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ZERO SETS—CONSEQUENCES FOR PRIMITIVE NEAR-RINGS

by S. D. SCOTT

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Throughout this paper all near-rings will be left distributive. We shall denote the zero-symmetric part of a near-ring N by N_0. The fact that the near-rings under consideration may not be zero-symmetric has important consequences for what follows, particularly the results of the last section.

Let N be a near-ring and V an N-group. With a non-empty subset S of N we may associate the zero subset $Z(S)$ of V. $Z(S)$ is simply the set of all v in V such that $v\sigma = 0$ for all σ in S. If S consists of a single element γ, $Z(S)$ will be written as $Z(\gamma)$. A subset Δ of V will be called a *zero subset* or Z-*closed*, if there exists a non-empty subset S of N such that $Z(S) = \Delta$.

In the first section of this paper we investigate zero subsets of an arbitrary N-group V. In the second section it is shown that for 2-primitive non-rings the Z-closed subsets yield a topology (c.f. the topology for N defined by Betsch [**3**, p. 111 and p. 116]). In the final section these notions are used, in the 2-primitive case, to study compatible near-rings and N-groups.

1. Z-closed subsets of an N-group

In this section some of the basic properties of zero subsets are developed. For example, a Galois correspondence between these subsets and annihilating right ideals is established.

Proposition 1.1. *Let V be an N-group. If S_i, $i \in I$, is a family of non-empty subsets of N, then*

$$\bigcap_{i \in I} Z(S_i) = Z\Big(\bigcup_{i \in I} S_i\Big).$$

Corollary 1. *Any intersection of Z-closed subsets of V is Z-closed.*

Corollary 2. *If S_1 and S_2 are subsets of N such that $S_1 \subseteq S_2$, then $Z(S_1) \supseteq Z(S_2)$.*

Proposition 1.2. *A Z-closed subset Z_1 of an N-group V has the property that $Z[(0: Z_1)] = Z_1$.*

Proof. Clearly $Z_1 \cdot (0: Z_1) = \{0\}$, and $Z_1 \subseteq Z[(0: Z_1)]$. However $Z_1 = Z(S)$, where S is a non-empty subset of N. Thus $Z_1 \cdot S = \{0\}$, and $S \subseteq (0: Z_1)$. From Corollary 2 of 1.1, $Z(S) = Z_1 \supseteq Z[(0: Z_1)]$. The proposition follows.

Corollary 1. *If S is a subset of V such that $(0: Z_1) \leqq (0: S)$, then $S \subseteq Z_1$.*

Proof. By Corollary 2 of 1.1, $Z[(0: Z_1)] \supseteq Z[(0: S)]$, and since $S \cdot (0: S) = \{0\}$, $S \subseteq Z[(0: S)]$. By 1.2, $Z[(0: Z_1)] = Z_1$ and the result follows.

Corollary 2. *If Z_1 and Z_2 are both Z-closed subsets of V, then $Z_1 = Z_2$ if, and only if,* $(0: Z_1) = (0: Z_2)$.

This follows since, if $(0: Z_1) = (0: Z_2)$, then by 1.2, $Z_1 = Z[(0: Z_1)] = Z[(0: Z_2)] = Z_2$.

Corollary 2 puts zero sets and their annihilators in one-one correspondence. The next proposition shows this correspondence is lattice inverting.

Proposition 1.3. *Let Z_1 and Z_2 be Z-closed subsets of an N-group $V \cdot Z_1 \subseteq Z_2$ if, and only if,* $(0: Z_1) \geqq (0: Z_2)$.

Proof. Clearly if $Z_1 \subseteq Z_2$, then $(0: Z_1) \geqq (0: Z_2)$. If $(0: Z_1) \geqq (0: Z_2)$, then $Z[(0: Z_1)] \subseteq Z[(0: Z_2)]$, By Corollary 2 of 1.1. From 1.2, $Z[(0: Z_i)] = Z_i$ for $i = 1, 2$, and the proposition holds.

Let N be a near-ring, V an N-group, Δ a subset of V and α an element of N. As in the case of functions we denote the sets $\{v\alpha: v \in \Delta\}$ and $\{v \in V: v\alpha \in \Delta\}$ by $\Delta\alpha$ and $\Delta\alpha^{-1}$, respectively.

