) = DT_\epsilon \leq N(C')$ (*cf.* [2], (3.8)). But $N(\xi) \leq N(b)$ for each $\xi \in \operatorname{Irr}(b)$, so $N(\xi) = N'(\xi)$, where $N'(\xi) = N(\xi) \cap N(C')$. The mappings $\operatorname{Ind}_{N(\xi)}^{N(C)}$ and $\operatorname{Ind}_{N(\xi)}^{N(C')}$ induce a height-preserving bijection between the sets $\operatorname{Irr}\big(b(C)\big)$ and $\operatorname{Irr}\big(b(C')\big)$. Thus $\mathrm{k}\big(b(C), h\big) = \mathrm{k}\big(b(C'), h\big)$ for all integers h. It suffices to show that

$$\text{(4.9)} \qquad \mathrm{k}(G, B, d) = \mathrm{k}\big(N(1 < \mathbb{Z}_2), B, d\big)$$

for all non-negative integers d.

Suppose $D \simeq SD_{2^{a+2}}$, so that $q \not= 3$. Let $C = C(1)^1$ and $C' = C(1)$. Then the Brauer correspondence induces a bijection between $\operatorname{Blk}\big(N(C) \mid B\big)$ and $\operatorname{Blk}\big(N(C') \mid B\big)$ such that for $b(C') \in \operatorname{Blk}\big(N(C') \mid B\big)$ both $b(C')$ and $b(C) = b(C')^{N(C)}$ have a same defect group Q. Let $b \in \operatorname{Blk}\big(C(C') \mid b(C')\big)$. A proof similar to above shows that $N(b) = QC(C) = N(C')$, and two mappings $\operatorname{Ind}_{N(\xi)}^{N(C)}$ and $\operatorname{Ind}_{N(\xi)}^{N(C')}$ induce a height-preserving bijection between $\operatorname{Irr}\big(b(C)\big)$ and $\operatorname{Irr}\big(b(C')\big)$, where $\xi \in \operatorname{Irr}(b)$. So it suffices to show (4.9).

Let $C = C(1)_1$ and $b(C) \in \operatorname{Blk}\big(N(C) \mid B\big)$. We may suppose $b(C) \subseteq \mathcal{E}_2\big(N(C), (s)\big)$. Thus $C_{N(C)}(s)$ is a regular subgroup L of $N(C) = \mathrm{SO}^+(4, q)$, where $L \simeq \mathrm{GL}(2, \epsilon q)$ or $\mathrm{GL}(2, -\epsilon q)$ according as $D \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ or $SD_{2^{a+2}}$. It follows by [6], Theorem 2.3 that $b(C) = \mathcal{E}_2\big(N(C), (s)\big)$ and

$$\text{(4.10)} \qquad \mathrm{k}\big(b(C), h\big) = \mathrm{k}(b_L, h)$$

for all non-negative integers h, where $b_L = B_0(L)$.

If $B = B_{1a}$ or B_{1b}, then $C_G(s) = L \simeq \mathrm{GL}(2, q)$ is a regular subgroup of G. Again by [6], Theorem 2.3,

$$\text{(4.11)} \qquad \mathrm{k}(B, h) = \mathrm{k}(b_L, h)$$

for all non-negative integers h. This implies (4.9).

If $B = B_{2a}$ or B_{2b}, then $C_G(s) = L \simeq \mathrm{U}(2, q)$ and $\pm R_L^G$ for some sign induces a height-preserving bijection between $\operatorname{Irr}(b_L)$ and $\mathcal{E}_2\big(G, (s)\big)$. Thus (4.11) and (4.9) hold.

(3) Suppose $B = B_3$, so that $\gcd(q,3) = 1$. A defect group D of B is isomorphic to $\mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ or $SD_{2^{a+2}}$ according as $\eta = \epsilon$ or $-\epsilon$.

Suppose $D \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$, so that $2^a \neq q - \epsilon$. A proof similar to that of (2) above shows that $N(D) \leq N(C)$, $N(D) \leq N(C')$, and (4.8) holds. Let $b \in \mathrm{Blk}\big(C(C') \mid b(C')\big)$. By Theorem 3.2 of [7], we may suppose

