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DUAL SPACE DERIVATIONS AND $H^2(L, F)$ OF MODULAR LIE ALGEBRAS

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0. Introduction. It is well-known that the classical vanishing results of the cohomology theory of Lie algebras depend on the characteristic of the underlying base field. The theorems of Cartan and Zassenhaus, for instance, entail that non-modular simple Lie algebras do not admit non-trivial central extensions. In contrast, early results by Block [**3**] prove that this conclusion loses its validity if the underlying base field has positive characteristic.

Central extensions of a given Lie algebra L, or equivalently its second cohomology group $H^2(L, F)$, can be conveniently described by means of derivations $\varphi:L \rightarrow L^*$. If

$$L = \bigoplus_{i=-r}^{s} L_i$$

is $\mathbf{Z}$-graded, then the L-module $\mathrm{Der}_F(L, L^*)$ of dual space derivations inherits the gradation from L. The derivations of degree $l \leqq -s - 1$, which are not effected by elements of L^*, were described in [**6**] by means of P-associative forms. The present paper is concerned with the case in which $l \geqq -s$.

According to general cohomology theory, every derivation $\varphi:L \rightarrow L^*$ is a restriction of a module homomorphism $\psi: U(L)^+ \rightarrow L^*$. We employ this connection in Section 2 in order to find several conditions for a given derivation $\varphi:L \rightarrow L^*$ to be inner on a subalgebra $K \subset L$. Theorem 2.1 may be used to study central extensions of simple Lie algebras with nondegenerate trace form by investigating the action of Casimir elements. In Sections 3 and 4 Theorem 2.3 is applied in order to determine the second cohomology groups of the graded Cartan type Lie algebras $W(n; \mathbf{m})$ and $K(n; \mathbf{m})$.

1. Subsidiary results. Throughout this section

$$L = \bigoplus_{i=-r}^{s} L_i$$

is assumed to be a finite dimensional graded Lie algebra over an

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algebraically closed field F. We shall be collecting basic facts pertaining to derivations $\varphi:L \to V$, where

$$V = \bigoplus_{j=-q}^{t} V_j$$

is a finite dimensional graded L-module.

Let $H \subset L_0$ be a nilpotent subalgebra with weight space decompositions

$$L = \bigoplus_{\alpha \in \Delta} L_{(\alpha)} \quad \text{and} \quad V = \bigoplus_{\beta \in \Lambda} V_{(\beta)},$$

respectively. As H is contained in L_0, we can find subsets $\Delta_i \subset \Delta$, $\Lambda_j \subset \Lambda$ such that

$$L_i = \bigoplus_{\alpha \in \Delta_i} L_i \cap L_{(\alpha)}; \ V_j = \bigoplus_{\beta \in \Lambda_j} V_j \cap V_{(\beta)}.$$

Hence L obtains the structure of a $\mathbf{Z} \times \mathrm{Map}(H, F)$-graded Lie algebra while V becomes a $\mathbf{Z} \times \mathrm{Map}(H, F)$-graded L-module, where $\mathrm{Map}(H, F)$ denotes the group of mappings from H into F.

Let $\mathrm{Der}_F(L, V)$ denote the space of derivations from L into V and let $\mathrm{Inn}_F(L, V)$ be the subspace of inner derivations, i.e., derivations of the form $\varphi(x) = x \cdot v$ for some $v \in V$. It is well-known that $\mathrm{Der}_F(L, V)$ inherits the $\mathbf{Z} \times \mathrm{Map}(H, F)$-gradation from L and V. The homogeneous derivations of degree (i_0, α_0) are given by the property

$$\varphi(L_i \cap L_{(\alpha)}) \subset V_{i+i_0} \cap V_{(\alpha+\alpha_0)}.$$

THEOREM 1.1. *Let L, H and V be as above and suppose that $\varphi:L \to V$ is a derivation of degree j. Then there exists a homogeneous derivation η of degree j from L into V satisfying $\eta(H) \subset V_{(0)}$ and $v \in V_j$ such that*

$$\varphi(x) = \eta(x) + x \cdot v \quad \forall x \in L.$$

Observe that the derivation η has the property

$$\eta(L_{(\alpha)}) \subset V_{(\alpha)} \quad \forall \alpha \in \Delta.$$

We shall refer to this fact by saying that η respects the root space decomposition.

For every L-module V we put

$$V^L := \{v \in V; L \cdot v = 0\}.$$

Let $U(L)$ denote the universal enveloping algebra of L and consider $U(L)^+$, the two-sided ideal generated by L. It is a result of general cohomology theory that for every derivation $\varphi:L \to V$ there exists a homomorphism $\psi:U(L)^+ \to V$ of $U(L)$-modules such that $\psi(x) = \varphi(x)\ \forall x \in L$ (cf. **[4]** p. 282, **[8]**).

LEMMA 1.2. *Assume that* $\mathrm{char}(F) = p > 0$. *Let* $\varphi:L \to V$ *be a derivation and suppose that* $e \in L$ *such that* $(\mathrm{ad}\, e)^{p^r} = 0$, $e^{p^r} \cdot V = 0$. *Then*

$$e^{p^r - 1} \cdot \varphi(e) \in V^L.$$

Proof. Let ψ be as indicated above. Then we obtain

$$e^{p^r - 1} \cdot \varphi(e) = e^{p^r - 1} \cdot \psi(e) = \psi(e^{p^r}).$$

Let z be an element of L. As e^{p^r} lies in the center $C(U(L)^+)$ of $U(L)^+$ we have

$$z \cdot \psi(e^{p^r}) = \psi(ze^{p^r}) = \psi(e^{p^r}z) = e^{p^r} \cdot \psi(z) = 0.$$

Consequently, $e^{p^r - 1} \cdot \varphi(e) \in V^L$.

In the special situation where $V = L^*$ the condition $(\mathrm{ad}\, e)^{p^r} = 0$ implies $e^{p^r} \cdot L^* = 0$. In addition,

$$(L^*)^L = \{f \in L^*;\ f([L, L]) = 0\}.$$

The L-module L^* inherits the $\mathbf{Z} \times \mathrm{Map}(H, F)$-gradation from L by virtue of

$$(L^*)_{(i,\alpha)} = \{f \in L^*;\ f(L_j \cap L_{(\beta)}) = 0 \text{ for } (j, \beta) \neq -(i, \alpha)\}$$

Thus we have

$$(L^*)_{(i,\alpha)} = (L^*)_i \cap (L^*)_{(\alpha)}$$

as well as

$$L^* = \bigoplus_{i=-s}^{r} (L^*)_i.$$

PROPOSITION 1.3. *Let*

$$L^* = \bigoplus_{\beta \in \Lambda} (L^*)_{(\beta)}$$

be the weight space decomposition relative to H. *Then the following statements hold*:

(1) $\Lambda = -\Delta$ *and* $(L^*)_{(\beta)} \cong (L_{(-\beta)})^* \ \forall \beta \in \Lambda$

(2) $\Lambda_i = -\Delta_{-i}$, $-s \leqq i \leqq r$.

Definition. The gradation $(L_i)_{-r \leqq i \leqq s}$ is said to be *standard* if

$$L_{i-1} = [L_{-1}, L_i],\ -r \leqq i \leqq -1.$$

LEMMA 1.4. *Suppose that* $L = U(L^-) \cdot L_s$, *where*

$$L^- := \sum_{i=-r}^{-1} L_i.$$

If L *has a standard gradation then* L *is generated by* $L_{-1} \oplus L_s$.

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Proof. Let G denote the subalgebra of L which is generated by $L_{-1} \oplus L_s$. Since the gradation of L is standard L^- is contained in G. Hence G is an $U(L^-)$-submodule of L containing L_s. Consequently,

$$L = U(L^-) \cdot L_s \subset U(L^-) \cdot G \subset G.$$

Definition. A derivation $\varphi : L \to L^*$ is said to be *skew* if

$$\varphi(x)(y) = -\varphi(y)(x) \quad \forall x, y \in L.$$

We consider the subalgebra

$$L^+ := \sum_{i=1}^{s} L_i$$

as well as

$$M := M(L) := [L^+, L^+].$$

Note that M is a graded subalgebra of L on which H operates. Hence for $k \geqq 1$ there is $\phi_k \subset \Delta_k$ such that

$$L_k = M_k + \bigoplus_{\alpha \in \phi_k} L_{(\alpha)} \cap L_k.$$

LEMMA 1.5. *Suppose that*

(a) $L = U(L^-) \cdot L_s$

(b) L_s *is an irreducible* L_0*-module.*

Let $\varphi : L \to L^*$ *be a homogeneous skew derivation of degree* l *where* $-2s \leqq l \leqq -s - 1$. *If* $- \Delta_s$ *is not contained in* $\phi_{-(s+l)}$ *then* $\varphi = 0$.

Proof. As φ annihilates L_0 it is homogeneous of degree $(l, 0)$ as well as a homomorphism of L_0-modules. Since $\varphi(L^-) = 0$ it suffices by virtue of (a) to show that φ vanishes on L_s. The assumptions concerning φ entail that φ annihilates $M_{-(s+l)}$. Since

$$\varphi(L_{-(s+l)}) \subset (L^*)_{-s}$$

and L_s is an irreducible L_0-module we either have

$$\varphi(L_{-(s+l)}) = (L^*)_{-s} \quad \text{or} \quad \varphi(L_{-(s+l)}) = 0.$$

The former alternative yields $-\Delta_s \subset \phi_{-(s+l)}$, a contradiction. Hence the latter case applies. As φ is skew, we obtain $\varphi(L_s) = 0$, as desired.