Proposition 1.4. *If V is an N-group, then $Z_1\alpha^{-1} = Z[\alpha(0: Z_1)]$, for any Z-closed subset Z_1 of V and α in the zero-symmetric part of N.*

Proof. As $Z_1\alpha^{-1}$ contains $\{0\}$ it is non-empty. If v is in $Z_1\alpha^{-1}$, then $v\alpha$ is in Z_1 and $v\alpha(0: Z_1) = \{0\}$. Thus v is in $Z[\alpha(0: Z_1)]$. If on the other hand v is in $Z[\alpha(0: Z_1)]$, then $v\alpha(0: Z_1) = \{0\}$, $(0: Z_1) \leqq (0: v\alpha)$ and $v\alpha$ is in Z_1 by Corollary 1 of 1.2. Thus v is in $Z_1\alpha^{-1}$ and the proposition follows.

2. Z-topologies

Let V be an N-group. By Corollary 1 of 1.1 the zero subsets of V are closed with respect to intersections. Also the zero 0_s of N_0 maps all elements of V to $\{0\}$ and therefore V is a zero set. From now on we shall regard the empty set $\varnothing$ as a zero subset of V. With this addition the zero subsets of V will form a topology (called the *Z-topology*) provided they are closed with respect to finite unions.

A near-ring N is 2-primitive on V if N_0 is 2-*primitive* on V and a *non-ring* if N_0 is a non-ring (see [**3**, p. 102]).

The next proposition makes use of the notion of abelian submodules and will be used to show that, for 2-primitive non-rings, a finite union of closed subsets is closed.

A submodule W of an N-group V will be called *abelian* if W^+ is commutative and $(w_1 + w_2)\alpha = w_1\alpha + w_2\alpha$ for all w_i, $i = 1, 2$, in W and α in N_0 (the zero-symmetric part of N).

Proposition 2.1. *Suppose an N-group V is a direct sum $V_1 \oplus V_2$, of submodules V_1 and V_2, where the submodule V_2 is minimal (as an N-submodule) and non-abelian (as an N_0-submodule). If W is any submodule of V, then either $V_2 \leqq W$ or $W \leqq V_1$.*

Proof. Assume $W \not\geqq V_1$, and $V_2 \not\geqq W$. As V_1 is a maximal submodule (N-submodule) of V, $W+V_1=V$. By [**3**, 2.23, p. 48],

$$[(V_1+W)\cap(V_2+W)]/W(=(V_2+W)/W)$$

is abelian. Since V_2 is minimal $V_2 \cap W=\{0\}$ and $(V_2+W)/W$ is N_0-isomorphic to V_2. This contradiction to V_2 being non-abelian completes the proof.

Theorem 2.2. *Suppose a near-ring N is 2-primitive on V and N_0 is a non-ring, then if Z_1 and Z_2 are Z-closed subsets of V so is $Z_1 \cup Z_2$.*

Proof. As the result is trivial if either Z_1 or Z_2 is empty we may assume this is not the case. Take $R=(0\colon Z_1)\cap(0\colon Z_2)$. Clearly $R=(0\colon Z_1\cup Z_2)$ and if it is shown that $Z(R)=Z_1\cup Z_2$, the result will follow. Since $(Z_1\cup Z_2)\cdot R=\{0\}$, $Z_1\cup Z_2\subseteq Z(R)$. Assume v is in $Z(R)$. As $vR=\{0\}$, $R\leqq(0\colon v)$. We may assume, $(0\colon Z_1)\not\leqq(0\colon v)$, otherwise by Corollary 1 of 1.2, v is in Z_1. Since $N/(0\colon v)$ is N-isomorphic to V, $(0\colon v)$ is a maximal right ideal of N and, $N=(0\colon v)+(0\colon Z_1)$. Let $\Gamma=(0\colon Z_1)\cap(0\colon v)$. The N-group

