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THE ENDOMORPHISM RING OF A FINITE-LENGTH MODULE

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Let M be an R-module of finite length. For a simple R-module A, let ℓ_A denote the nuber of times the isomorphism type of A appears in a composition chain of M, and let σ denote the maximum of the ℓ_A, A ranging over all simple submodules of M. Let S be the endomorphism ring of M. We show that the Loewy length of S is bounded by σ.

It is well-known that the endomorphism ring S of a finite-length module M_R over any ring R is semi-primary, that means the factor ring of S modulo its radical J is semisimple artinian, and J is nilpotent. The smallest number m with the property $J^m = 0$ is called the Loewy length of S. Let ℓ denote the length of M_R. Then the estimate $m \leqslant \ell$ holds. According to a remark of Bourbaki ([**1**, Chapter 8, Section 2, exercise 3]), this result is due to A. Rosenberg.

For any simple module A_R, let ℓ_A be the number of times the isomorphism type of A_R appears as a composition factor in a composition chain of M_R. Let h denote the maximum of the numbers ℓ_A, A_R ranging over all simple R-modules. Improving the estimate given above, Smalø [**4**] showed that the inequality $m \leqslant h$ holds.

In this paper, we will prove the estimate $m \leqslant \sigma$, where σ is the maximum of the numbers ℓ_A, A_R ranging only over all simple submodules of M_R. Note that all of the numbers ℓ_A, ℓ, h, σ are invariants of M_R by the Jordan– Hölder Theorem. An analogous result for infinite cardinals was proved in [**3**, Satz 4], under more general assumptions on M_R, including not only finite-length modules, but also certain semi-artinian modules which have perfect endomorphism rings. As the methods in [**3**] are rather technical, it might be useful to provide a simple proof for the estimate $m \leqslant \sigma$ in the finite-length case. This is the aim of the present note.

THEOREM. *Let M_R be a finite-length module over any ring R. Let S be the endomorphism ring of M_R. Then the Loewy length of S is bounded by the number σ.*

PROOF: We need the following lemma (compare [**2**, Lemma 4], which may be of interest in its own right.

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LEMMA. *Let ${}_SX_R$ be a bimodule, where S is a semi-primary ring and X_R has finite length. Let X_R have a composition factor isomorphic to some simple module A_R. Then the socle of ${}_SX$, considered as a right R-module, has also a composition factor isomorphic to A_R.*

PROOF: Let $J = \operatorname{Rad}(S)$. Recall that $\operatorname{Soc}({}_SX) = \operatorname{ann}_X(J)$, where ann_X denotes the right annihilator in X. Choose $x \in X$ and $U \subseteq X$ with A_R isomorphic to xR/U. Assume at first $\operatorname{ann}_X(J) \cap xR \not\subseteq U$. Then there are R-isomorphisms $A \cong (\operatorname{ann}_X(J) \cap xR + U)/U \cong \operatorname{ann}_X(J) \cap xR / \operatorname{ann}_X(J) \cap xR \cap U$, and the assertion follows. Assume now $\operatorname{ann}_X(J) \cap xR \subseteq U$. As X_R is artinian, there is a finite subset $\{f_1, \ldots, f_k\}$ of J such that $\operatorname{ann}_X(J) \cap xR = \operatorname{ann}_X(f_1, \ldots, f_k) \cap xR$. Then the map $g\colon xR \to \Pi_{i=1}^k f_i xR$, $g(xr) = (f_1xr, \ldots, f_kxr)$ has kernel $\operatorname{ann}_X(J) \cap xR \subseteq U$. Therefore, the image of g has a composition factor isomorphic to A_R, hence one of the f_ixR and JX have a composition factor isomorphic to A_R.

The Loewy length of JX, considered as a left S-module, is one less than that of ${}_SX$. The Lemma follows by induction over the S-Loewy length of the bimodule in question. ∎

PROOF OF THE THEOREM: Let f be a nonzero element of J^{m-1}, m denoting the Loewy length of S. Let A_R be a simple submodule of $M/\operatorname{Ker}(f)$. As $M/\operatorname{Ker}(f)$ embeds in M_R, the module A_R, up to isomorphism, is a simple submodule of M_R.

For any bimodule ${}_SX_R$ and any subset T of S, again let $\operatorname{ann}_X(T)$ denote the right annihilator of T in X. Note that $\operatorname{ann}_X(T)$ is a $S-R$-bimodule. By the choice of f, the inclusion $\operatorname{ann}_M(J^{m-1}) \subseteq \operatorname{Ker}(f)$ holds, thus A_R is a composition factor of $M/\operatorname{ann}_M(J^{m-1})$. Consider now the ascending Loewy chain (=chain of iterated socles) of ${}_SM$, this is the chain $0 = \operatorname{ann}_M(J^0) \subset \operatorname{ann}_M(J) \subset \ldots \subset \operatorname{ann}_M(J^{m-1}) \subset \operatorname{ann}_M(J^m) = M$.

As A_R appears in the top factor module of this chain, we conclude that A_R is a composition factor of each $S-R$-bimodule $X_i = M/\operatorname{ann}_M(J^i)$, $0 \leqslant i \leqslant m-1$. By our Lemma, the module $\operatorname{ann}_{X_i}(J)$ has a composition factor isomorphic to A_R. Using the identity $\operatorname{ann}_{X_i}(J) = \operatorname{ann}_M(J^{i+1})/\operatorname{ann}_M(J^i)$ and looking again at the ascending Loewy chain of ${}_SM$, we see that A_R appears at least m times as a composition factor in M_R, and we conclude that $m \leqslant \ell_A \leqslant \sigma$. ∎

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