Remark. Let $\deg : L \to L$ denote the degree derivation of L, i.e.,

$$\deg(x) = ix \quad \forall x \in L_i.$$

If $\deg = \operatorname{ad} h$ for some element h of H, then $-\Delta_s$ is not contained in $\phi_{-(s+l)}$ for $l \not\equiv 0 \bmod(p)$.

We finally give two results which will enable us to construct certain outer derivations.

LEMMA 1.6. *Suppose that* $L = U(L^-) \cdot L_s$ *and let* $\varphi: L \to L^*$ *be a homogeneous linear map of degree* $l > 2r - s$. *Assume furthermore that* L^- *is generated by some subset* A. *If*

$$\varphi([x, y]) = x \cdot \varphi(y) - y \cdot \varphi(x) \quad \forall x \in A \quad \forall y \in L$$

then φ *is a derivation.*

Proof. It is well-known that

$$N := \{x \in L; \varphi([x, y]) = x \cdot \varphi(y) - y \cdot \varphi(x) \quad \forall y \in L\}$$

is a subalgebra of L. Since A is by assumption contained in N, we see that $L^- \subset N$. Thus N is a $U(L^-)$-submodule of L. As $L = U(L^-) \cdot L_s$ we shall conclude the proof by verifying the inclusion $L_s \subset N$.

Let x be an element of L_s. Then

$$[x, y] \in \sum_{i \geqq s-r} L_i$$

for every element y of L. Since $l > 2r - s$, φ annihilates $[x, y]$. By the same token we have $\varphi(x) = 0$. Let $y \in L_j$, then

$$x \cdot \varphi(y) \in (L^*)_{s+j+l}.$$

Since $s + j + l > r$ we obtain $x \cdot \varphi(y) = 0$. Consequently, x lies in N, as desired.

LEMMA 1.7. *Suppose that* $L = U(L^-) \cdot L_s$ *and let* $\varphi: L \to L^*$ *be a skew homomorphism of* L^-*-modules of degree* $-s$. *If*

$$\varphi([x, y]) = x \cdot \varphi(y) - y \cdot \varphi(x) \quad \forall x \in L_s \quad \forall y \in L_0,$$

then φ *is a derivation.*

Proof. Let N be defined as in the proof of the preceding lemma. Since $L^- \subset \ker \varphi$ and φ is L^--linear, L^- is contained in N. As in (1.6) it will therefore be sufficient to show that $L_s \subset N$.

Let $x \in L_s$. If $y \in L^-$ then

$$\varphi([x, y]) = -y \cdot \varphi(x) = x \cdot \varphi(y) - y \cdot \varphi(x).$$

Suppose that $y \in L_i$ for $i > 0$. Then

$$[x, y] = 0 \quad \text{and} \quad x \cdot \varphi(y) - y \cdot \varphi(x) \in (L^*)_i.$$

For $z \in L_{-i}$ we obtain observing that $z \in L^-$:

$$\begin{aligned}(x \cdot \varphi(y))(z) - (y \cdot \varphi(x))(z) &= -\varphi(y)([x, z]) + \varphi(x)([y, z]) \\ &= \varphi([x, z])(y) + \varphi(x)([y, z]) \\ &= -(z \cdot \varphi(x))(y) + \varphi(x)([y, z]) \\ &= \varphi(x)([z, y]) + \varphi(x)([y, z]) = 0.\end{aligned}$$

By virtue of our assumption concerning φ, we obtain the desired inclusion.

2. The canonical map $\Phi_1:H^1(L, L^*) \rightarrow H^1(K, L^*)$. Throughout this section K is assumed to be a subalgebra of the finite dimensional Lie algebra L. Let $S:U(L) \rightarrow U(L)$ denote the antipode map of $U(L)$, i.e., the antihomomorphism of $U(L)$ satisfying

$$S(x) = -x \quad \forall x \in L.$$

Observe that the $U(L)$-module structure of L^* induced by the representation $L \rightarrow gl(L^*)$ is given by

$$(u \cdot f)(x) = f(S(u) \cdot x) \quad \forall u \in U(L) \quad \forall x \in L.$$

In the sequel we shall study the canonical map

$$\Phi_1:H^1(L, L^*) \rightarrow H^1(K, L^*)$$

which is induced by the restriction map

$$\mathrm{Der}_F(L, L^*) \rightarrow \mathrm{Der}_F(K, L^*).$$

THEOREM 2.1. *Let $V \subset L$ be a subspace such that $L = U(K)^+ \cdot V$ and assume that there is an element*

$$c_0 \in C_{U(L)^+}(U(K)^+),$$

the centralizer of $U(K)^+$ in $U(L)^+$, which acts on V as the identity via the adjoint representation. Then

$$\Phi_1:H^1(L, L^*) \rightarrow H^1(K, L^*)$$

is trivial.

Proof. Let $\varphi:L \rightarrow L^*$ be a derivation and let

$$\psi:U(L)^+ \rightarrow L^*$$

be the homomorphism of $U(L)$-modules which extends φ. We first prove: If

$$(*) \qquad \sum_{i=1}^{m} u_i v_i = 0, \quad u_i \in U(K)^+, \quad v_i \in V$$

then

$$\sum_{i=1}^{m} \psi(S(u_i))(v_i) = 0.$$

Since c_0 commutes with every element of $U(L)^+$ we obtain

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$$\sum_{i=1}^{m} \psi(S(u_i))(v_i) = \sum_{i=1}^{m} \psi(S(u_i))(c_0 v_i)$$

$$= \sum_{i=1}^{m} (S(c_0) \cdot \psi(S(u_i)))(v_i)$$

$$= \sum_{i=1}^{m} \psi(S(c_0)S(u_i))(v_i)$$

$$= \sum_{i=1}^{m} (S(u_i) \cdot \psi(S(c_0)))(v_i)$$

$$= \psi(S(c_0))\left(\sum_{i=1}^{m} u_i \cdot v_i\right) = 0.$$

Now we define $f \in L^*$ by means of

$$f\left(\sum_{i=1}^{m} u_i \cdot v_i\right) := \sum_{i=1}^{m} \psi(S(u_i))(v_i) \quad u_i \in U(K)^+ \quad v_i \in V \quad m \geqq 1.$$

The assumptions of the theorem ensure in conjunction with (*) that f is a well-defined linear mapping. Let a be an element of K. Then

$$\varphi(a)\left(\sum_{i=1}^{m} u_i \cdot v_i\right) = \sum_{i=1}^{m} \psi(S(u_i)a)(v_i) = \sum_{i=1}^{m} \psi(S(S(a)u_i))(v_i)$$

$$= f\left(\sum_{i=1}^{m} S(a)u_i \cdot v_i\right) = (a \cdot f)\left(\sum_{i=1}^{m} u_i \cdot v_i\right).$$

Hence $\varphi(a) = a \cdot f$ and $\Phi_1 = 0$.

COROLLARY 2.2. *Let L be simple and suppose that F is algebraically closed. If the center $C(U(L)^+)$ operates non-trivially on L, then $H^1(L, L^*) = 0$.*

Proof. By virtue of Schur's Lemma there exists a homomorphism

$$\lambda : C(U(L)^+) \to F$$

such that

$$c \cdot x = \lambda(c)x \quad \forall x \in L \quad \forall c \in C(U(L)^+).$$

As L is simple, we find an element v such that

$$L = U(L)^+ \cdot Fv.$$

By assumption there is $c_0 \in C(U(L)^+)$ such that $\lambda(c_0) = 1$. Hence (2.1) applies and we obtain the asserted result.

THEOREM 2.3. *Let $V \subset L$ be a subspace such that*

$$L = U(K)^+ \cdot V \oplus V.$$

Suppose there is a basis $\{e_1, \ldots, e_n\}$ of K and a subset $J \subset N_0^n$ such that

(a) $\operatorname{ann}_{U(K)^+}(L) = \langle \{e^b;\ b \notin J\} \rangle$,

where

$$e^b := e_1^{b_1} \cdot e_2^{b_2} \ldots e_n^{b_n} \quad b = (b_1, \ldots, b_n) \quad \textit{and}$$

$$\operatorname{ann}_{U(K)^+}(L) := \{u \in U(K)^+;\ u \cdot L = 0\}.$$

(b) *There is a basis $\{v_1, \ldots, v_m\}$ of V such that*

$$\{e^a \cdot v_j;\ a \in J \quad 1 \leqq j \leqq m\}$$

is a basis of L over F.

Then the following statements hold:

(1) *Every derivation $\varphi: L \to L^*$ satisfying*

$$\ker(\operatorname{ad} e_i) \subset \ker \varphi(e_i) \quad 1 \leqq i \leqq n$$

defines an element of $\ker \Phi_1$.

(2) *Suppose there is $\mu \in N_0^n$* such that

$$J = \{b \in N_0^n;\ b \leqq \mu\}.$$

Then

$$\ker(\operatorname{ad} e_i) \subset \varphi(e_i) \textit{ if and only if } e_i^{\mu_i} \cdot \varphi(e_i) = 0.$$

(3) *If* $\operatorname{char}(F) = p > 0$, $\mu_i = p^{k_i} - 1 \quad 1 \leqq i \leqq n$ *and* $L = [L, L]$ *then the canonical mapping*

$$\Phi_1: H^1(L, L^*) \to H^1(K, L^*)$$

is trivial.

