On the stationary eccentricity of a system of conics through four given points.

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Let the equation of the four point system be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where h is a variable parameter and a, b, g, f and c are constants. The four fixed points are thus given by $y = 0$, $ax^2 + 2gx + c = 0$ and $x = 0$, $by^2 + 2fy + c = 0$. Since the quadrilateral is real $g^2 > ac$ and $f^2 > bc$: it is convex if $\frac{c}{a}$ and $\frac{c}{b}$ have the same sign, *i.e.* if ab is positive, and concave if ab is negative.

First let us consider the eccentricity of $ax^2 + 2hxy + by^2 = 1$, when the conic is a hyperbola. In this case $h^2 > ab$, and if the equation referred to principal axes be $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$, we have the invariant relations

$$\frac{1}{\alpha^2} - \frac{1}{\beta^2} = \frac{a + b - 2h\cos\omega}{\sin^2\omega} \text{ and } -\frac{1}{\alpha^2\beta^2} = \frac{ab - h^2}{\sin^2\omega}.$$

Whence

$$2e^2\sin^2\omega = \frac{K(K + a + b - 2h\cos\omega)}{h^2 - ab},$$

where K is the positive square root of $(a + b - 2h\cos\omega)^2 + 4(h^2 - ab)\sin^2\omega$. The eccentricity of

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ or } ax^2 + 2hxy + by^2 + \frac{\Delta}{C} = 0$$

is obtained by writing $a \cdot \frac{h^2 - ab}{\Delta}$ for a, etc.

Thus

$$2e^2\sin^2\omega = \frac{K^2}{h^2 - ab} + \frac{h^2 - ab}{\Delta}(a + b - 2h\cos\omega)\frac{\text{pos.}\sqrt{\frac{(h^2 - ab)^2}{\Delta^2}.K^2}}{\frac{(h^2 - ab)^2}{\Delta^2}.(h^2 - ab)}$$

$$= \frac{K^2}{h^2 - ab} \pm \frac{a + b - 2h\cos\omega}{h^2 - ab}K,$$

the positive or negative sign being taken according as Δ is positive or negative.

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Hence the eccentricity is discontinuous when h passes through infinity, and when Δ changes sign. There are in general three discontinuities, which occur when the hyperbola degenerates into a pair of straight lines.

If we put $\lambda = \dfrac{a+b-2h\cos\omega}{\text{pos. } \sqrt{h^2-ab}}$ we have

$$2e^2\sin^2\omega = \lambda^2 + 4\sin^2\omega \pm \lambda\sqrt{\lambda^2+4\sin^2\omega},$$

the upper or lower sign being taken according as Δ is positive or negative.

Hence except at a discontinuity

$$2\sin^2\omega \, . \, \frac{de^2}{dh} = \frac{\pm(\lambda \pm \sqrt{\lambda^2+4\sin^2\omega})^2}{\sqrt{\lambda^2+4\sin^2\omega}} \, . \, \frac{-1}{(h^2-ab)^{\frac{3}{2}}} \, . \, \{(a+b)h - 2ab\cos\omega\}.$$

Thus there is only one value of h, viz. $\dfrac{2ab\cos\omega}{a+b}$ for which the eccentricity is stationary.

Now when h has this value

$$h^2 - ab = -\frac{ab}{(a+b)^2}(a^2 - 2ab\cos\omega + b^2), \text{ which has the same sign as } -ab.$$

Hence if ab is positive, *i.e.* if the quadrilateral is convex, the hyperbolas of the system have no stationary eccentricity.

If the quadrilateral is concave, ab is negative and $h = \dfrac{2ab\cos\omega}{a+b}$ gives a hyperbola with stationary eccentricity: to find whether e^2 is a maximum or minimum we must examine the sign of $\dfrac{d^2e^2}{dh^2}$.

When $(a+b)h - 2ab\cos\omega = 0$, we have

$$2\sin^2\omega \, . \, \frac{d^2e^2}{dh^2} = \frac{\pm(\lambda \pm \sqrt{\lambda^2+4\sin^2\omega})^2}{\sqrt{\lambda^2+4\sin^2\omega}} \, . \, \frac{-(a+b)}{(h^2-ab)^{\frac{3}{2}}} \, .$$

Therefore $\dfrac{d^2e^2}{dh^2}$ has the same sign as $-(a+b)\Delta$, so that e^2 is a maximum or minimum according as $h = \dfrac{2ab\cos\omega}{a+b}$ makes $-(a+b)\Delta$ positive or negative.

