Notes on the Extension of Aitken's Theorem (for Polynomial Interpolation) to the Everett Types.

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1. These notes are intended to be read in connexion with Dr A. C. Aitken's paper, *Proc. Edinburgh Math. Soc.* (2) 1 (1929), 199-203. It is proposed to show (by a simple line of direct algebraic demonstration which is also applicable to the original formula) that Aitken's Theorem can be extended to the Everett types, *i.e.* the types which include two sets of terms—one set involving $u(0)$ and the resultant of generalised operations on $u(0)$, and the other set involving $u(1)$ and the resultant of similar operations on $u(1)$.

2. Let λ_r be an operator which reduces the degree of a polynomial, $P(x)$, by two, and eliminates constants and terms in x.

3. Let Λ be that form of the inverse operator, λ^{-1}, that produces a $P(x)$ divisible by x and $x-1$, *i.e.* by $x(x-1)$. This will be called Condition (A). Then $\Lambda . \lambda P(x)$ will reproduce $P(x)$ as far as terms in x^2, but may differ from $P(x)$ by terms of the form $\alpha x + \beta$. In practice Λ will usually be the resultant of two inverse θ-operations, as defined by Aitken, *loc. cit.*, as for example $\Lambda = D^{-1}\Delta^{-1}$ or $= \Delta^{-2}$; but Λ is not necessarily so separable into two inverse θ-operations.

4. *Everett Type I.* Here the data are $(1, \lambda_1, \lambda_2\lambda_1, \ldots$ down to $\lambda_n \ldots \lambda_1)$ operating on $u(0)$ and $u(1)$. $P(x)$ is of degree $2n+1$. Take the fifth degree as an example. Put $z = x - 1$. Consider the following Scheme:

	Value of terms of degree < 2. $x=0$	$x=1,\ z=0.$
$u(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$	a_0	$a_1 + a_0$
$\lambda_1 u(x) = b_3x^3 + b_2x^2 + b_1x + b_0$	b_0	$b_1 + b_0$
$\lambda_2\lambda_1 u(x) = c_1x + c_0$	c_0	$c_1 + c_0$

Since $c_0 = \lambda_2 \lambda_1 u(0)$ and $c_1 + c_0 = \lambda_2 \lambda_1 u(1)$, we have

$$c_1 = \lambda_2 \lambda_1 u(1) - \lambda_2 \lambda_1 (u_0).$$

Substituting in $\lambda_2 \lambda_1 u(x)$, we have

$$\begin{aligned}\lambda_2 \lambda_1 u(x) = c_1 x + c_0 &= x\{\lambda_2 \lambda_1 u(1) - \lambda_2 \lambda_1 u(0)\} + \lambda_2 \lambda_1 u(0)\\ &= x \lambda_2 \lambda_1 u(1) - (x-1) \lambda_2 \lambda_1 u(0), \quad \ldots\ldots(1)\end{aligned}$$

which is an expression of Everett Type I.

Operate on (1) with Λ_2: then (see (A) above) we have

$$\lambda_1 u(x) = \Lambda_2 . \lambda_2 \lambda_1 u(x) + \alpha x + \beta \quad \ldots\ldots\ldots\ldots\ldots(2)$$

and similarly

$$\begin{aligned}\lambda_1 u(z) &= \Lambda_2 . \lambda_2 \lambda_1 u(z) + \alpha z + \beta\\ &= \Lambda_2 . \lambda_2 \lambda_1 u(z) + \alpha x + (\alpha + \beta) \ldots\ldots\ldots\ldots(3)\end{aligned}$$

Put $x = 0$ in (2); then bearing in mind the definition of Λ in (A) we see that (2) reduces to $\lambda_1 u(0) = \beta$. Similarly, putting $z = 0$ in (3) we get $\lambda_1 u(1) = \alpha + \beta$. Hence $\alpha = \lambda_1 u(1) - \lambda_1 u(0)$, and

$$\begin{aligned}\alpha x + \beta &= x\{\lambda_1 u(1) - \lambda_1 u(0)\} + \lambda_1 u(0)\\ &= x \lambda_1 u(1) - (x-1) \lambda_1 u(0), \quad \ldots\ldots\ldots\ldots(4)\end{aligned}$$

another expression of Everett Type I.