Proof. (1) In analogy with the proof of (2.1) we consider for a given derivation $\varphi: L \to L^*$ a module homomorphism $\psi: U(L)^+ \to L^*$ such that $\psi|_L = \varphi$. Assume that

$$\ker(\operatorname{ad} e_i) \subset \ker \varphi(e_i) \quad 1 \leqq i \leqq n.$$

We shall first prove that

$$\psi(S(\operatorname{ann}_{U(K)^+}(L))) = 0.$$

To that end let b not be contained in J. Since $e^b \in U(K)^+$ then there exists $j \in \{1, \dots, n\}$ such that

$$e^b = e_j e^{b-\epsilon_j}$$

where

$$\epsilon_j := (\delta_{ij})_{1 \leqq i \leqq n}.$$

Let x be an element of L. Then

$$y := e^{b-\epsilon_j} \cdot x \in \ker(\mathrm{ad}\, e_j)$$

and we obtain

$$\begin{aligned}\psi(S(e^b))(x) &= -\psi(S(e^{b-\epsilon_j})e_j)(x) = -(S(e^{b-\epsilon_j}) \cdot \psi(e_j))(x) \\ &= -\psi(e_j)(y) = -\varphi(e_j)(y) = 0.\end{aligned}$$

The assertion now follows from condition (a).

We proceed by verifying the validity of (*) of the proof of (2.1). According to the PBW-Theorem every element u of $U(K)$ may be expressed in the form

$$u = \sum_{a \in N_0^n} \alpha(a) e^a.$$

Put

$$\bar{u} := \sum_{a \in J} \alpha(a) e^a$$

as well as

$$\tilde{u} := \sum_{a \notin J} \alpha(a) e^a.$$

Then

$$u = \bar{u} + \tilde{u} \quad \text{and} \quad \tilde{u} \in \mathrm{ann}_{U(K)^+}(L).$$

Suppose that

$$\sum_{i=1}^{m} u_i \cdot v_i = 0 \quad u_i \in U(K)^+.$$

Then

$$0 = \sum_{i=1}^{m} \bar{u}_i \cdot v_i$$

and condition (b) shows that $\bar{u}_i = 0 \quad \forall i$. Consequently,

$$\sum_{i=1}^{m} \psi(S(u_i))(v_i) = \sum_{i=1}^{m} \psi(S(\tilde{u}_i))(v_i) = 0$$

on account of the first part of the proof.

We define $f \in L^*$ by means of

$$f(v) = 0 \quad \forall v \in V,$$

$$f\left(\sum_{i=1}^{m} u_i \cdot v_i\right) = \sum_{i=1}^{m} \psi(S(u_i))(v_i) \quad u_i \in U(K)^+.$$

The above deliberations then warrant that f is well-defined. Let a be an element of $U(K)^+$. Then we obtain

$$(a \cdot f)(v_i) = f(S(a) \cdot v_i) = \psi(a)(v_i)$$

as well as

$$(a \cdot f)\left(\sum_{i=1}^{m} u_i \cdot v_i\right) = f\left(\sum_{i=1}^{m} S(a)u_i \cdot v_i\right) = \sum_{i=1}^{m} \psi(S(S(a)u_i))(v_i)$$

$$= \sum_{i=1}^{m} \psi(S(u_i)a)(v_i) = \sum_{i=1}^{m} \psi(a)(u_i \cdot v_i)$$

$$= \psi(a)\left(\sum_{i=1}^{m} u_i \cdot v_i\right).$$

Hence

$$a \cdot f = \psi(a) = \varphi(a) \quad \forall a \in K.$$

(2) We shall show that

$$\ker(\operatorname{ad} e_i) = e_i^{\mu_i} \cdot L \quad 1 \leqq i \leqq n.$$

By assumption, $e_i^{\mu_i} \cdot L$ is contained in the kernel of ad e_i. In order to prove the converse inclusion, we give L the structure of a filtered $U(K)$-module. Recall that

$$U(K)_{(k)} := \langle \{e^b; |b| \leqq k\} \rangle \quad |b| := \sum_{i=1}^{n} b_i$$

defines the canonical filtration on $U(K)$. We put

$$L_{(k)} := U(K)_{(k)} \cdot V.$$

Then

$$L = \bigcup_{k \geqq 0} L_{(k)} \quad \text{and}$$

$$L_{(k)} = \langle \{e^b \cdot v_j;\ |b| \leqq k \quad b \leqq \mu \quad 1 \leqq j \leqq m\} \rangle.$$

We shall prove by induction on k that

$$\ker(\operatorname{ad} e_i) \cap L_{(k)} \subset e_i^{\mu_i} \cdot L.$$

If $x \in \ker(\operatorname{ad} e_i) \cap L_{(0)}$, then

$$x = \sum_{j=1}^{m} \alpha_j v_j \quad \text{and} \quad 0 = \sum_{j=1}^{m} \alpha_j e_i \cdot v_j.$$

If $\mu_i = 0$ then $e_i^{\mu_i} \cdot L = L$ and there is nothing to be shown. Otherwise (b) implies that $\alpha_j = 0 \quad 1 \leqq j \leqq m$, and $x = 0$.

Now suppose that $k \geqq 1$. We recall that

(**)
$$e_i^{r_i} e^a \equiv e^{a + r_i \epsilon_i} \bmod U(K)_{(|a| + r_i - 1)}.$$

Let

$$x = \sum_{j=1}^{m} \sum_{0 \leqq a \leqq \mu} \alpha(a, j) e^a \cdot v_j$$

be an element of $\ker(\operatorname{ad} e_i) \cap L_{(k)}$. Then

$$0 \equiv e_i \cdot x \equiv \sum_{j=1}^{m} \sum_{0 \leqq a \leqq \mu, |a| = k} \alpha(a, j) e_i e^a \cdot v_j$$

$$\equiv \sum_{j=1}^{m} \sum_{0 \leqq a \leqq \mu, |a| = k} \alpha(a, j) e^{a + \epsilon_i} \cdot v_j \bmod L_{(k)}.$$

It follows from condition (b) that $\alpha(a, j) = 0$ for $|a| = k \quad 0 \leqq a \leqq \mu - \epsilon_i$. As a result (**) implies that

$$x \equiv \sum_{j=1}^{m} \sum_{0 \leqq a \leqq \mu, a_i = \mu_i, |a| = k} \alpha(a, j) e^a \cdot v_j$$

$$\equiv \sum_{j=1}^{m} \sum_{0 \leqq a \leqq \mu, a_i = \mu_i, |a| = k} \alpha(a, j) e_i^{\mu_i} e^{a - \mu_i \epsilon_i} \cdot v_j \bmod L_{(k-1)}.$$

Hence there is an element y such that

$$x - e_i^{\mu_i} \cdot y \in L_{(k-1)}.$$

Since

$$\operatorname{ad} e_i (x - e_i^{\mu_i} \cdot y) = 0$$

the induction hypothesis applies and we see that

$$x - e_i^{\mu_i} \cdot y \in e_i^{\mu_i} \cdot L.$$

The statement just proved readily implies that

$$\ker(\operatorname{ad} e_i) \subset \ker \varphi(e_i) \quad \text{if and only if} \quad e_i^{\mu_i} \cdot \varphi(e_i) = 0.$$

(3) By virtue of (1) and (2) it suffices to show that

$$e_i^{p^{k_i}-1} \cdot \varphi(e_i) = 0 \quad 1 \leqq i \leqq n$$

for every derivation φ. As

$$(\operatorname{ad} e_i)^{p^{k_i}}(L) = 0 \quad \text{and} \quad e_i^{p^{k_i}} \cdot L^* = 0,$$

(1.2) entails that

$$e_i^{p^{k_i}-1} \cdot \varphi(e_i) \in (L^*)^L.$$

Since $L = [L, L]$ it follows that $(L^*)^L = 0$ and we obtain the desired result.

We conclude this section by considering the situation in which

$$L = \bigoplus_{i=-r}^{s} L_i$$

is graded and $K = L^-$. Note that if $L = U(L^-) \cdot L_s$ then

$$L = U(L^-)^+ \cdot L_s \oplus L_s,$$

i.e., the general assumption of (2.3) holds.

PROPOSITION 2.4. *Suppose that $L = U(L^-) \cdot L_s$ and let $\varphi:L \to L^*$ be a homogeneous derivation of degree l. Then*:

(1) *If $l > r - s$ and φ defines an element of* $\ker \Phi_1$, *then φ is inner.*

(2) *If $l = r - s$, φ is skew and defines an element of* $\ker \Phi_1$, *then φ is inner.*

Proof. By assumption there is $f \in (L^*)_l$ such that

$$\varphi(x) = x \cdot f \quad \forall x \in L^-.$$

We consider the derivation $\varphi_1 := \varphi - \operatorname{ad} f$ and observe that φ_1 is skew whenever φ is skew. We also note that

$$\varphi_1(L^-) = 0, \quad \deg \varphi_1 = l.$$

Hence $\ker \varphi_1$ is a $U(L^-)$-submodule of L. If (1) applies then φ_1 annihilates L_s. Otherwise $\varphi_1(L_s) \subset (L^*)_r$ and we obtain $\varphi(L_s) = 0$ from the skewness of φ_1 in combination with $\varphi_1(L_{-r}) = 0$. In either case we see that $L_s \subset \ker \varphi_1$, thus $\varphi_1 = 0$.