$$N/\Gamma=[(0\colon v)+(0\colon Z_1)]/\Gamma.$$

can be written as a direct sum

$$(0\colon v)/\Gamma\oplus(0\colon Z_1)/\Gamma.$$

But $(0\colon Z_1)/\Gamma$ is N-isomorphic to

$$[(0\colon v)+(0\colon Z_1)]/(0\colon v)(=N/(0\colon v))$$

which in turn is N-isomorphic to $vN(=V)$. Since N_0 is a non-ring, $(0\colon Z_1)/\Gamma$, is a non-abelian minimal submodule (N-submodule) of N/Γ. By 2.1, either

$$[\Gamma+(0\colon Z_2)]/\Gamma\geqq(0\colon Z_1)/\Gamma$$

or

$$(0\colon v)/\Gamma\geqq[\Gamma+(0\colon Z_2)]/\Gamma.$$

It follows that either

$$\Gamma+(0\colon Z_2)\geqq(0\colon Z_1) \quad \text{or} \quad (0\colon v)\geqq\Gamma+(0\colon Z_2).$$

If $\Gamma+(0\colon Z_2)\geqq(0\colon Z_1)$, then

$$(0\colon Z_1)\cap(0\colon v)+(0\colon Z_2)\geqq(0\colon Z_1)$$

and

$$(0\colon Z_1)=(0\colon Z_1)\cap(0\colon v)+(0\colon Z_2)\cap(0\colon Z_1).$$

But $(0\colon Z_1)\cap(0\colon Z_2)=R$, and as $R\leqq(0\colon v)$, $(0\colon Z_1)\leqq(0\colon v)$. In this case it follows from Corollary 1 of 1.2 that v is in Z_1. It can therefore be assumed that, $(0\colon v)\geqq\Gamma+(0\colon Z_2)$. Consequently $(0\colon v)\geqq(0\colon Z_2)$, and in this case v is in Z_2. The proof is complete.

It follows from this theorem that for 2-primitive non-rings the Z-topology exists.

Theorem 2.3. *If the non-ring N is 2-primitive on V, then the maps of V into V induced by elements of N are continuous with respect to the Z-topology.*

Proof. If α is an element of N, then the map of V into V induced by α is continuous if $Z_1\alpha^{-1}$ is closed for any Z-closed subset Z_1 of V. As in 1.4, $Z_1\alpha^{-1} = Z[\alpha(0: Z_1)]$ and the theorem follows.

3. Primitive compatible near-rings

Let N be a zero-symmetric near-ring with identity and V a unitary N-group. The N-group V is said to be *compatible* if, for any v in V and α in N, there exists κ in N such that $(v+w)\alpha - v\alpha = w\kappa$ for all w in V (see [**4**, §6]). Compatible near-rings are just those zero-symmetric near-rings N having an identity and a faithful compatible N-group. Examples of compatible N-groups and near-rings are plentiful. If one takes any group, V and S any semigroup of endomorphisms of V that contain the inner automorphisms of V, then the near-ring $N(S)$ of maps of V into V generated by S, has V as a faithful compatible $N(S)$-group. Another example is to take an Ω-group V, and $P_0(V)$ the near-ring of zero-symmetric polynomial maps from V to V (see [**4**, §6] or [**3**, pp. 215–216]). Alternatively one may start with a topological group V (additive but not necessarily commutative) and consider $C_0(V)$, the near-ring of all continuous maps from V to V taking 0 to 0. V is then a compatible $C_0(V)$ group. In the case where $V = \mathbb{R}$ (the reals) the subnear-rings $C_0^{(n)}(\mathbb{R})$, or $C_0^{(\infty)}(\mathbb{R})$, of those functions through (0, 0) with continuous nth derivative, or which are infinitely differentiable, are again compatible on $\mathbb{R}$.

Let N_1 be a zero-symmetric near-ring and V a unitary faithful N_1-group. Let N be the near-ring of maps of V to V generated by N_1 and the constant maps of V to V. V is a compatible N_1-group precisely when N_0 (the zero-symmetric part of N) coincides with N_1 i.e. compatible near-rings are just those zero-symmetric near-rings to which the constants may be adjoined without the zero-symmetric part changing or, alternatively, they are zero-symmetric parts of unitary near-rings admitting all constants.