Substituting from (1) and (4) in (3), we get

$$\begin{aligned}\lambda_1 u(x) &= \Lambda_2 \{x\lambda_2 \lambda_1 u(1) - (x-1) \lambda_2 \lambda_1 u(0)\} + x\lambda_1 u(1) - (x-1) \lambda_1 u(0)\\ &= \left.\begin{aligned}&x . \lambda_1 u(1) + \Lambda_2 x . \lambda_2 \lambda_1 u(1)\\ &\quad - (x-1) \lambda_1 u(0) - \Lambda_2 (x-1) . \lambda_2 \lambda_1 u(0)\end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots(5)\end{aligned}$$

which again is an expression of Everett Type I.

5. Operating with Λ_1 on (5) and proceeding as before, we shall find

$$u(x) = \Lambda_1 . \lambda_1 u(x) + x u(1) - (x-1) u(0), \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

and finally, substituting from (5) in (6) and collecting terms,

$$\left.\begin{aligned}u(x) = x\, u(1) + \Lambda_1 x . \lambda_1 u(1) + \Lambda_1 \Lambda_2 x . \lambda_2 \lambda_1 u(1)\\ - (x-1) u(0) - \Lambda_1 (x-1) . \lambda_1 u(0) - \Lambda_1 \Lambda_2 (x-1) . \lambda_2 \lambda_1 u(0)\end{aligned}\right\}, \quad (7)$$

the required expansion for $u(x)$, or $P(x)$, in Everett Type I.

6. It is evident that, beginning always at the bottom and working upwards line by line, the same process will apply however

many lines are involved, *i.e.* whatever the degree of $P(x)$. Thus the general expansion for a $P(x)$ of degree $2n+1$ is evidently found by continuing (7) for $(n+1)$ terms on each line.

As an example, let $\lambda_1 = \lambda_2 = \lambda_3 = \ldots\ldots = d^2/dx^2 = D^2$. Then $\Lambda x = x^3/6 + Ax$, where A is a constant of integration to be fixed in conformity with Condition (A). This requires $A = -1/6$, and so $\Lambda x = x(x^2-1)/6$. Similarly

$$\Lambda^2 x = (3x^5 - 10x^3 + 7x)/360 = x(x^2-1)(3x^2-7)/360.$$

In this case (but see the general warning in para. 8, *infra*) we can get the corresponding values of $\Lambda(x-1)$ and $\Lambda^2(x-1)$ by putting $(x-1)$ for x in the values already found. Hence we have the following formula for $u(x)$ in terms of $u(0)$ and $u(1)$ and their differential coefficients of even order:

$$\left.\begin{aligned} u(x) = & xu(1) + x(x^2-1)u''(1)/6 + x(x^2-1)(3x^2-7)u^{IV}(1)/360 + \ldots \\ & - zu(0) - z(z^2-1)u''(0)/6 - z(z^2-1)(3z^2-7)u^{IV}(0)/360 - \ldots \end{aligned}\right\} (8)$$

where for compactness z is written for $(x-1)$.

7. *Everett Type II.* In this type there is one θ-operator preceding any number of λ-operators, and the data are $u(0)$ and $(\theta_1, \lambda_2\theta_1, \ldots$ down to $\lambda_n\lambda_{n-1}\ldots\lambda_2\theta_1)$ operating on $u(0)$ and $u(1)$. The degree of $P(x)$ is $2n$.