LEMMA 2.5. *Suppose that $H \subset L_0$ is a nilpotent subalgebra and assume that $L = U(L^-) \cdot L_s$. Let φ be a skew derivation of degree $(l, 0)$ which*

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defines an element of $\ker \Phi_1$. *Then the following statements hold*:

(1) *If* $-s < l \leqq r - s - 1$, *then* φ *is inner.*

(2) *If* $l = -s$ *and* $\Delta_s \cap \Delta_0 = \emptyset$, *then* φ *is inner.*

Proof. Since φ defines an element of the kernel of Φ_1 there is $f \in (L^*)_{(0)} \cap (L^*)_l$ such that $\varphi_1 := \varphi - \operatorname{ad} f$ is a homogeneous skew derivation of degree $(l, 0)$ which annihilates L^-. Consequently, $\ker \varphi_1$ is a $U(L^-)$-submodule of L. We note that

$$\varphi_1(L_s) \subset (L^*)_{s+l}$$

where $0 \leqq s + l \leqq r - 1$. If $s + l > 0$, then

$$\varphi_1(x)(y) = -\varphi(y)(x) = 0 \quad \forall x \in L_s \quad \forall y \in L_{-(s+l)}$$

i.e., $\varphi_1(L_s) = 0$. If $s + l = 0$ then

$$\varphi_1(L_s) \subset (L^*)_0.$$

Since $\Delta_s \cap -\Delta_0 = \emptyset$ we obtain $\varphi_1(L_s) = 0$. In either case we conclude that

$$L = U(L^-) \cdot L_s \subset \ker \varphi_1.$$

Hence φ is an inner derivation.

3. The cohomology groups $H^2(W(n; \mathbf{m}), F)$. Let $n \geqq 1$ be an integer and consider $\mathbf{m} := (m_1, \dots, m_n) \in N^n$. We put

$$\tau := (p^{m_1} - 1, \dots, p^{m_n} - 1)$$

where $p > 2$ denotes the characteristic of the underlying base field F. For n-tuples $a, b \in N_0^n$ we write

$$a + b := (a_i + b_i)_{1 \leqq i \leqq n}$$

and $a \leqq b$ if $a_i \leqq b_i \quad 1 \leqq i \leqq n$. Let $A(n)$ denote the algebra of divided powers over F. For $a \in N_0^n$ we define

$$x^{(a)} := x_1^{(a_1)} \dots . x_n^{(a_n)}.$$

It is known that

$$A(n; \mathbf{m}) := \langle \{x^{(a)}; 0 \leqq a \leqq \tau\} \rangle$$

is a subalgebra of $A(n)$ of dimension $np^{|\mathbf{m}|}$. Let

$$\partial_j : A(n; \mathbf{m}) \to A(n; \mathbf{m})$$

denote the derivation of $A(n; \mathbf{m})$ which is defined by

$$\partial_j(x^{(a)}) := x^{(a - \epsilon_j)}.$$

Then

$$W(n;\ \mathbf{m}) := \sum_{j=1}^{n} A(n;\ \mathbf{m})\partial_j$$

is a graded Lie algebra (cf. [9] for details). We put

$$h_j := x^{(\epsilon_j)}\partial_j$$

and observe that $H := \langle \{h_j;\ 1 \leqq j \leqq n\} \rangle$ is a Cartan subalgebra of $W(n;\ \mathbf{m})_0$. Let

$$W(n;\ \mathbf{m}) = \bigoplus_{\alpha \in \Delta} W(n;\ \mathbf{m})_\alpha$$

denote the corresponding root space decomposition. Since

$$[h_i,\ x^{(a)}\partial_j] = (a_i - \delta_{ij})x^{(a)}\partial_j,$$

every element a gives rise to a root by means of

$$a(h_i) = a_i \quad 1 \leqq i \leqq n{:}\Delta = \{a - \epsilon_j;\ 0 \leqq a \leqq \tau\}.$$

Note that

$$x^{(a)}\partial_j \in W(n;\ \mathbf{m})_{a-\epsilon_j}.$$

LEMMA 3.1. *Let* $\varphi{:}W(n;\ \mathbf{m}) \to W(n;\ \mathbf{m})^*$ *be a derivation. Then there exists* $f \in W(n;\ \mathbf{m})^*$ *such that*

$$\varphi(x) = x \cdot f \quad \forall x \in W(n;\ \mathbf{m})_{-1}.$$

Proof. We put $L := W(n;\ \mathbf{m})$ as well as $K := W(n;\ \mathbf{m})_{-1}$. Then $\{\partial_1, \ldots, \partial_n\}$ is a basis of the subalgebra K. Consider $V := W(n,\ \mathbf{m})_s$ with basis $\{x^{(\tau)}\partial_1, \ldots, x^{(\tau)}\partial_n\}$. The simplicity of L entails

$$L = U(K)^+ \cdot V \oplus V.$$

For $a \in N_0^n$ we put

$$\partial^a := \partial_1^{a_1} \ldots . \partial_n^{a_n}.$$

Since

$$\partial^a \cdot x^{(\tau)}\partial_j = x^{(\tau - a)}\partial_j$$

we see that

$$\{\partial^a \cdot x^{(\tau)}\partial_j;\ a \in J \quad 1 \leqq j \leqq n\},$$

where $J := \{a;\ 0 \leqq a \leqq \tau\}$, is a basis for L over F. Clearly,

$$\partial^a \in \mathrm{ann}_{U(K)^+}(L) \quad \text{for } a \notin J.$$

Suppose that

$$u = \sum_{0<a} \alpha(a)\partial^a \in \mathrm{ann}_{U(K)^+}(L).$$

Then we obtain

$$0 = u \cdot x^{(\tau)}\partial_1 = \sum_{0<a\leqq\tau} \alpha(a)x^{(\tau-a)}\partial_1.$$

Thus $\alpha(a) = 0$ for $a \leqq \tau$ and $u \in \langle \{\partial^a; a \notin J\} \rangle$. Consequently, (2.3) applies and we obtain the asserted result.

THEOREM 3.2. *The following statements hold*:

(1) *If $p > 3$, then the second cohomology group $H^2(W(n; \mathbf{m}), F)$ is trivial for $n > 1$ and one dimensional for $n = 1$.*

(2) *If $p = 3$, then the second cohomology group $H^2(W(n; \mathbf{m}), F)$ is trivial for $n > 2$, two dimensional for $n = 2$, and $m_1 - 1$ dimensional for $n = 1$.*

Proof. Let $Z^2(L, F)$ denote the space of 2-cocycles of L. It was shown in [**6**] that the mapping which associates with $f \in Z^2(L, F)$ the skew derivation $x \to f(x, \ )$ induces an isomorphism between $H^2(L, F)$ and the vector space of skew outer derivations from L into L^*. We put $L := W(n; \mathbf{m})$ and note that

$$U(L^-) \cdot L_s = L.$$

Let $\varphi: L \to L^*$ be a homogeneous skew derivation of degree l.

(a) $l \geqq r - s = 1 - s$. We apply (3.1) and (2.4) consecutively in order to see that φ is inner.

(b) $l = -s$. Note that

$$\Delta_0 = \{\epsilon_i - \epsilon_j; 1 \leqq i, j \leqq n\} = -\Delta_0$$

as well as

$$\Delta_s = \{\tau - \epsilon_k; 1 \leqq k \leqq n\}.$$

Suppose that $\Delta_0 \cap \Delta_s \neq \emptyset$. Then there are i, j, k such that $\epsilon_i - \epsilon_j = \tau - \epsilon_k$. If $k \notin \{i, j\}$, then $-2 \equiv 0 \bmod(p)$, a contradiction. If $k = j$, then $\epsilon_i = \tau$, which is also impossible. The case $k = i$ readily yields $n = 2$ and $p = 3$. Hence if $n \neq 2$ or $p \neq 3$ then φ is inner by virtue of (1.1) and (2.5).

In order to treat the case $n = 2$, $p = 3$ we introduce the linear mapping

$$\alpha_\tau: A(n; \mathbf{m}) \to A(n; \mathbf{m}) \quad \alpha_\tau\left(\sum_{a\leqq\tau} \beta(a)x^{(a)}\right) := \beta(\tau).$$

It is easily seen that in our particular situation

$$\lambda(x^{(a)}\partial_i, x^{(b)}\partial_j) := (i + j)\alpha_\tau(x^{(a)}x^{(b)})$$

defines a nondegenerate L^--associative form on L. The linear mappings

$$D_i: L \to L \quad D_i(x^{(a)}\partial_j) = (i + j)x^{(a-\epsilon_{3-i})}\partial_i \quad 1 \leqq i \leqq 2$$

are skew homomorphisms of L^--modules of degree -1. Consequently,

$$\varphi_i : L \to L^* \quad \varphi_i(x)(y) := \lambda(D_i(x), y) \quad 1 \leqq i \leqq 2$$

is a skew homomorphism of L^--modules of degree $-s$. A case by case analysis shows that the condition of (1.7) is satisfied. Hence the φ_i are skew derivations. Suppose that φ respects the root space decomposition. Then

$$\varphi(x^{(\tau)}\partial_j)(x^{(\epsilon_l)}\partial_k) = 0$$

unless $k = j$, $l = 3 - j$. Since

$$\varphi_i(x^{(\tau)}\partial_j)(x^{(\epsilon_l)}\partial_k) = -i\delta_{ij}\delta_{ik}\delta_{i(3-i)},$$

we obtain for

$$r_i = (-i)^{-1}\varphi(x^{(\tau)}\partial_i)(x^{(\epsilon_{3-i})}\partial_i)$$

that $\varphi - r_1\varphi_1 - r_2\varphi_2$ annihilates L_s. Hence

$$\varphi = r_1\varphi_1 + r_2\varphi_2.$$

Note that φ_1, φ_2 define linearly independent elements of $H^1(L, L^*)$.