Let us next consider the ellipses of the system. In this case $ab - h^2$ is positive, so that ab is positive and the quadrilateral is convex. A slight modification of the preceding work gives us

$$2e^2\sin^2\omega = -(\lambda^2 - 4\sin^2\omega) \pm \lambda\sqrt{\lambda^2 - 4\sin^2\omega},$$

where $\lambda = \frac{a+b-2h\cos\omega}{\text{pos.}\sqrt{ab-h^2}}$, and the upper or lower sign is to be taken according as Δ is positive or negative.

There is only one value of h, viz. $\frac{2ab\cos\omega}{a+b}$ for which e^2 is stationary, and $\frac{d^2e^2}{dh^2}$ has the same sign as $-(a+b)\Delta$.

Now $a\Delta = \text{BC} - \text{F}^2$ and $b\Delta = \text{AC} - \text{G}^2$ where A, etc., are the co-factors of a, etc., in Δ.

$\therefore$ $a\Delta$ and $b\Delta$ are both negative, for A and B are negative and C is positive.

Hence $-(a+b)\Delta$ is positive and e^2 a minimum.

To sum up:—If the quadrilateral is convex the ellipses of the system have a minimum eccentricity, but the hyperbolas no stationary eccentricity; if the quadrilateral is concave there are no ellipses belonging to the system, and the hyperbolas have a maximum or minimum eccentricity according as $h = \frac{2ab\cos\omega}{a+b}$ makes $(a+b)\Delta$ positive or negative.

In the case of the concave quadrilateral it may be observed that when the eccentricity is a maximum the asymptotic angle is obtuse, and when a minimum acute.

For when $h = \frac{2ab\cos\omega}{a+b}$, $a+b$ has the same sign as $a+b-2h\cos\omega$.

Now the invariant relation

$$-\frac{\text{C}}{\Delta} \cdot \frac{a+b-2h\cos\omega}{\sin^2\omega} = \frac{1}{a^2} - \frac{1}{\beta^2}$$

shows (since C is negative) that $(a+b-2h\cos\omega)\Delta$ has the same sign as $\frac{1}{a^2} - \frac{1}{\beta^2}$.

Hence the eccentricity is a maximum or minimum according as $\beta^2 >$ or $< a^2$, *i.e.*, according as the asymptotic angle is obtuse or acute.

The question that now arises is the geometrical interpretation of the sign of $(a+b)\Delta$ for the hyperbola of stationary eccentricity. We can put $c = 1$ without losing generality: if the four fixed points are (α_1, o), (α_2, o), (o, β_1), (o, β_2) we have

$$\frac{2g}{a} = -(\alpha_1 + \alpha_2),\ \frac{1}{a} = \alpha_1\alpha_2,\ \frac{2f}{b} = -(\beta_1 + \beta_2),\ \frac{1}{b} = \beta_1\beta_2.$$

Now $\Delta = -\Pi(h - fg \pm \sqrt{(f^2-b)(g^2-a)}$

$$= -\Pi\left\{h - fg \pm \frac{ab}{4}(a_1 - a_2)(\beta_1 - \beta_2)\right\}$$

Also for the stationary eccentricity $h = \dfrac{2\cos\omega}{a_1a_2 + \beta_1\beta_2}$

$$\therefore (a+b)\Delta = -\frac{a_1a_2+\beta_1\beta_2}{a_1a_2\beta_1\beta_2}\left(\frac{2\cos\omega}{a_1a_2+\beta_1\beta_2} - \frac{a_1\beta_1+a_2\beta_2}{2a_1a_2\beta_1\beta_2}\right)\left(\frac{2\cos\omega}{a_1a_2+\beta_1\beta_2} - \frac{a_1\beta_2+a_2\beta_1}{2a_1a_2\beta_1\beta_2}\right).$$

Let the angular points of the quadrilateral be $P(o, \beta_2)$, $Q(a_2, o)$, $R(a_1, o)$ and $S(o_1, \beta_1)$. A the intersection of QR and PS is the origin : QS meets PR at B and RS meets PQ at C (Figure IV). The quantities a_1, β_1 and β_2 are positive, while a_2 is negative.