If a Scheme similar to that in § 4 be written down for this case it will be seen that—except for the top line, which gives $u(x)$—the scheme is of the same form as in Type I. Hence $\theta_1(x)$ may be expressed as in Type I, by (7). Applying to this expression for $\theta_1(x)$ the inverse operator $\Theta_1 = \theta_1^{-1}$, we shall produce all the terms of the top line, $u(x)$, except the constant term, which is equal to $u(0)$, given in the data. Thus the required expression for $u(x) \equiv P(x)$ in Everett Type II is as follows :

$$\left.\begin{aligned} u(0) + & \Theta_1 x \,.\, \theta_1 u(1) + \Theta_1\Lambda_2 x \,.\, \lambda_2\theta_1 u(1) + \Theta_1\Lambda_2\Lambda_3 x \,.\, \lambda_3\lambda_2\theta_1 u(1) + \ldots \\ & - \Theta_1 z \,.\, \theta_1 u(0) - \Theta_1\Lambda_2 z \,.\, \lambda_2\theta_1 u(0) - \Theta_1\Lambda_2\Lambda_3 z \,.\, \lambda_3\lambda_2\theta_1 u(0) - \ldots, \end{aligned}\right\} (9)$$

where again z is written for $(x-1)$.

As an example of Type II, put

$$\theta = d/dx = D; \;\; \lambda_1 = \lambda_2 = \ldots = d^2/dx^2 = D^2.$$

Apply the last formula to $u'(x)$, then integrate both sides, introducing the constant $u(0)$. No other constants of integration are needed in applying the operation $\Theta = \theta^{-1} = D^{-1}$ to the R.H.S., because it is a

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condition that $\Theta . P(x)$, or $\theta^{-1} P(x)$ is divisible by x. (*cf.* Aitken, *loc. cit.*) We thus get the following formula for $u(x)$ in terms of $u(0)$ and the odd differential coefficients of $u(0)$ and $u(1)$:

$$\left.\begin{array}{l} u(0) + x^2 u'(1)/2! + x^2(x^2-2)u'''(1)/4! + x^2(x^4-5x^2+7)u^V(1)/6! + \ldots \\ \qquad - x(x-2)\,u'(0)/2! - x^2(x-2)^2 u'''(0)/4! \\ \qquad\qquad - x^2(x^4-6x^3+10x^2-8)\,u^V(0)/6! + \ldots \end{array}\right\} \quad (10)$$

8. It must be specially noted that in operating by $\Lambda_1 \Lambda_2 \ldots \Lambda_r$ on x and $(x-1)$ the condition of divisibility by $x(x-1)$—see (A), para. 3—must be satisfied at each stage, *and separately for the inverse function of x and $x-1$*. It must not be assumed that the inverse function of $(x-1)$ can necessarily be found by putting $(x-1)$ for x in the corresponding inverse function of x. For example, if $\lambda \equiv \Delta^2$ and $\Lambda \equiv \Delta^{-2}$, $\Lambda x = x(x-1)(x-2)/3!$, but $\Lambda(x-1)$ will not be $(x-1)(x-2)(x-3)/3!$: it will be

$$(x^3 - 6x^2 + 5x)/3! = x(x-1)(x-5)/3!$$

9. [*Added* 14*th November* 1929.] The correction for Condition (A) may be found by a simple rule. If $P(x)$, $Q(x)$ and $R(x)$ are polynomials, and $\lambda^{-1} P(x)$, *not* corrected for Condition (A), is taken as $Q(x) = x(x-1)R(x) + ax + b$, the required value of $\Lambda . P(x)$ will be $Q(x) - (ax+b)$. Now evidently $b = Q(0)$ and $(a+b) = Q(1)$, or $a = Q(1) - Q(0)$, and so we have

$$\Lambda . P(x) = Q(x) - x[Q(1) - Q(0)] - Q(0). \qquad \ldots\ldots(11)$$

We may thus obtain $\Lambda . P(x)$ in the form $x(x-1)R(x)$. Putting $x-1$ for x in this we get $(x-1)(x-2)R(x-1)$; and applying (11) we find that the adjustment for Condition (A) is $2(x-1)R(-1)$. This vanishes if $R(x)$ is divisible by $(x+1)$, *i.e.* if $\Lambda . P(x)$ is divisible by $x(x+1)(x-1)$.

This condition is satisfied in the example of § 6, but not in the example of § 8.

Note. Paragraphs 8 and 9 apply equally to Type I and Type II.

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