(c) $-2s \leqq l \leqq -s - 1$. We consider the following identities:

$$(1) \qquad [x^{((a_j+1)\epsilon_j)}\partial_j, x^{(a-a_j\epsilon_j)}\partial_j] = -x^{(a)}\partial_j \quad 0 \leqq a_j \leqq \tau_j$$

$$(2) \qquad [x^{(2\epsilon_j)}\partial_j, x^{(a-\epsilon_j)}\partial_j] = \frac{1}{2}a_j(a_j - 3)x^{(a)}\partial_j$$

$$(3) \qquad [x^{(\epsilon_k+\epsilon_j)}\partial_j, x^{(a-\epsilon_k)}\partial_j] = a_k(a_j - 1)x^{(a)}\partial_j \quad k \neq j.$$

We shall employ the above equations in order to show that

$$M(L)_q = L_q \quad \text{for } q \geqq 3 \quad q \not\equiv -1 \bmod(p)$$

$$L_q = M(L)_q + \sum_{j=1}^{n} \sum_{b_i \equiv 0 \bmod(p) \forall i} Fx^{(b)}\partial_j \quad q \geqq 3, q \equiv -1 \bmod(p).$$

Let $x^{(a)}\partial_j$ be an element of L_q. Then $q = |a| - 1$ and formula (1) entails that $x^{(a)}\partial_j \in M(L)_q$ unless $a_j = 0$ or $a_j \geqq |a| - 1$ or $a_j = p^{m_j} - 1$. The first alternative implies in conjunction with the assumption $q \geqq 3$ that $n \geqq 2$. Then we apply (3) in order to obtain the desired result. If $a_j \geqq |a| - 1$ then $a = \epsilon_k + a_j\epsilon_j \quad k \neq j$ or $a = a_j\epsilon_j$. In the former case we see by virtue of (3) that $x^{(a)}\partial_j \in M(L)_q$ unless $a_j \equiv 1 \bmod(p)$. As $3 \leqq q = a_j$ equation (2) implies that $x^{(a)}\partial_j \in M(L)_q$. If $a = a_j\epsilon_j$, then, by (2), we may assume that $a_j \equiv 0, 3 \bmod(p)$. Only the case in which $a_j \equiv 3 \bmod(p) \quad p \neq 3$ needs to be considered. Since $q \geqq 3$ the identity

$$[x^{(3\epsilon_j)}\partial_j, x^{((a_j-2)\epsilon_j)}\partial_j] = -2x^{(a_j\epsilon_j)}\partial_j$$

yields the desired result. The remaining case $a_j = p^{m_j} - 1$ can be treated by applying (2).

Let $n \geqq 2$. If $l \leqq -s - 3$ then $-(s + l) \geqq 3$ and the above shows that in (1.5) $\phi_{-(s+l)}$ may be chosen as follows:

(a) $\quad \phi_{-(s+l)} = \phi \quad (s + l) \not\equiv 1 \bmod(p)$

(b) $\quad \phi_{-(s+l)} = \{-\epsilon_j;\ 1 \leqq j \leqq n\} \quad (s + l) \equiv 1 \bmod(p).$

Hence $-\Delta_s$ is not contained in $\phi_{-(s+l)}$ and φ vanishes by virtue of (1.5). Next we assume that $l \in \{-s - 2, -s - 1\}$. We may choose

$$\phi_2 = \{3\epsilon_k - \epsilon_j;\ 1 \leqq j, k \leqq n\}.$$

Since $n \geqq 2$ we have $-\Delta_s \subset \phi_2$ as well as $-\Delta_s \subset \Delta_1$. Hence (1.5) applies and $\varphi = 0$.

Now suppose that $n = 1$. It was shown in [**6**] that the bilinear form

$$\lambda: W(1; \mathbf{m}) \times W(1; \mathbf{m}) \rightarrow F \quad \lambda(x^{(a)}\partial, x^{(b)}\partial) = \alpha_\tau(x^{(a)}x^{(b)})$$

is nondegenerate and L^--associative. By virtue of (3.3) and (3.7) of [**6**] we see that the skew derivations of degree $1 \leqq -s - 1$ are in one to one correspondence with those L^--module homomorphisms $D:L \rightarrow L$ of degree k; $-(s + 1) \leqq k \leqq -2$ satisfying the following properties:

(a) $D(L_s) \perp M(L)_{-(s+l)}$

(b) D is skew with respect to λ

(c) $k \equiv -3 \bmod(p), \quad k = l + s - 1.$

Note that D is uniquely determined by its action on L_s. Suppose that $l \leqq -s - 3$. If $s + l \not\equiv 1 \bmod(p)$, then we obtain

$$M(L)_{-(s+l)} = L_{-(s+l)} \quad \text{and}$$

$$D(L_s) \subset L_{2s+l-1} \cap L^{\perp}_{-(s+1)} = \{0\}.$$

Hence $D = 0$. If $s + l \equiv 1 \bmod(p)$, then $k \equiv 0 \bmod(p)$ and (c) entails that $D = 0$ unless $p = 3$. Suppose this to be the case. Then

$$D(x^{(\tau)}\partial)$$

$$= \alpha x^{(\tau+k)}\partial \quad k \equiv 0 \bmod(p) \quad 1 - p^{m_1} \leqq k \leqq -4 \quad \alpha \in F.$$

Hence $D = \alpha(\mathrm{ad}\ \partial)^{-k}$. Let $q \equiv -1 \bmod(p)$. It can be easily seen from the equation

$$[x^{(a)}\partial, x^{(b)}\partial] = \left[\binom{a + b - 1}{a} - \binom{a + b - 1}{a - 1}\right]x^{(a+b-1)}\partial$$

that $M(L)_q = L_q$ unless $q = 3^r - 1$. We now conclude from (a) that $D = 0$ unless there is $r \in \{2, \ldots, m_1 - 1\}$ such that $k = -3^r$. One therefore obtains $m_1 - 2$ linearly independent elements of $H^1(L, L^*)$ for $m_1 \geqq 2$.

We finally assume that $l \in \{-s - 2, -s - 1\}$. If $l = -s - 1$ then $k \equiv -2 \bmod(p)$ and (c) forces D to vanish. In the remaining case $k = -3$ which implies $D = \alpha(\mathrm{ad}\ \partial)^3$, $\alpha \in F$.

4. The cohomology groups $H^2(K(2r + 1; \mathbf{m}), F)$. Throughout this section we shall assume that $p > 3$. We put $n := 2r + 1$ and consider the algebra $A(n; \mathbf{m})$ with Lie product

$$\text{(I)} \qquad \langle x^{(a)}, x^{(b)} \rangle = \{x^{(a)}, x^{(b)}\} + \left(\|b\| \binom{a + b - \epsilon_n}{b} - \|a\| \binom{a + b - \epsilon_n}{a} \right) x^{(a+b-\epsilon_n)}$$

where

$$\|a\| := |a| + a_n - 2, \quad \{x^{(a)}, x^{(b)}\} := \sum_{j=1}^{2r} \sigma(j) x^{(a-\epsilon_j)} x^{(b-\epsilon_{j'})}$$

$$\sigma(j) = 1 \quad j \leqq r \quad \sigma(j) = -1 \quad j \geqq r + 1$$

$$j' = j + r \quad j \leqq r \quad j' = j - r \quad j \geqq r + 1.$$

Since

$$\langle x^{(\epsilon_n)}, x^{(b)} \rangle = \|b\| x^{(b)},$$

we see that $H := Fx^{(\epsilon_n)}$ is an abelian subalgebra of $A(n; \mathbf{m})$ which operates on $A(n; \mathbf{m})$ by semisimple endomorphisms. The algebra $A(n; \mathbf{m})$ is graded by means of

$$A(n; \mathbf{m})_i = \sum_{\|a\|=i} Fx^{(a)} \quad A(n; \mathbf{m}) = \bigoplus_{i=-2}^{s} A(n; \mathbf{m})_i \quad s = \|\tau\|.$$

We recall the definition of the contact algebra:

$$K(n; \mathbf{m}) = \begin{cases} A(n; \mathbf{m}) & n + 3 \not\equiv 0 \bmod(p) \\ \sum_{a<\tau} Fx^{(a)} & n + 3 \equiv 0 \bmod(p). \end{cases}$$

Hence H is an abelian subalgebra of $K(n; \mathbf{m})_0$ and

(II) $\quad \Delta_i \cap \Delta_j \neq \emptyset$ if and only if $i \equiv j \bmod(p)$.

For $i \in \{1, \ldots, n\}$ we consider the linear mappings

$$\alpha_i: \begin{cases} A(n; \mathbf{m}) \to F \\ \sum_{a \leqq \tau} \beta(a) x^{(a)} \to \beta(\tau - \tau_{i'} \epsilon_{i'}) \end{cases}$$

where $n' := n$. It is known that

$$A(n; \mathbf{m}) = U(A(n; \mathbf{m})^-) \cdot A(n; \mathbf{m})_s.$$

Formula (I) readily entails the validity of

$$\text{(III)} \quad \langle x^{(\epsilon_j)}, x^{(a)} \rangle = \sigma(j) x^{(a-\epsilon_{j'})} + (a_j + 1) x^{(a+\epsilon_j-\epsilon_n)} \quad 1 \leqq j \leqq 2r.$$

$$\text{(IV)} \quad \langle 1, x^{(a)} \rangle = 2x^{(a-\epsilon_n)}.$$

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LEMMA 4.1. *Suppose that* $n + 3 \equiv 0 \bmod(p)$.