Now $a_1a_2 + \beta_1\beta_2$ is AQ.AR + AS.AP, and since the condition which discriminates between a maximum and minimum must be symmetrical, we should expect the other factors of $(a+b)\Delta$ to be multiples of BR.BP + BS.BQ and CQ.CP + CS.CR.

The equations of QS and PR are

$$\frac{x}{a_2} + \frac{y}{\beta_1} = 1 \text{ and } \frac{x}{a_1} + \frac{y}{\beta_2} = 1.$$

Hence for the point B

$$x = \frac{a_1a_2(\beta_1 - \beta_2)}{a_1\beta_1 - a_2\beta_2}$$

Thus $\text{BR.BP} = -x(a_1 - x)\dfrac{\text{RP}^2}{\text{RA}^2} = -\dfrac{a_2\beta_1(a_1-a_2)(\beta_1-\beta_2)(a_1^2+\beta_2^2-2a_1\beta_2\cos\omega)}{(a_1\beta_1 - a_2\beta_2)^2}$

and $\text{BS.BQ} = x(x - a_2)\dfrac{\text{QS}^2}{\text{QA}^2} = -\dfrac{a_1\beta_2(a_1-a_2)(\beta_1-\beta_2)(a_2^2+\beta_1^2-2a_2\beta_1\cos\omega)}{(a_1\beta_1 - a_2\beta_2)^2}$

$$\therefore \text{BR.BP} + \text{BS.BQ} = -\frac{(a_1-a_2)(\beta_1-\beta_2)}{(a_1\beta_1-a_2\beta_2)^2}\left\{(a_1a_2+\beta_1\beta_2)(a_1\beta_1+a_2\beta_2) - 4a_1a_2\beta_1\beta_2\cos\omega\right\}$$

and similarly

$$\text{CQ.CP} + \text{CS.CR} = \frac{(a_1-a_2)(\beta_1-\beta_2)}{(a_1\beta_2-a_2\beta_1)^2}\left\{(a_1a_2+\beta_1\beta_2)(a_1\beta_2+a_2\beta_1) - 4a_1a_2\beta_1\beta_2\cos\omega\right\}$$

Hence (a_2 being negative) the sign of $(a+b)\Delta$ is opposite to that of (AQ.AR + AS.AP)(BR.BP + BS.BQ)(CP.CQ + CS.CR).

Now let us suppose that P, Q and R are fixed, and that S moves so that AQ.AR + AS.AP = 0. Let PSA meet the circle PQR at S′. Then S′A = AS. The locus of S can be plotted from this property : its general shape is quite obvious. The three loci of this nature intersect at the orthocentre of PQR. If S lies in one of the three regions bounded by two of these curves, *i.e.* in one of the shaded regions in Figure I., $(a+b)\Delta$ is negative and the system has a minimum eccentricity. If S lies outside the shaded regions the eccentricity is a maximum.

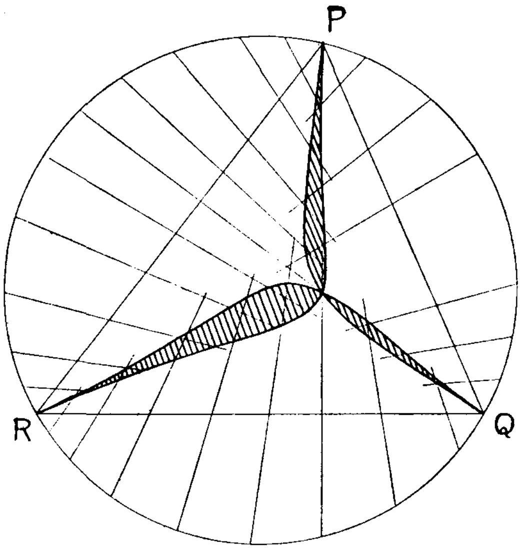
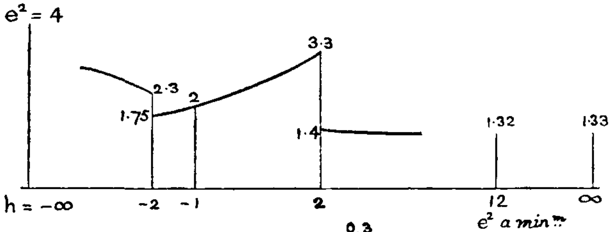


Fig I

Figure II. is the graph of e^2 for the system of points depicted. The eccentricity is a minimum when $h = 12$.