Let N be a near-ring (necessarily zero-symmetric with identity) which has a faithful compatible N-group V (necessarily unitary). As the N-subgroups of V are precisely the N-submodules, N is 0-primitive on V if, and only if, it is 2-primitive. Accordingly, in this case, we say V is minimal and call N a primitive compatible near-ring. Also, if N is a non-ring, then by Theorem 2.2 the Z-closed subsets of V form a topology on V. This topology has the disadvantage that the zero of V is contained in every closed subset. However, we may adjoin the constants to N to obtain a near-ring N' with $N'_0 = N$. The zero sets of V obtained from N' give rise to a new topology (such closed subsets will be called *Z^t-closed*). Also as $N \subseteq N'$, it follows from 2.3 that the elements of N are continuous with respect to this topology. If Δ is a Z^t-closed subset of V, then as $\rho = 1 + \lambda_v$ (for given v in V, λ_v is defined as the element of N' with the property that $u\lambda_v = v$ for all u in V) is in N', $\Delta\rho^{-1}$ is, by 2.3, Z^t-closed i.e. $\Delta - v = \{u - v : u \in \Delta\}$ is Z^t-closed for any v in V. Thus right translations (also left) of Z^t-closed subsets are Z^t-closed and a similar statement is true for the Z^t-open subsets of V. If v is taken to be in Δ, then $\{0\}$ is contained in $\Delta - v$. Now $\Delta - v = Z(S)$, where S is a non-empty subset of N' ($\Delta = \varnothing$ excluded). By 1.1

$$\Delta - v = \bigcap_{i \in I} Z(\gamma_i),$$

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where γ_i, $i \in I$, are the elements of S. Thus each γ_i is such that $0\gamma_i = 0$, and in the zero-symmetric part N, of N'. It follows that $\Delta - v$ is Z-closed and right (or left) translations of the Z-closed subsets of V are precisely the Z^t-closed subsets of V. Also, the above argument shows that whenever a Z^t-closed subset Δ of V contains $\{0\}$, it is Z-closed. Furthermore as 1 is in N' (or N) and $Z(1) = \{0\}$, $\{0\}$ is a Z^t-closed (and Z-closed) subset of V. Also, if Γ is Z^t-closed, then $\Gamma(-1) = \{-v: v \in \Gamma\}$, is Z^t-closed and the group V with the Z^t-topology is not far from being a topological group (translations and inversion being continuous functions) and in particular a T_0-group ($\{0\}$ being closed).

From now on we shall be dealing with zero-symmetric near-rings with identity and, as we are considering only the case of V a minimal compatible N-group, the Z^t-topology may be used together with the property that Z^t-closed subsets containing $\{0\}$ are Z-closed.

Proposition 3.1. *If N is a non-ring and V a faithful compatible minimal N-group, then for every v in V, $\{v\}$ is Z^t-closed and any finite subset of V is Z^t-closed.*

Proof. As $\{0\}$ is Z-closed, $v + \{0\} = \{v\}$ is also. A finite union of closed subsets is closed and the proposition holds.

If under the conditions of 3.1 the Z^t-topology is discrete N is said to be *discrete.* That this is indeed a property of N follows from the next proposition.

Proposition 3.2. *If a primitive compatible non-ring N is discrete, then it has a minimal right ideal.*

Proof. Let V be a minimal compatible N-group in which the Z^t-topology is discrete. If v is a non-zero element of V, then $\Delta_v = V \backslash \{v\}$ is a Z^t-closed subset of V containing $\{0\}$. Thus Δ_v is Z-closed. The right ideal $(0: \Delta_v)$, of N is non-zero as $Z(\{0\}) = V$ but $Z[(0: \Delta_v)] = \Delta_v$, by 1.2. If α is in $(0: v) \cap (0: \Delta_v)$, then as $\Delta_v \cup \{v\} = V$, $V\alpha = \{0\}$, and $\alpha = 0$. It follows that the map of $(0: \Delta_v)$ onto V taking ρ in $(0: \Delta_v)$ to $v\rho$, is an N-isomorphism and $(0: \Delta_v)$ is a minimal right ideal of N.