(1) $$\varphi_i : A(n; \mathbf{m}) \to A(n; \mathbf{m})^* \quad \varphi_i(x^{(a)})(x^{(b)}) = \alpha_i(x^{(a-\epsilon_i)}x^{(b)})$$

is a skew derivation of degree $1 = p^{m_{i'}} - s \quad 1 \leqq i \leqq 2r$

(2) *Let* deg *denote the degree derivation of* $A(n; \mathbf{m})$. *Then*

$$\varphi_n : A(n; \mathbf{m}) \to A(n; \mathbf{m})^* \quad \varphi_n(f)(g) = \alpha_n(\deg(f)g)$$

is a skew derivation of degree $l = 2p^{m_n} - s$.

Proof. (1) The linear map φ_i is clearly skew and of the indicated degree. Since $A(n; \mathbf{m})^-$ is generated by

$$A(n; \mathbf{m})_{-1} = \sum_{i=1}^{2r} Fx^{(\epsilon_i)}$$

and $p^{m_{i'}} > 4$ it suffices by virtue of (1.6) to verify for $1 \leqq j \leqq 2r$ the equation

(*) $$\varphi_i(\langle x^{(\epsilon_j)}, x^{(a)}\rangle)(x^{(b)}) = (x^{(\epsilon_j)} \cdot \varphi_i(x^{(a)}))(x^{(b)}) - (x^{(a)} \cdot \varphi_i(x^{(\epsilon_j)}))(x^{(b)}).$$

Formula (III) implies that the left hand side equals

$$\sigma(j)\alpha_i(x^{(a-\epsilon_i-\epsilon_{j'})}x^{(b)}) + (a_j + 1)\alpha_i(x^{(a-\epsilon_i+\epsilon_j-\epsilon_n)}x^{(b)})$$

while the right hand side coincides with

$$-\sigma(j)\alpha_i(x^{(a-\epsilon_i)}x^{(b-\epsilon_{j'})}) - (b_j + 1)\alpha_i(x^{(a-\epsilon_i)}x^{(b+\epsilon_j-\epsilon_n)}) + \delta_{ij}\alpha_i(\langle x^{(a)}, x^{(b)}\rangle).$$

The proof can now be completed by considering the following cases:

1. $i \neq j$.
1.1. $a + b - \epsilon_{j'} - \epsilon_i = \tau - \tau_{i'}\epsilon_{i'}$.
1.2. $a + b + \epsilon_j - \epsilon_i - \epsilon_n = \tau - \tau_{i'}\epsilon_{i'}$.
1.3. Neither 1.1 nor 1.2 hold.
2. $i = j$.
2.1. $a + b - \epsilon_i - \epsilon_{i'} = \tau - \tau_{i'}\epsilon_{i'}$.
2.2. $a + b - \epsilon_n = \tau - \tau_{i'}\epsilon_{i'}$.
2.3. Neither 2.1 nor 2.2 hold.

For the sake of brevity, we shall only dwell on the most intricate case 2.2. Then

$$\varphi_i(\langle x^{(\epsilon_i)}, x^{(a)}\rangle)(x^{(b)}) = (a_i + 1)\binom{\tau - \tau_{i'}\epsilon_{i'}}{b} = (-1)^{|b|}(a_i + 1),$$

while the right hand side of (*) equals

$$-(b_i + 1)\binom{\tau - \tau_{i'}\epsilon_{i'}}{a - \epsilon_i} + \left(||b||\binom{a + b - \epsilon_n}{b}\right)$$

$$- \|a\|\binom{a + b - \epsilon_n}{a}\Big).$$

By virtue of our present assumptions we also have

$$1 \equiv \|\tau - \tau_{i'}\epsilon_{i'}\| \equiv \|a + b - \epsilon_n\| \equiv \|a\| + \|b\| \bmod(p)$$
$$a_i + b_i \equiv -1 \bmod(p)$$
$$(-1)^{|a|} = -(-1)^{|b|}.$$

Hence the last expression coincides with

$$-(b_i + 1)(-1)^{|a|-1} + (-1)^{|b|}(\|b\| + \|a\|)$$
$$= (-1)^{|b|}(-b_i - 1 + 1) = (a_i + 1)(-1)^{|b|}$$

as desired.

(2). The mapping φ_n is obviously skew and of degree $2p^{m_n} - s$. It is therefore sufficient to check (*) for φ_n. Let

$$B := \{x^{(a)} \in A(n; \mathbf{m}); a_n = 0\}.$$

Then B is a subalgebra of $A(n; \mathbf{m})$ and

$$\lambda(x^{(a)}, x^{(b)}) = \alpha_n(x^{(a)}x^{(b)})$$

defines an associative form on B. Consequently, (*) holds whenever $a_n = b_n = 0$. In accordance with (III) only the case in which

$$a + b + \epsilon_j - \epsilon_n = \tau - \tau_n\epsilon_n$$

needs to be investigated. Note that

(a) $\quad a_j + b_j \equiv -2 \bmod(p)$

(b) $\quad \|a\| + \|b\| \equiv 1 \bmod(p)$

(c) $\quad a_n = 1, b_n = 0$ or $a_n = 0, b_n = 1$.

If the latter alternative of (c) applies then the left hand side of (*) is readily seen to vanish. For the right hand side we obtain

$$-(b_j + 1)\,\|a\|\binom{\tau - \tau_n\epsilon_n}{a} - \Big(\|b\|\binom{a + b - \epsilon_n}{b} - \|a\|\binom{a + b - \epsilon_n}{a}\Big)$$

$$= -(b_j + 1)\,\|a\|\,(-1)^{|a|} + \|a\|\,(-1)^{|a|-a_j}\binom{p^{m_j} - 2}{a_j}.$$

Since

$$\binom{p^m - 1 - l}{r} \equiv (-1)^r\binom{l + r}{l} \bmod(p)$$

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the last expression coincides with

$$(a_j + 1)\,\|a\|\,(-1)^{|a|} - \|a\|\,(-1)^{|a|-a_j}(-1)^{a_j}\binom{a_j+1}{1} = 0.$$

The other alternative can be treated similarly.

If $n + 3 \equiv 0 \bmod(p)$ then

$$(A(n;\,\mathbf{m})^*)^{A(n;\mathbf{m})} = F\alpha_\tau.$$

It therefore follows from (1.2) and the proof of the ensuing result that there exist $r_i \in F$ such that

$$(x^{(\epsilon_i)})^{\tau_{i'}} \cdot \varphi_i(x^{(\epsilon_i)}) = r_i\alpha_\tau \quad 1 \leqq i \leqq 2r \quad \text{and} \quad 1^{\tau_n} \cdot \varphi_n(1) = r_n\alpha_\tau.$$

By applying $x^{(\tau)}$ to the above equations we see that

$$r_i = 1 \quad 1 \leqq i \leqq 2r \quad \text{and} \quad r_n = -2.$$

LEMMA 4.2. *Let $\varphi:A(n;\,\mathbf{m}) \to A(n;\,\mathbf{m})^*$ be a derivation. Then the following statements hold*:

(1) *If $n + 3 \not\equiv 0 \bmod(p)$ then there exists $f \in A(n;\,\mathbf{m})^*$ such that*

$$\varphi(x) = x \cdot f \quad \forall x \in A(n;\,\mathbf{m})^-.$$

(2) *If $n + 3 \equiv 0 \bmod(p)$ then there are $r_1, \dots, r_n \in F$, $f \in A(n;\,\mathbf{m})^*$ such that*

$$\varphi(x) = \sum_{i=1}^{n} r_i\varphi_i(x) + x \cdot f \quad \forall x \in A(n;\,\mathbf{m})^-.$$

Proof. We put $L := A(n;\,\mathbf{m})$, $K := L^-$. The general assumption of (2.3) is valid for $V := Fx^{(\tau)}$. Let $e_i := x^{(\epsilon_i)}$ $1 \leqq i \leqq 2r$ $e_n := 1$ as well as $\tau' := (\tau_{1'}, \dots, \tau_{n'})$. We shall show that $J := \{b;\, b \leqq \tau'\}$ satisfies the requirements of (2.3).