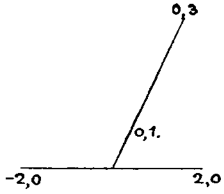


Fig II

Figure III. is the graph of e^2 for another system. The eccentricity is a maximum when $h=0$. There is no discontinuity at $h=\infty$, for the straight lines into which the hyperbola degenerates are at right angles, so that the eccentricities of the hyperbola and its conjugate are equal.

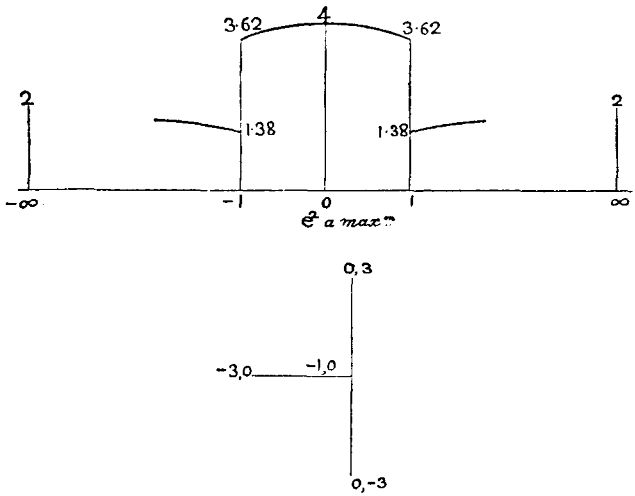


Fig III

The geometrical solution of the problem for the concave quadrilateral is interesting.

Any straight line is met by a system of four-point conics in a range of pairs of points in involution. This is also true of the line at infinity. Hence the parallels through any point, S say, to the asymptotes of the system form a pencil of pairs of lines in involution. Through S draw pSp', qSq', rSr' parallel to the sides of PQR. Then SP, Sp is a line pair of the pencil, since PA, QR constitute one of the conics of the system. Similarly SR, Sr is another pair, and SB, Sq another. Thus Pp, Rr, B∞ are pairs of points in involution on the transversal PR. B is the centre of the

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involution and $BP.Bp = BR.Br = -q^2$, say. In the work that follows lengths along PR carry their own signs, while other lengths are supposed positive.

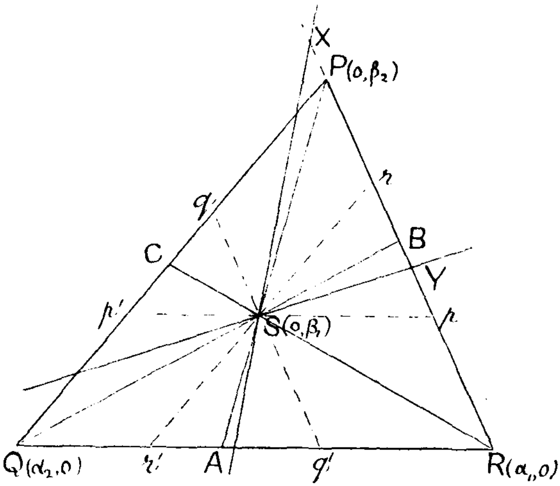

Fig IV

Let XY be a point pair such that the angle XSY is stationary, and X′Y′ an adjacent pair.

Then $X\widehat{S}X' = Y\widehat{S}Y'$, so that $\dfrac{XX'\sin X'}{SX} = \dfrac{YY'\sin Y}{SY'}$.

Now $BX.BY = BX'.BY' = -q^2 \quad \therefore \quad \dfrac{XX'}{BX} = -\dfrac{YY'}{BY}$.

Also $\sin X : \sin Y = \dfrac{1}{SX} : \dfrac{1}{SY}$.

Hence when $X\widehat{S}Y$ is stationary

$$\frac{BX}{SX^2} = -\frac{BY}{SY^2}.$$

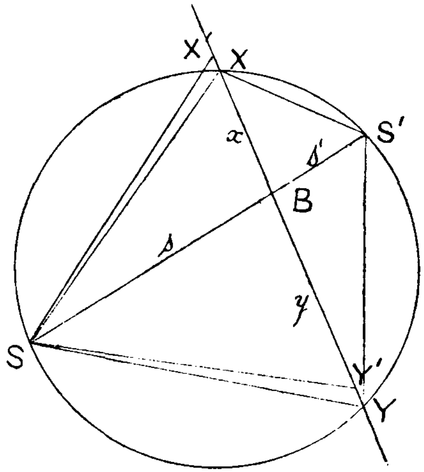

Fig V

Let SB produced meet the circle SXY at S'. Then since SB.BS' $= q^2$, S' is a fixed point. For BX, BY, SB, BS' write x, y, s, s' respectively.