If the conditions of 3.2 hold then, up to N-isomorphism, V is unique for being a compatible minimal N-group. This follows from the fact that any two minimal N-groups are N-isomorphic to $(0: \Delta_v)$. It therefore makes sense to say N is discrete. Furthermore $(0: \Delta_v)$ exists for any $v \neq 0$ in V and the sum $\sum_{v \in V^*} (0: \Delta_v)$, where $V^* = V \backslash \{0\}$, is direct (note that if $u \neq 0$ in V is such that $(0: \Delta_u) = (0: \Delta_v)$ then, by 1.2, $\Delta_u = \Delta_v$ and $u = v$).

Corollary. *If a primitive compatible non-ring N is discrete and has maximal condition on right ideals, then V is finite.*

This follows from the fact that the direct sum $\sum_{v \in V^*} (0: \Delta_v)$, is finite and $(0: \Delta_u) = (0: \Delta_v)$, if, and only if $u = v$. Furthermore, in this case $N \cong M_0(V)$. This is so, since for each $v \neq 0$ in V, $(0: \Delta_v)$ has order $|V|$, the order of the direct sum is $|V|^{|V|-1}$, and being faithful on V, N represents all functions of V to V fixing 0.

Theorem 3.3. *A primitive compatible non-ring N is discrete if, and only if, it has a minimal right ideal.*

Proof. Suppose N is primitive on the compatible N-group V. The fact that N has a minimal right ideal, when N is discrete, follows directly from 3.2.

Suppose N has a minimal right ideal R. Now $R \nsubseteq (0: v)$, for some $v \neq 0$ in V (otherwise $R=\{0\}$). As $(0: v)$ is a maximal right ideal of N (c.f. [**3**, 7.22, p. 199]) $N=R+(0: v)$, and, since R is minimal, this sum is direct. Also $vR=vN(=V)$, and since $(0: v) \cap R=\{0\}$, R is N-isomorphic to V from the isomorphism theorems. Thus $N=(0: v) \oplus R$, where R is a minimal non-abelian submodule of N. Let $u \neq 0$ be an element of V. It follows readily from 2.1 and the maximality of $(0: u)$ that either $(0: u)=(0: v)$, or $R \leqq (0: u)$. Suppose $(0: u)=(0: v)$. By 3.1 $\Delta_1=\{0, u\}$ and $\Delta_2=\{0, v\}$ are Z^t-closed subsets of V. Both Δ_1 and Δ_2 contain $\{0\}$ and are therefore Z-closed. Furthermore $(0: \Delta_1)=(0: \Delta_2)$ and by 1.2

$$\Delta_1=Z[(0: \Delta_1)]=Z[(0: \Delta_2)]=\Delta_2$$

which means $u=v$. Thus $R \leqq (0: u)$, for all $u \neq v$ and $Z(R)=V \backslash \{v\}$. As $Z(R)$ is Z-closed it follows that $\{v\}$ is Z-open (and therefore Z^t-open). Any translation of a Z^t-open subset of V is Z^t-open and thus all one element subsets of V are Z^t-open. As a non-empty subset of V is a union of such, it is Z^t-open. The theorem follows.

For a non-ring N which is compatible and primitive on V, it frequently happens that the Z^t-open subsets of V (other than $\varnothing$) are dense in V. A topology having this property we shall call *sparse.*

Theorem 3.4. *Suppose the non-ring N is primitive and compatible on V. If N is simple, then the Z^t-topology is either sparse or discrete.*

Proof. First suppose there exists a proper Z^t-closed subset Δ of V containing a Z^t-open subset Γ, with $\{0\} \subset \Gamma$, and where any finite intersection of the form

$$\Gamma\alpha_1^{-1} \cap \Gamma\alpha_2^{-1} \cap \ldots \cap \Gamma\alpha_r^{-1}$$

with $\alpha_1, \ldots, \alpha_r$, elements of N, differs from $\{0\}$. We shall show that if this is so, then N cannot be simple. First note that since $\{0\}$ is in Γ and $0\alpha_i=0$, $i=1, \ldots, r$, a finite intersection as above must contain $\{0\}$ as a proper subset. Take $\alpha \neq 0$ in $(0: \Delta)$. This is possible since Δ contains $\{0\}$, is Z-closed, and $(0: \Delta) > (0: V)(=\{0\})$ by 1.3. Now, the subset $N\alpha(=\{\eta\alpha: \eta \in N\})$ of N has the property that the right ideal $R(N\alpha)$ of N generated by $N\alpha$ is an ideal. But