(a) $\operatorname{ann}_{U(K)^+}(L) = \langle \{e^b;\, b \notin J\} \rangle.$

Suppose that $b \notin J$. Then there is $i \in \{1, \dots, n\}$ such that $b_i \geqq p^{m_{i'}}$. If $i = n$ then

$$e^b \cdot x^{(a)} = 0 \quad \forall 0 \leqq a \leqq \tau$$

as $(\operatorname{ad} 1)^{p^{m_n}} = 0$ (cf. (IV)). Suppose that $i \leqq 2r$. It will suffice to show that

$$e_i^{p^{m_{i'}}} \cdot x^{(a)} = 0 \quad 0 \leqq a \leqq \tau.$$

To that end we introduce the mapping

$$D_K:A(n;\,\mathbf{m}) \to W(n;\,\mathbf{m})$$

which is defined by means of

$$D_K(f) = \sum_{j=1}^{2r} \sigma(j) d_j(f) d_{j'} + f d_n$$

where

$$d_j = \partial_j + \sigma(j) x^{(\epsilon_{j'})} \partial_n \quad 1 \leqq j \leqq 2r \quad d_n = 2\partial_n.$$

It is known that D_K is a homomorphism of Lie algebras. The following equalities hold within the restricted algebra $\mathrm{Der}_F(A(n;\mathbf{m}))$ of derivations of the associative algebra $A(n;\mathbf{m})$. Observe that

$$D_K(e_i) = \sigma(i)\partial_{i'} + x^{(\epsilon_i)}\partial_n.$$

Since the two summands commute, we obtain

$$D_K(e_i)^{p^{m_{i'}}} = \sigma(i)\partial_{i'}^{p^{m_{i'}}} + (x^{(\epsilon_i)}\partial_n)^{p^{m_{i'}}}.$$

Noting that

$$\partial_{i'}^{p^{m_{i'}}} = 0 = (x^{(\epsilon_i)}\partial_n)^{p^{m_{i'}}}$$

we conclude

$$\begin{aligned} D_K(e_i^{p^{m_{i'}}} \cdot x^{(a)}) &= D_K((\mathrm{ad}\, e_i)^{p^{m_{i'}}}(x^{(a)})) = (\mathrm{ad}\, D_K(e_i))^{p^{m_{i'}}}(D_K(x^{(a)})) \\ &= (\mathrm{ad}\, D_K(e_i)^{p^{m_{i'}}})(D_K(x^{(a)})) = 0. \end{aligned}$$

The injectivity of D_K now yields the desired result.

Conversely, suppose that

$$u = \sum_{0<b} \alpha(b) e^b$$

is an element of $\mathrm{ann}_{U(K)^+}(L)$. Then the above shows that

$$v := \sum_{0<b\leqq\tau'} \alpha(b) e^b \in \mathrm{ann}_{U(K)^+}(L).$$

Assume that $v \neq 0$ and put

$$k := \min\{b_n;\ \alpha(b) \neq 0\}.$$

Formulas (III) and (IV) then entail

$$\begin{aligned} 0 = v \cdot x^{(\tau+(k-\tau_n)\epsilon_n)} &= \sum_{0<b\leqq\tau'_1 b_n=k} \alpha(b) e^b \cdot x^{(\tau+(k-\tau_n)\epsilon_n)} \\ &= 2^k \sum_{0<b\leqq\tau'_1 b_n=k} \alpha(b)\gamma(b) x^{(\tau-\tau_n\epsilon_n-\widetilde{b})} \end{aligned}$$

where

$$\widetilde{b} = (b_{1'}, \ldots, b_{(2r)'}, 0), \quad \gamma(b) \in \{1, -1\}.$$

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This shows that $\alpha(b) = 0$ whenever $b_n = k$, a contradiction. Hence

$$u \in \langle \{e^b; b \notin J\} \rangle.$$

(b) It follows from (a) that $\{e^b \cdot x^{(\tau)}; 0 \leqq b \leqq \tau'\}$ generates $A(n; \mathbf{m})$. Since

$$\dim_F A(n; \mathbf{m}) = p^{|\mathbf{m}|}$$

condition (b) of (2.3) holds.

(1) If $n + 3 \not\equiv 0 \bmod(p)$ then $L = [L, L]$ and (3) of (2.3) yields the desired result.

(2) If $n + 3 \equiv 0 \bmod(p)$ then we have

$$[L, L] = \sum_{a<\tau} Fx^{(a)}.$$

By (1.2) there are $r_1, \ldots, r_n \in F$ such that

$$(x^{(\epsilon_i)})^{\tau_{i'}} \cdot \varphi(x^{(\epsilon_i)}) = r_i\alpha_\tau \quad 1 \leqq i \leqq 2r$$

and

$$1^{\tau_n} \cdot \varphi(1) = -2r_n\alpha_\tau.$$

We consider the derivation

$$\tilde{\varphi} := \varphi - \sum_{j=1}^{n} r_j\varphi_j.$$

The remark preceding (5.2) reveals in conjunction with

$$e_i^{\tau_{i'}} \cdot \varphi_j(e_i) = 0 \quad \text{for } i \neq j$$

that

$$e_i^{\tau_{i'}} \cdot \tilde{\varphi}(e_i) = 0 \quad 1 \leqq i \leqq n.$$

The assertion now follows from a consecutive application of (2) and (1) of (2.3).

LEMMA 4.3. *Suppose that* $n + 3 \equiv 0 \bmod(p)$ *and let*

$$\varphi: K(n; \mathbf{m}) \to K(n; \mathbf{m})^*$$

be a skew derivation of degree $l \geqq 4 - s$. *Then there exists a homogeneous skew derivation*

$$\tilde{\varphi}: A(n; \mathbf{m}) \to A(n; \mathbf{m})^*$$

of degree l *which extends* φ.

Proof. We consider the linear map

$$\tilde{\varphi}: A(n; \mathbf{m}) \to A(n; \mathbf{m})^*$$

which is given by

$$\tilde{\varphi}(x^{(a)})(x^{(b)}) := \begin{cases} \varphi(x^{(a)})(x^{(b)}) & a, b < \tau \\ 0 & \text{otherwise.} \end{cases}$$

As $A(n; \mathbf{m}) = K(n; \mathbf{m}) \oplus Fx^{(\tau)}$, $\tilde{\varphi}$ is a well-defined skew linear mapping of degree l. We propose to verify

$$(*) \qquad \tilde{\varphi}([x, y]) = x \cdot \tilde{\varphi}(y) - y \cdot \tilde{\varphi}(x).$$

Only two cases need to be considered:

1) $\qquad x = x^{(\tau)}$.

2) $\qquad x, y \in K(n; \mathbf{m})$.

Let $x = x^{(\tau)}$ and note that

$$[x^{(\tau)}, A(n; \mathbf{m})] \subset K(n; \mathbf{m})_{s-1} \oplus K(n; \mathbf{m})_s.$$

As $l \geqq 4 - s$ we see that

$$\tilde{\varphi}([x^{(\tau)}, y]) \in \sum_{i \geqq 3} (A(n; \mathbf{m})^*)_i = 0.$$

Since

$$y \cdot \tilde{\varphi}(x^{(\tau)}) = 0 \quad \text{and}$$

$$x^{(\tau)} \cdot \tilde{\varphi}(y) \in \sum_{i \geqq 4-s+s+1-2} (A(n; \mathbf{m})^*)_i = 0$$

the validity of (*) follows.

Next, we assume that $x, y \in K(n; \mathbf{m})$. It obviously suffices to verify

$$\tilde{\varphi}([x, y])(x^{(\tau)}) = (x \cdot \tilde{\varphi}(y))(x^{(\tau)}) - (y \cdot \tilde{\varphi}(x))(x^{(\tau)}).$$

The left hand side vanishes by definition. Suppose that

$$x \in K(n; \mathbf{m})_i, \quad y \in K(n; \mathbf{m})_j.$$

Then the right hand side of (*) is easily seen to be contained in $(A(n; \mathbf{m})^*)_{i+j+l}$. Hence the right hand side of the last equation vanishes unless $i + j + l = -s - 1$. Since this implies that

$$-s - 1 = i + j + l \geqq i + j + 4 - s \geqq -s \quad (i, j \geqq -2)$$

we are done.

Formula (I) implies the validity of the following rules:

$$(1) \qquad \langle x^{(\epsilon_i + \epsilon_n)}, x^{(b - \epsilon_i)} \rangle = b_i(\|b\| - 1 - b_n)x^{(b)} + \sigma(i)(b_n + 1)x^{(b - \epsilon_i - \epsilon_{i'} + \epsilon_n)}$$

$$(2)\qquad \langle x^{(2\epsilon_i+\epsilon_n)}, x^{(b-2\epsilon_i)}\rangle = \frac{1}{2}b_i(b_i - 1)(\,\|b\| - 2 - 2b_n)x^{(b)} + \sigma(i)(b_i - 1)(b_n + 1)x^{(b-\epsilon_i-\epsilon_{i'}+\epsilon_n)}$$

$$(3)\qquad \det\begin{pmatrix} b_i(\,\|b\| - 1 - b_n) & \sigma(i)(b_n + 1) \\ \frac{1}{2}b_i(b_i - 1)(\,\|b\| - 2 - 2b_n) & \sigma(i)(b_i - 1)(b_n + 1)\end{pmatrix} = \frac{1}{2}\sigma(i)b_i(b_i - 1)(b_n + 1)\,\|b\|$$

$$(4)\qquad \langle x^{(\epsilon_i+\epsilon_n)}, x^{(b-\epsilon_i)}\rangle + \langle x^{(\epsilon_{i'}+\epsilon_n)}, x^{(b-\epsilon_{i'})}\rangle = (b_i + b_{i'})(\,\|b\| - 1 - b_n)x^{(b)}$$

$$(5)\qquad \langle x^{(2\epsilon_n)}, x^{(b-\epsilon_n)}\rangle = b_n(\,\|b\| - 1 - b_n)x^{(b)}$$

$$(6)\qquad \langle x^{(3\epsilon_n)}, x^{(b-2\epsilon_n)}\rangle = b_n(b_n - 1)\left(\frac{1}{2}\|b\| - \frac{2}{3} - \frac{2}{3}b_n\right)x^{(b)}$$

$$(7)\qquad \langle x^{(\epsilon_i+\epsilon_j+\epsilon_n)}, x^{(b-\epsilon_i-\epsilon_j)}\rangle = b_ib_j(\,\|b\| - 2 - 2b_n)x^{(b)} + \sigma(i)b_j(b_n + 1)x^{(b-\epsilon_i-\epsilon_{i'}+\epsilon_n)} + \sigma(j)b_i(b_n + 1)x^{(b-\epsilon_j-\epsilon_{j'}+\epsilon_n)}\quad i \neq j.$$

LEMMA 4.4. *Suppose that* $k \geqq 3$. *Then*

$$M(A(n; \mathbf{m}))_k = A(n; \mathbf{m})_k \quad \textit{for } k \not\equiv 0, -2 \bmod(p).$$

Proof. We put $M := M(A(n; \mathbf{m}))$. Let $x^{(b)}$ be an element of $A(n; \mathbf{m})_k$. We first suppose that

$$\|b\| - 1 - b_n \not\equiv 0 \bmod(p).$$

If $b_n \not\equiv 0 \bmod(p)$, then, as $\|b\| \geqq 3$, $x^{(b)} \in M_k$ by virtue of (5). Hence we assume that $b_n \equiv 0 \bmod(p)$. According to (4), we see that $x^{(b)} \in M_k$ unless

$$b_i + b_{i'} \equiv 0 \bmod(p) \quad 1 \leqq i \leqq 2r.$$

This, however, entails that $\|b\| \equiv -2 \bmod(p)$.

