Proceedings of the Edinburgh Mathematical Society (1996) 39, 403–408

BOOK REVIEWS

BLOOM, W. R. and HEYER, H. *Harmonic analysis of probability measures on hypergroups* (de Gruyter Studies in Mathematics Vol. 20, de Gruyter, Berlin, New York 1995) vi+601pp., 3 11 012105 0, about £140.

One would think that any theory motivated by the group concept but in which one was from the outset expressly denied any possibility of taking inverses would be at best a feeble shadow of the group-theoretic case. On the contrary: that in such a setting the authors of this book have been able to expound at length a well-integrated and mature theory in which it is the achievements rather than the limitations of the theory that catch the eye is a tribute partly to the richness of the group concept and its associated machinery (group representations, harmonic analysis, measure algebras, etc.) and partly to the ingenuity of the many authors who have worked in the area.

The term hypergroup originates in algebraic work of F. Marty in 1935 and H. S. Wall in 1937. In its modern topological form the concept is due, almost simultaneously, to C. F. Dunkl in 1973, R. I. Jewett in 1975 and R. Spector in 1975; it is Jewett's axiom scheme that has proved most influential subsequently. Write K for a locally compact Hausdorff space acting as base space, ε_x for the Dirac probability measure at $x \in K$. Were K a group, one could pass from K to the convolution algebra $M^b(K)$ of bounded measures on K by $\varepsilon_x * \varepsilon_y = \varepsilon_{x.y}$. The principal hypergroup axiom is that one has a continuous map $(x, y) \to \varepsilon_x * \varepsilon_y$ to a measure $\varepsilon_x * \varepsilon_y$ of compact support; one also needs an involution $x \to x^-$ on K respecting this convolution operation $*$. One may then pass from the sparsely structured base space K to the more richly structured measure algebra $M^b(K)$ and work there. For instance, the group operation of (left) translation is replaced by the hypergroup operation of generalized (left) translation,

$$(T^x f)(y) := \int_K f(z)(\varepsilon_x * \varepsilon_y)(dz), \quad y \in K,$$

studied by Delsarte in 1938, Levitan in 1945 and Bochner in 1956.

The authors begin (Ch. 1) with hypergroups and their measure algebras. In particular, they consider the important case of double coset hypergroups $G//H$ with G a locally compact group and H a compact subgroup. They follow (Ch. 2) with the dual $K^\wedge$ of a commutative hypergroup K, the set of equivalence classes of irreducible representations of K. Chapter 3 deals extensively with the principal special classes of hypergroup: polynomial hypergroups in one and several variables (see below), one-dimensional hypergroups, Sturm–Liouville hypergroups.

The authors then turn to positive definite and negative definite functions (Ch. 4), obtaining hypergroup forms of Bochner's theorem, the Lévy continuity theorem and the Lévy–Khintchine formula. They consider convolution semigroups and the embedding therein of infinitely divisible measures in Ch. 5. Then the transience-recurrence dichotomy and the hypergroup form of the Chung–Fuchs criterion follow in Ch. 6, by potential-theoretic methods. Strong laws of large numbers and central limit theorems for random walks and Markov chains on hypergroups are given in Ch. 7 (see the remarks on randomized sums below), following work by Hm. Zeuner, M. Voit and others. Brief remarks on structure theory for hypergroups and stationary random fields for hypergroups follow in Ch. 8. There is a full bibliography, of some six hundred items. Full notes are given at the ends of chapters; I recommend that these be used sectionwise rather than chapterwise. There is a list of examples of hypergroups, classified into fourteen main categories.

To return to one of the main motivating categories, polynomial hypergroups: these are hypergroups on Z_+, arising when the convolution is of the form

$$\varepsilon_x * \varepsilon_y = \sum_{k \geq 0} c(m,n,k)\varepsilon_k,$$

with $c(m,n,k)$ non-negative and summing to 1, when the $c(m,n,k)$ are the linearization coefficients

$$Q_m Q_n = \sum_{k \geq 0} c(m,n,k) Q_k$$

of a system of orthogonal polynomials (Q_n). There are many examples; the positivity of the linearization coefficients of the Jacobi polynomials, for example, has been studied at length by R. Askey and G. Gasper in the 1970s. Further work here is due to R. Lasser and others.

Another classical instance of hypergroups arises in work of Kingman of 1963 on random walks with spherical symmetry. Here one forms "randomized sums" S_n, where the lengths of vectors S_{n-1} and X_n are random, the angle between them is also random, all three are independent, and the angle distribution is chosen to make the relevant convolution associative. The relevant Fourier transform (radialization of the Euclidean Fourier transform) involves a Bessel-function kernel; the convolution operation is given via the cosine rule for Euclidean triangles. There are analogues for random walk on spheres (cosine rule for spherical triangles; Gegenbauer polynomials: the reviewer in 1972) and hyperbolic spaces (Zeuner in 1986). At the time of my work I was aware of the spherical, Euclidean and hyperbolic cases being instances of the general theory of symmetric spaces (of compact, Euclidean and non-compact types) and Gelfand pairs [**4**], the characters (Bessel functions or Gegenbauer polynomials) being zonal spherical functions, and the Bessel and Gegenbauer addition theorems (the relevant formulae from special-function theory) being the relevant instances of Harish–Chandra's formula for zonal spherical functions. It is very satisfying now to see these old friends given a thorough and integrated treatment in §3.4, where they appear as the three main examples (Bingham, Bessel–Kingman and hyperbolic) of one-dimensional hypergroups. Like Molière's M. Jourdain, who had been speaking prose all his life without knowing it, I—like a good many others—had been using hypergroups without being able to put my finger on this subtle but ubiquitous concept.