$$R(N\alpha)=\sum_{\eta \in N} R(\eta\alpha),$$

where $R(\eta\alpha)$ is the right ideal of N generated by the element $\eta\alpha$. If λ is in $R(N\alpha)$, then $\lambda=\lambda_1+\ldots+\lambda_k$, where λ_i is in $R(\eta_i\alpha)$, for some η_i, $i=1, \ldots, k$, in N and positive integer k. But

$$\Delta\eta_i^{-1}\eta_i\alpha \subseteq \Delta\alpha=\{0\},$$

and therefore

$$\Delta\eta_i^{-1}R(\eta_i\alpha)=\{0\},$$

for $i=1,\ldots,k$. Hence $\Delta\eta_i^{-1}\lambda_i=\{0\}$, for $i=1,\ldots,k$. If

$$\Delta_1=\Delta\eta_1^{-1}\cap\Delta\eta_2^{-1}\cap\ldots\cap\Delta\eta_k^{-1},$$

then as Δ_1 is contained in each $\Delta\eta_i^{-1}$, $i=1,\ldots,k$, $\Delta_1\lambda_i=\{0\}$ for $i=1,\ldots,k$, and $\Delta_1\lambda=\{0\}$. Now

$$\Gamma_1=\Gamma\eta_1^{-1}\cap\Gamma\eta_2^{-1}\cap\ldots\cap\Gamma\eta_k^{-1}$$

is, by 2.3, a Z^t-open subset of V and from our assumptions $\Gamma_1\subseteq\Delta_1$, and Γ_1 contains $\{0\}$ as a proper subset. Since $\Gamma_1\lambda=\{0\}$, λ has zero values on a subset of V properly containing $\{0\}$. Clearly $R(N\alpha)\neq\{0\}$ (α is in $N\alpha$) and, since N is simple, $R(N\alpha)=N$. On taking $\lambda=1$ we arrive at a contradiction viz. 1 takes zero values on a subset of V properly containing $\{0\}$. It follows that either:

(a) no proper Z^t-closed subset Δ of V contains an open subset containing $\{0\}$; or

(b) if such an open subset Γ of V exists some finite intersection

$$\Gamma\alpha_1^{-1}\cap\Gamma\alpha_2^{-1}\cap\ldots\cap\Gamma\alpha_r^{-1}$$

is zero.

Suppose (a) holds. If Δ_2 is a proper Z^t-closed subset of V containing a non-empty Z^t-open subset Γ_2, then for v in Γ_2, $-v+\Gamma_2$ is a Z^t-open subset of V containing $\{0\}$, $-v+\Delta_2$ is a proper Z^t-closed subset and $-v+\Gamma_2\subseteq -v+\Delta_2$. This contradicts (a). In this case it follows that the smallest closed subset of V containing Γ_2 is V and Z^t is sparse.

If (b) holds, then as the $\Gamma\alpha_i^{-1}$ are, by 2.3, Z^t-open subsets of V, $\{0\}$ is open. Translation will yield the fact that any one element subset of V is open. Any subset (being a union of such) is therefore open. The theorem follows.

If N is simple and satisfies the conditions of 3.4 and has maximal condition on right ideals, it follows by the Corollary of 3.2 that either V is finite (and $N\cong M_0(V)$) or Z^t is sparse. However, the next theorem shows a stronger result than this is possible.

Theorem 3.5. *Suppose the non-ring N is primitive and compatible on V. If N has maximal condition on right ideals, then either V is finite (in which case $N\cong M_0(V)$) or the Z^t-topology is sparse.*

Proof. The proof is in some respects similar to that of 3.4. Suppose there exist proper Z^t-closed and open subsets Δ and Γ of V having the properties stated in the proof of 3.4. As any finite intersection

$$\Delta\alpha_1^{-1}\cap\Delta\alpha_2^{-1}\cap\ldots\cap\Delta\alpha_r^{-1} \tag{1}$$