$$\mathrm{Irr}(b) = \{\phi(ys) : y \in O_2(T_\epsilon)\},$$

where ϕ is a isomorphism from T_ϵ to $\mathrm{Irr}(T_\epsilon)$. It follows by [2], (3.6), (3.7) and (3.8) that $\mathrm{Irr}(b)$ has exactly one character $\xi = \phi(s)$ such that $\big(N(\xi) \colon T_\epsilon\big) = 6$, $3(2^a - 1)$ characters ξ such that $\big(N(\xi) \colon T_\epsilon\big) = 2$, and $2^{2a} - 1 - 3(2^a - 1)$ characters ξ such that $N(\xi) = T_\epsilon$. By Clifford theory, $\mathrm{Irr}\big(b(C)\big)$ contains 1 character of degree 4 and 2 characters of degree 2 covering $\phi(s)$, $2(2^a - 1)$ of degree 6, and $\frac{1}{6}\big(2^{2a} - 1 - 3(2^a - 1)\big)$ of degree 12. So $\mathrm{k}\big(b(C), 0\big) = 2(2^a - 1) + 2 = 2^{a+1}$, $\mathrm{k}\big(b(C), 1\big) = 1 + \frac{1}{6}\big(2^{2a} - 1 - 3(2^a - 1)\big)$, and $\mathrm{k}\big(b(C), h\big) = 0$ for $h \neq 0, 1$.

Let $N'(b) = N(b) \cap N(C')$ and $N'(\xi) = N(\xi) \cap N(C')$. Then $N'(b) = DT_\epsilon$, and $\mathrm{Irr}(b)$ has 2^a characters ξ such that $\big(N'(\xi) \colon T_\epsilon\big) = 2$, and $2^{2a} - 2^a$ characters ξ such that $N'(\xi) = T_\epsilon$. By Clifford theory, $\mathrm{Irr}\big(b(C')\big)$ has 2^{a+1} characters of degree 2 and $\frac{1}{2}(2^{2a} - 2^a)$ of degree 4. So $\mathrm{k}\big(b(C'), 0\big) = 2^{a+1}$, $\mathrm{k}\big(b(C'), 1\big) = \frac{1}{2}(2^{2a} - 2^a)$, and $\mathrm{k}\big(b(C'), h\big) = 0$ for $h \neq 0, 1$. It follows that

$$(4.12) \qquad \begin{aligned} &(-1)^{|C|+1}\,\mathrm{k}\big(N(C), B, d\big) + (-1)^{|C'|+1}\,\mathrm{k}\big(N(C'), B, d\big) \\ &\quad = \begin{cases} 1 + \frac{1}{6}(2^{2a} + 2 - 3 \times 2^a) - \frac{1}{2}2^a(2^a - 1) & \text{if } d = \mathrm{d}(B) - 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Suppose $D \simeq SD_{2^{a+2}}$, so that $q \neq 3$. Let $C = C(1)^1$ and $C' = C(1)$. A proof similar to that of (2) above shows that the Brauer correspondence induces a bijection between $\mathrm{Blk}\big(N(C) \mid B\big)$ and $\mathrm{Blk}\big(N(C') \mid B\big)$ such that $b(C')$ and $b(C) = b(C')^{N(C)}$ have a same dihedral group $Q = D_8$, where $b(C') \in \mathrm{Blk}\big(N(C') \mid B\big)$. It follows by [2], (3.6), (3.7) and (3.8) and Clifford theory that $\mathrm{Irr}\big(b(C)\big)$ has 1 character of degree 4, 2 of degree 2, and 2 of degree 6, and $\mathrm{Irr}\big(b(C')\big)$ has one character of degree 4 and four characters of degree 2. Thus

$$\mathrm{k}\big(b(C), h\big) = \mathrm{k}\big(b(C'), h\big)$$

for integers $h \geq 0$, so it suffices to show (4.9).

Finally, let $C = C(1)_1$ and $L = C_{N(C)}(s)$. Then $\mathrm{Blk}\big(N(C) \mid B\big) = \{b(C)\}$, and $L \simeq \mathrm{GL}(2, \epsilon q)$ or $\mathrm{GL}(2, -\epsilon q)$ according as $D \simeq \mathbb{Z}_{2^a} \wr \mathbb{Z}_2$ or $SD_{2^{a+2}}$. Let $b_L = B_0(L)$ and $B_\eta = B_{1a}$ or B_{2a} according as $\eta = \epsilon$ or $-\epsilon$. As shown in the proof of (2) above

$$\mathrm{k}(B_\eta, h) = \mathrm{k}(b_L, h) = \mathrm{k}\big(b(C), h\big)$$

for integers $h \geq 0$. If $\eta = -\epsilon$, then by (4B) (b) and (4B) (c), $\mathrm{k}(B, h) = \mathrm{k}(B_\eta, h)$, so that $\mathrm{k}(B, h) = \mathrm{k}\big(b(C), h\big)$ and Dade's conjecture follows. If $\eta = \epsilon$, then by (4B) (b) and (4B) (c), $\mathrm{k}(B, h) = \mathrm{k}(B_\eta, h)$ except when $h = 1$, in which case

$$\mathrm{k}(B, h) - \mathrm{k}\big(b(C), h\big) = \begin{cases} 1 + \frac{1}{6}(2^{2a} + 2 - 3 \times 2^a) - \frac{1}{2}2^a(2^a - 1) & \text{if } h = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus Dade's conjecture follows by (4.12), and this proves (4D).

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