A slight extension of Euler's Theorem on Homogeneous Functions.

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Euler's Theorem may be looked upon as the result of a certain operator acting on a special kind of function. This function may depend on any number of variables, but for convenience it is usual to consider three, viz., x, y, z. A function is homogeneous in x, y, z and of the nth degree if it can be put into the form

$$x^n f\left(\frac{y}{x}, \quad \frac{z}{x}\right).$$

If we adopt for conciseness the following notation, viz.,

$$\Delta_1 = x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} = xd_1 + yd_2 + zd_3 \quad \text{say}$$

$$\Delta_2 = (xd_1 + yd_2 + zd_3)^2$$

and generally $\Delta_p = (xd_1 + yd_2 + zd_3)^p$ where the multinomial function is to be expanded and then interpreted as an operator, we may state Euler's theorem thus

$$\Delta_p u = n(n-1)(n-2)\ldots(n-p+1)u,$$

where u is a homogeneous function of x, y, z of degree n.

In this note we wish to express the result of the same operator acting on any function of u, say $U \equiv F(u)$.

Now

$$\Delta_1 \ U = (xd_1 + yd_2 + zd_3)F(u) = F'(u)\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right\} = nuF'(u)$$

$$= nu\frac{d}{du}F(u) = F_1(u), \text{ say.}$$

$$\Delta_1{}^2 U = \Delta_1 F_1(u) = nu\frac{d}{du}F_1(u) = \left(nu\frac{d}{du}\right)^2 F(u),$$

and generally

$$\Delta_1^m U = \left(nu\frac{d}{du}\right)^m F(u).$$

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If we make the substitution $u = e^{n\theta}$, this gives

$$\Delta_1^m F(u) = \left(\frac{d}{d\theta}\right)^m F(e^{n\theta}) = \delta^m F(e^{n\theta}), \quad \text{where } \delta \equiv \frac{d}{d\theta}.$$

Hence it follows that any rational integral function of Δ_1 may be expressed in terms of δ.

Thus $$\phi(\Delta_1)F(u) = \phi(\delta)F(e^{n\theta}).$$

Now let $\Delta_p U$ be denoted by U_p.

We have $$\Delta_1 U_p = U_{p+1} + pU_p.$$

For we have first to apply the operator $xd_1 + yd_2 + zd_3$ to the various powers of d_1, d_2, d_3 in the expression Δ_p, and this produces U_{p+1}; then to the various powers of x, y, z, and this, by Euler's theorem, gives pU_p.

Thus $$U_{p+1} = (\Delta_1 - p)U_p.$$

Again $$U_p = (\Delta_1 - \overline{p-1})U_{p-1}$$

and $$U_1 = \Delta_1 U$$

$$\begin{aligned} \therefore \quad U_{p+1} &= (\Delta_1 - p)(\Delta_1 - \overline{p-1})\ldots\Delta_1 U \\ &= \phi(\Delta_1)F(u) = \phi(\delta)F(e^{n\theta}) \\ &= \delta(\delta - 1)\ldots(\delta - p)F(e^{n\theta}). \end{aligned}$$

Now put $\lambda = e^\theta$ and we get by a well-known theorem

$$U_{p+1} = \lambda^{p+1}\frac{d^{p+1}}{d\lambda^{p+1}}F(\lambda^n).$$

This is the result desired.

Suppose, for example, $F(u) = u$

then $$U_p = \lambda^p \frac{d^p}{d\lambda^p}(\lambda^n) = n(n-1)\ldots(n-p+1)\lambda^n$$

$$= n(n-1)(n-p+1)u, \quad \text{Euler's result.}$$

Again take $$F(u) \equiv \log u = \log\lambda^n = n\log\lambda$$

$$U_p = (-1)^{p+1}(p-1)\,!$$

The case where u is homogeneous and of the first degree gives the result

$$U_p = \lambda^p \frac{d^p}{d\lambda^p}F(\lambda) = u^p F^{(p)}(u).$$

Other examples might be written down.

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