(α_i $i=1,\ldots,k$, elements of N) contains

$$\Gamma\alpha_1^{-1}\cap\Gamma\alpha_2^{-1}\cap\ldots\cap\Gamma\alpha_r^{-1} \tag{2}$$

which in turn properly contains $\{0\}$, it follows that a finite intersection of the form (1) cannot be zero. However, if $v\neq 0$ is in

$$\Delta\alpha_1^{-1}\cap\Delta\alpha_2^{-1}\cap\ldots\cap\Delta\alpha_r^{-1}\quad(=\Delta_1 \text{ say}),$$

then $vN=V$ and there exists α_{r+1} in N such that $v\alpha_{r+1}=w$, where $w\notin\Delta$. If $\Delta\alpha_{r+1}^{-1}\supseteq\Delta_1$, then as

$$\Delta\supseteq(\Delta\alpha_{r+1}^{-1})\alpha_{r+1}\supseteq\Delta_1\alpha_{r+1},$$

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and w is in $\Delta_1\alpha_{r+1}$, it would follow that w is in Δ. Thus

$$\Delta\alpha_1^{-1}\cap\Delta\alpha_2^{-1}\cap\ldots\cap\Delta\alpha_r^{-1}\cap\Delta\alpha_{r+1}^{-1}\quad(=\Delta_2\text{ say})$$

is a proper subset of Δ_1. Repeating this process with Δ_1 replaced by Δ_2 we obtain a proper subset Δ_3 of Δ_2 which is again an intersection of the form (1). Also Δ_1, Δ_2 and Δ_3 contain an intersection of the form (2), containing $\{0\}$ properly. In this way we construct Z^t-closed subsets

$$\Delta_1\supset\Delta_2\supset\Delta_3\supset\ldots$$

where the inclusions are proper and each Δ_i, $i=1,2,\ldots$, contains zero (the Δ_i being Z^t-closed by 2.3). As the Δ_i, $i=1,2,\ldots$, contain $\{0\}$ they are Z-closed and, by 1.3,

$$(0\colon\Delta_1)<(0\colon\Delta_2)<\ldots.$$

From this contradiction to the maximal condition it follows that either (a) or (b) of the proof of 3.4 holds. From which, as in 3.4, we conclude that either Z^t is discrete or sparse. The result can now be deduced from the Corollary of 3.2 and the comment following it.

An example will now be given illustrating the applicability of the preceding results.

Example. Let V be a simple Ω-group (written additively) and k a positive integer, then we may form polynomials $(x_1, x_2, \ldots, x_k)\alpha$ over V in k indeterminates. k such polynomials $\alpha_1, \ldots, \alpha_k$, determine a map of $V\oplus V\oplus\ldots\oplus(=W)$, taken k times, into W. This map is simply given by

$$(v_1,\ldots,v_k)(\alpha_1,\ldots,\alpha_k)=((v_1,\ldots,v_k)\alpha_1,\ldots,(v_1,\ldots,v_k)\alpha_k),$$

for $(v_1,\ldots,v_k)$ in W (see [**2**, Ch. 3, p. 75]). Furthermore two such maps $(\alpha_1,\ldots,\alpha_k)$ and $(\beta_1,\ldots,\beta_k)$ may be composed according to the formula

$$(\alpha_1,\ldots,\alpha_k)(\beta_1,\ldots,\beta_k)=((\alpha_1,\ldots,\alpha_k)\beta_1,\ldots,(\alpha_1,\ldots,\alpha_k)\beta_k)$$

with addition simply given by

$$(\alpha_1,\ldots,\alpha_k)+(\beta_1,\ldots,\beta_k)=(\alpha_1+\beta_1,\ldots,\alpha_k+\beta_k).$$

Thus a near-ring N' is obtained having W as an N'-group. The zero-symmetric part N of N' has W as a faithful compatible N-group and the simplicity of V ensures that W is minimal. As this near-ring is frequently a non-ring e.g. if V is a field, the polynomial maps yield a non-ring (see also [**4**, Thm 8.4]), and the results of this section are applicable. The Z^t-topology is therefore available for W. In the case of polynomials over a field this is none other than the usual Zariski topology (see [**5**, Ch. 8] also [**1**, p. 132]). Moreover, if this near-ring satisfies maximal condition, as is the case of polynomials over a field, then by 3.5 it follows that either W and therefore V is finite, or the Z^t-topology is sparse.

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University of Auckland
Private Bay
Auckland
New Zealand