Next, we assume

$$\|b\| - 1 - b_n \equiv 0 \bmod(p).$$

Since $\|b\| \not\equiv 0 \bmod(p)$,

$$\frac{1}{2}\|b\| - \frac{2}{3} - \frac{2}{3}b_n \not\equiv 0 \bmod(p).$$

If $b_i \equiv 0 \bmod(p)$ $\quad 1 \leqq i \leqq 2r$ then

$$0 \equiv \|b\| - 1 - b_n \equiv b_n - 3 \bmod(p).$$

Thus $b_n \geqq 8$ and $\|b\| \geqq 5$. Then $x^{(b)} \in M_k$ on account of (6). Suppose that there is $i_0 \leqq 2r$ such that

$$b_{i_0} \not\equiv 0 \bmod(p).$$

Since $b_n + 1$ does not vanish, we obtain according to (3) that $x^{(b)} \in M_k$ unless

$$b_{i_0} \equiv 1 \bmod(p).$$

For $j \in \{1, \ldots, 2r\}$ we consider

$$c := b - \epsilon_j - \epsilon_{j'} + \epsilon_n.$$

Then either $x^{(c)} = 0$ or $0 \leqq c \leqq \tau$. The first part of the proof then ensures that $x^{(c)}$ is contained in M_k. We now apply (7) in order to see that $x^{(b)} \in M_k$ unless

$$b_j \equiv 0 \bmod(p) \qquad \forall j \neq i_0.$$

Then

$$0 \equiv \|b\| - 1 - b_n \equiv b_n - 2 \bmod(p).$$

This entails that $\|b\| \geqq 5$ and the proof may be concluded by applying (6).

THEOREM 4.5. *The second cohomology group* $H^2(K(n; \mathbf{m}), F)$ *is* $n + 1$*-dimensional if* $n + 3 \equiv 0 \bmod(p)$, $|\mathbf{m}| - n$*-dimensional if* $n + 5 \equiv 0 \bmod(p)$, *and trivial otherwise.*

Proof. Suppose first that $n + 3 \equiv 0 \bmod(p)$. According to [**6**], the mapping

$$\varphi_{n+1}:K(n; \mathbf{m}) \rightarrow K(n; \mathbf{m})^* \quad \varphi_{n+1}(x^{(a)})(x^{(b)}) = \alpha_\tau(x^{(a-\epsilon_n)}x^{(b)})$$

is a skew derivation of degree $-s - 1$. We shall prove that the vector space V of skew derivations from $K(n; \mathbf{m})$ to $K(n; \mathbf{m})^*$ decomposes

$$V = \bigoplus_{i=1}^{n+1} F\varphi_i \oplus \mathrm{Inn}_F(K(n; \mathbf{m}), K(n; \mathbf{m})^*),$$

where the φ_i $1 \leqq i \leqq n$ are considered elements of

$$\mathrm{Der}_F(K(n; \mathbf{m}), K(n; \mathbf{m})^*).$$

Let φ be an element of V and suppose that $\deg \varphi = l$.

(a) $l \geqq -s$. We may assume by virtue of (1.1) that φ respects the root space decomposition. Hence $\varphi = 0$ or $l \equiv 0 \bmod(p)$. As $n + 3 \equiv 0 \bmod(p)$ we have $s \equiv -1 \bmod(p)$ and $\varphi = 0$ for $l = -s, 1 - s, 2 - s, 3 - s$. Hence we assume that $l \geqq 4 - s$. According to (4.3) φ may be extended to a skew derivation

$$\tilde{\varphi}: A(n;\, \mathbf{m}) \to A(n;\, \mathbf{m})^*.$$

Owing to (4.2) there are $r_1, \ldots, r_n \in F$ and $f \in A(n;\, \mathbf{m})^*$ such that

$$\tilde{\varphi}(x) = \sum_{i=1}^{n} r_i \varphi_i(x) + x \cdot f \quad \forall x \in A(n;\, \mathbf{m})^-.$$

Put $g := f|_{K(n;\mathbf{m})}$. Then

$$\varphi(x) = \sum_{i=1}^{n} r_i \varphi_i(x) + x \cdot g \quad \forall x \in K(n;\, \mathbf{m})^-.$$

Hence

$$\varphi - \sum_{i=1}^{n} r_i \varphi_i \in \mathrm{Inn}_F(K(n;\, \mathbf{m}), K(n;\, \mathbf{m})^*)$$

by virtue of (2.4).

(b) $l \leqq -s - 1$. As in (a) we may assume that $\varphi = 0$ or $l \equiv 0 \bmod(p)$. Consequently, $l \leqq -s - 3$ or $l = -s - 1$. In the former case we have $-(s + l) \geqq 3$ as well as $-(s + l) \not\equiv 0, -2 \bmod(p)$. Hence (4.4) implies in combination with (1.5) that $\varphi = 0$. If $l = -s - 1$ we consider the bilinear symmetric form

$$\lambda: K(n;\, \mathbf{m}) \times K(n;\, \mathbf{m}) \to F \quad \lambda(x^{(a)}, x^{(b)}) = \alpha_\tau(x^{(a)}x^{(b)}).$$

Note that $\mathrm{rad}(\lambda) = F1$ and put

$$\tilde{V} := K(n;\, \mathbf{m})/F1, \quad \pi: K(n;\, \mathbf{m}) \to \tilde{V}$$

canonical projection. Furthermore, let ρ denote the bilinear form on $\tilde{V}$ which is induced by λ. According to (3.3) and (3.7) of **[6]** there exists a unique skew homomorphism $D: \tilde{V} \to \tilde{V}$ of

$$P := K(n;\, \mathbf{m})^- \oplus \langle \{x^{(\epsilon_i + \epsilon_j)};\ 1 \leqq i, j \leqq 2r\} \rangle$$

-modules such that

$$\varphi(x)(y) = \rho(D(\pi(x)), \pi(y)) \quad \forall x, y \in K(n;\, \mathbf{m}).$$

The mapping D is uniquely determined by $D(\pi(x^{(\tau - \epsilon_1)}))$. Since $\deg D = -2$ a direct computation ensures the existence of $\alpha \in F$ with

$$D(\pi(x^{(\tau - \epsilon_1)})) = \alpha\pi(x^{(\tau - \epsilon_1 - \epsilon_n)}).$$

Hence

$$D(v) = \frac{1}{2}\alpha \cdot 1 \cdot v \quad \forall v \in V \quad \text{and} \quad \varphi = \frac{1}{2}\alpha\varphi_{n+1}.$$

Using the gradation of $\mathrm{Der}_F(K(n;\, \mathbf{m}), K(n;\, \mathbf{m})^*)$ one readily shows that the given sum is direct.

(2) $n + 5 \equiv 0 \bmod(p)$. This case was treated in [**6**].

(3) $n + 3 \not\equiv 0 \bmod(p)$, $n + 5 \not\equiv 0 \bmod(p)$.

(a) $l \geqq -s$. According to (4.2) there is $f \in K(n; \mathbf{m})^*$ such that

$$\varphi(x) = x \cdot f \quad \forall x \in K(n; \mathbf{m})^-.$$

As $s \equiv 0 \bmod(p)$ we may apply (2.4) and (2.5) in order to see that φ is inner.

(b) $l \leqq -s - 1$. As φ respects the root space decomposition we have $\varphi = 0$ or $l \equiv 0 \bmod(p)$. Suppose that $l \leqq -s - 3$. Then (4.4) and (1.5) entail the vanishing of φ unless $s \equiv 0, 2 \bmod(p)$. Since

$$s \equiv \|\tau\| \equiv -(n + 3)\bmod(p),$$

this is by virtue of our present assumption impossible. If $-(s + l) \in \{1, 2\}$ (3.3) and (3.7) of [**6**] provide a skew homomorphism

$$D:K(n; \mathbf{m}) \rightarrow K(n; \mathbf{m})$$

of P-modules of degree $k \in \{-3, -4\}$ such that

$$\varphi(x)(y) = \lambda(D(x), y).$$

An elementary computation then reveals that $D = 0$.

The central extensions of the hamiltonian algebras were obtained in [**6**]. The treatment of the special algebras requires a modification of the methods set forth in Section 2.

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Note added in proof. An extensive generalization of Theorem 2.1 has meanwhile appeared in the author's paper *On the cohomology of associative algebras and Lie algebras*, Proc. Amer. Math. Soc. *99* (1987), 415-420.

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