The book is written rather in the style of the second author's monumental treatise on the group case [**5**]. The coverage is clear and thorough throughout. Perhaps one aspect not much emphasized here is the link mentioned above with Gelfand pairs; the survey by Heyer [**6**] would serve as a useful complement, or preliminary, to the book in this regard. (The authors reluctantly omit discrete Gelfand pairs, for background on which see e.g., [**3**], [**7**].) Another complement to the book is provided by the extensive literature on group representations and special functions; see e.g., [**8**], [**2**]. Note in particular the role of addition formulae such as the Bessel and Gegenbauer cases above [**9**, Ch. XI], [**1**, Ch. 4].

In sum: this book provides a definitive account of an extensive and important area. It is clearly destined to be the standard work in its field and we are indebted to Walter Bloom and Herbert Heyer for their labour of love in writing it.

N. H. BINGHAM

REFERENCES

1. R. Askey, *Orthogonal polynomials and special functions* (SIAM, 1975).

2. R. Askey *et al.*, ed., *Special functions: group-theoretical aspects and applications* (Reidel, 1984).

3. P. Diaconis, *Group representations in probability and statistics* (Lecture Notes—Monographs **11**, Inst. Math. Statistics, 1988).

4. S. Helgason, *Differential geometry and symmetric spaces* (Academic Press, 1962).

5. H. HEYER, *Probability measures on locally compact groups* (Springer-Verlag, 1977).

6. H. HEYER, Convolution semigroups of probability measures on Gelfand pairs, *Exposition. Math.* **1** (1983), 3–45.

7. I. G. MACDONALD, *Symmetric functions and Hall polynomials* (2nd edn) (Clarendon Press, 1995).

8. N. YA. VILENKIN, *Special functions and the theory of group representations* (Transl. Math. Monographs **22**, Amer. Math, Soc., 1968).

9. G. N. WATSON, *A treatise on Bessel functions* (2nd edn), (Cambridge Univ. Press, 1944).

RAMSAY, A. and RICHTMYER, R. D., *Introduction to hyperbolic geometry* (Universitext, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong 1995), xii+287 pp., soft-cover, 3 540 94339 0, £30.

This is indeed a very nice book on hyperbolic geometry. It clarifies the axiomatic and logical development of the subject, describes its traditional geometrical features and rounds off with a differential geometric setting.

After a useful introduction Chapter 1 deals with the axiomatic method (based on Hilbert's ideas), summarises the relevant properties of the real numbers and discusses categoricalness. The choice of parallel axiom, distinguishing Euclidean and hyperbolic geometry, is introduced. In Chapter 2 'neutral geometry' and the usual 'neutral theorems' are studied. A quite thorough discussion of the hyperbolic plane H^2 is given in Chapter 3, including asymptotic features, isometries, tilings and horocycles. Three-dimensional hyperbolic space H^3 is the subject of Chapter 4 and again an instructive section on isometries is included. In Chapter 5 the differential geometry of surfaces is introduced and the discussion includes metrics, parallel transport and geodesics, vectors and tensors and the relation between isometries and preservation of the metric (line element). Here H^2, as such a surface, is described and continued further in Chapter 6. In Chapter 7 the classical models of H^2 are clearly described and its isometries interpreted within them. The categoricalness of the axioms is established. Isometries are revisited in Chapter 8 and the link with fractional linear transformations and $SL(2,\mathbb{R})$ is shown for H^2 and compared to the isometry group of the Euclidean plane. The differential geometry of H^3 is studied in Chapter 9 where the idea of a manifold is introduced. Lorentz metrics are introduced in Chapter 10 and the corresponding Lorentz and Poincaré groups are discussed. Special relativity is briefly mentioned and the 'symmetries' of Maxwell's equations used to introduce Lorentz transformations. The realisation of H^2 in 3-dimensional Minkowski space is shown, as is the relation between the isometries of H^2 and the associated Lorentz transformations. A similar discussion of H^3 is presented. Chapter 11, the final chapter, is devoted to straightedge and compass constructibility in H^2.

This book represents a most commendable attempt to introduce hyperbolic geometry in a little under 300 pages. It clarifies the subject axiomatically, describes it differentially and exhibits it within Minkowski space. There is much discussion of isometries and comparisons with Euclidean space are usually at hand. The book is well laid out with no shortage of diagrams and with each chapter prefaced with its own useful introduction. The mathematical prerequisites are developed *ab initio*. Also well written, it makes pleasurable reading.

G. S. HALL

SEYDEL, R. *Practical bifurcation and stability analysis: from equilibrium to chaos* (2nd edition) (Interdisciplinary Applied Mathematics, Vol. 5, Springer-Verlag, Berlin–Heidelberg–New York–London–Paris–Tokyo–Hong Kong 1994), xv+407 pp., 3 540 94316 1, £34.50.

According to the Preface this book is a new version of a previous text, *From Equilibrium to*