From similar triangles $\dfrac{S'X^2}{SY^2} = \dfrac{x^2}{s^2}$.

But $\dfrac{SY^2}{SX^2} = -\dfrac{y}{x}$.

$$\therefore \quad \frac{S'X^2}{SX^2} = -\frac{xy}{s^2} = \frac{ss'}{s^2} = \frac{s'}{s};$$

and similarly $\dfrac{S'Y^2}{SY^2} = \dfrac{s'}{s}$.

Hence XY is the intercept on PR by the circle which is the locus of a point whose distances from S and S' are in the subduplicate ratio of s to s'. Consequently there is only one position of XY for which the angle XSY is stationary.

We now have to determine when the stationary angle is a maximum and when a minimum.

If $s > s'$, then $SX > SX'$ and $SY > SY'$.

$\therefore S\widehat{S'}X > S'\widehat{S}X$ and $S\widehat{S'}Y > S'\widehat{S}Y \quad \therefore X\widehat{S'}Y > X\widehat{S}Y$.

Thus $X\widehat{S}Y \gtrless 90^\circ$ according as $s \lessgtr s'$, *i.e.* $s^2 \lessgtr q^2$.

Let us take the case $s > s'$ so that $X\widehat{S}Y > 90^\circ$. When X approaches B and Y approaches ∞ the angle XSY tends to SBX, and when X approaches ∞ and Y approaches B the angle XSY tends to SBY. But one of the angles SBX, SBY must be at least as great as 90°. Hence the stationary value of the angle XSY is in this case a minimum.

In the same way we can show that when $s < s'$, *i.e.* $s^2 < q^2$, the stationary angle is a maximum and obtuse.

$$\text{Now} \quad \frac{Bp}{BR} = \frac{BS}{BQ} \quad \therefore \quad BP.Bp = BP.BR.\frac{BS}{BQ} = \frac{BP.BR}{BS.BQ}.BS^2$$

$$\therefore \quad \frac{BS.BQ}{BP.BR} = \frac{BS^2}{BP.Bp} = \frac{s^2}{-q^2}.$$

Therefore the parallels through S to the asymptotes of the hyperbola of stationary eccentricity intercept on PR a length which subtends a maximum or minimum angle at S according as $BR.BP + BS.BQ$ is negative or positive.

It remains to be seen in which of the angles between the asymptotes the hyperbola lies. Of the four points P, Q, R, S one (either P, Q or R) will lie on one branch, and the other three on the other branch. Of these three S will be the point situated between the other two. Say PSR lie on one branch: then the parallels to SP and SR through the centre lie in those angles between the asymptotes in which the hyperbola does not lie, while the parallel to SQ lies within the angles in which the conic is actually situated. Hence the asymptotic angle is that angle formed by SX and SY which contains one and only one of the lines PSA, QSB, RSC.

Now if X lies in the interval $P\infty$, Y lies in Bp. In this case XS and YS produced meet QR. Hence the angle subtended at S by the XY interval on QR is equal to the angle XSY. Also SX meets PQ and YS produced meets PQ, so that the XY interval on

PQ subtends at S an angle supplementary to XSY. The angle subtended by the interval on PQ is the angle which contains one and only one of the lines PSA etc., and is in this case the asymptotic angle.

A consideration of the other two possible cases, viz. X in the interval Pr and X in the interval rB, will show that the XY intervals on the three sides of PQR subtend at S angles, two of which are equal and the third supplementary to them, and that it is the third which is the asymptotic angle.

The condition which discriminates between a maximum and minimum can now be evolved. We may observe that since the three XSY angles cannot be all acute or all obtuse, the three quantities AQ.AR + AS.AP, etc., cannot all have the same sign. Denote them by D_P, D_Q, D_R.

Then $D_P . D_Q . D_R$ positive means two D's negative and one positive,
i.e., two XSY angles obtuse and one acute,
i.e., the asymptotic angle acute and a minimum;

and $D_P . D_Q . D_R$ negative means two D's positive and one negative,
i.e., two XSY angles acute and one obtuse,
i.e., the asymptotic angle obtuse and a maximum.