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THE MAXIMAL SPECTRAL TYPE OF A RANK ONE TRANSFORMATION

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ABSTRACT. In this paper it is shown that the maximal spectral type of a general rank one transformation is given by a kind of generalized Riesz product, with possibly some discrete measure.

1. **Introduction.** The purpose of this note is to show that the maximal spectral type of a general rank one transformation is given by a kind of generalized Riesz product, with possibly some discrete measure. For weakly mixing rank one transformations our description of the maximal spectral type is exact. It is known that spectra of certain transformations can be written as Riesz products or variants of them (see B. Host, J.-F. Méla, and F. Parreau [5] and references therein), but it does not seem to be sufficiently well known that the maximal spectral type of a general rank one transformation can be written this way, although this fact is mentioned without proof in S. Ferenczi's thesis (Paris 1991). The necessary ingredients for the proof are contained in J. R. Baxter [1] and B. Host, J.-F. Méla, and F. Parreau [5]. One can view a rank one transformation as a suitable tower over a general adding machine and calculate the maximal spectral type as in [5], but it is easier to do the calculations directly with reference to the stacks obtained by cutting and stacking. We denote the transformation by T.

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2. **Main calculations.** We will assume that the reader is familiar with the method of cutting and stacking to construct a rank one transformation. A nice account is given in Friedman [3].

Divide the unit interval Ω_0 into m_1 equal parts, add spacers, and form a stack of a certain height (say h_1) in the usual fashion. This is the first stage of our construction. At the k-th stage, divide the stack obtained at the $(k-1)$-st stage into m_k equal parts, add spacers, and obtain a new stack (say of height h_k) in the usual fashion. If during the k-th stage of construction the number of spacers put above the j-th column of the $(k-1)$-st stack is $a_j^{(k)}$, $0 \le a_j^{(k)} < \infty$, $1 \le j \le m_k$, then we have

$$h_k = m_k h_{k-1} + \sum_{j=1}^{m_k} a_j^{(k)}.$$

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Proceeding thus we get a rank one transformation on a certain measure space $(X, \mathcal{B}, m)$ which may be finite or σ-finite depending on the number of spacers added. Since we are concerned with general rank one transformations, no control is necessary over the manner in which the spacers are added except that we add only finitely many spacers at each stage.

For each $k = 1, 2, 3, \ldots$, let Ω_k denote the base of the stack at the end of the k-th stage of construction. Note that $m(\Omega_k) = \frac{1}{m_1} \cdot \frac{1}{m_2} \cdots \frac{1}{m_k}$. For $A, B \subseteq X$, write $R_i(x, A, B)$ to denote the i-th entry time into B of the point $x \in A$, with the convention that $R_1(x, A, B) > 0$, even if $x \in A \cap B$. We note that $R_i(x, \Omega_k, \Omega_{k-1})$ is independent of $x \in \Omega_k$ for $i = 1, 2, \ldots, m_k - 1$. We therefore write $R_i(x, \Omega_k, \Omega_{k-1}) = R_{i,k}$, $i = 1, 2, \ldots, m_k - 1$. Note that

$$R_{i,k} = ih_{k-1} + a_1^{(k)} + \cdots + a_i^{(k)}, \quad 1 \leq i \leq m_k - 1.$$

We have

$$\Omega_{k-1} = \Omega_k \cup T^{R_{1,k}}\Omega_k \cup \cdots \cup T^{R_{m_k-1,k}}\Omega_k$$

(1) $$\mathbf{1}_{\Omega_{k-1}} = P_k(U)\mathbf{1}_{\Omega_k}$$

where

$$P_k(U) = I + U^{-R_{1,k}} + \cdots + U^{-R_{m_k-1,k}}$$

and U is the unitary operator on $L^2(X, \mathcal{B}, m)$ defined by $Uf = f \circ T$, $f \in L^2(X, \mathcal{B}, m)$. Note that in terms of stack heights and spacer lengths

$$P_k(U) = I + \sum_{j=1}^{m_k-1} U^{-(jh_{k-1}+a_1^{(k)}+\cdots+a_j^{(k)})}.$$

Iterating (1) we get

$$\mathbf{1}_{\Omega_0} = \Big(\prod_{j=1}^{k} P_j(U)\Big)\mathbf{1}_{\Omega_k}.$$

Let us write $f_k = \big(m(\Omega_k)\big)^{-1/2}\mathbf{1}_{\Omega_k}$, $k = 0, 1, 2, 3, \ldots$. Note that $m(\Omega_0) = 1$ so that $f_0 = \mathbf{1}_{\Omega_0}$. We have

$$f_0 = \Big[\big(m(\Omega_k)\big)^{1/2} \prod_{j=1}^{k} P_j(U)\Big] f_k.$$

If σ_k denotes the measure on S^1 defined by

$$(U^n f_k, f_k) = \int_{S^1} z^n \, d\sigma_k, \quad n \in \mathbb{Z}, \ k = 0, 1, 2, 3, \ldots$$

then we see that

$$d\sigma_0 = \Big(\Big|\prod_{j=1}^{k} P_j(z)\Big|^2 \cdot m(\Omega_k)\Big) \, d\sigma_k, \quad k = 1, 2, 3, \ldots$$

or

(2) $$d\sigma_0 = \Big(\prod_{j=1}^{k} \frac{1}{m_j} |P_j(z)|^2\Big) \, d\sigma_k, \quad k = 1, 2, 3, \ldots .$$

Since the coefficient of $d\sigma_k$ can vanish at only finitely many points (being a trigonometric polynomial) we see that the non-atomic parts of σ_0 and σ_k are mutually absolutely continuous. Moreover $\bigvee_{k=1}^{\infty} \sigma_k$ is the maximal spectral type of U, whence σ_0 and the maximal spectral type σ of U have their non-atomic parts mutually absolutely continuous, *i.e.* in the same measure class.

Let us replace $d\sigma_k$ by dz in the right hand side of (2) above and let ρ_k denote the resulting measure:

$$d\rho_k = \Big(\prod_{j=1}^{k} \frac{1}{m_j}|P_j(z)|^2\Big)\, dz, \quad k = 1, 2, 3, \ldots .$$

We have:

THEOREM. *σ_0 is the weak limit of the ρ_k. In other words, for each $n \in \mathbb{Z}$, we have $\hat{\rho}_k(n) \longrightarrow \hat{\sigma}_0(n)$ as $k \longrightarrow \infty$.*

PROOF. Let N_k denote the set of integers consisting of zero together with the entry times $R_1(x, \Omega_k, \Omega_0)$, $R_2(x, \Omega_k, \Omega_0) \cdots$ which are less than the height h_k of the k-th stack. Note that under this constraint $R_i(x, \Omega_k, \Omega_0)$ is independent of $x \in \Omega_k$. We further have

$$\Omega_0 = \bigcup_{s \in N_k} T^s \Omega_k, \quad \text{a disjoint union,}$$

$$\mathbf{1}_{\Omega_0} = \sum_{s \in N_k} U^{-s} \mathbf{1}_{\Omega_k},$$

$$f_0 = \Big(\sum_{s \in N_k} U^{-s} f_k\Big)\Big(m(\Omega_k)\Big)^{1/2} = Q_k(U) f_k$$

where

$$Q_k(U) = \Big(\sum_{s \in N_k} U^{-s}\Big)\Big(m(\Omega_k)\Big)^{1/2}.$$

Clearly $Q_k(z) = \big(\prod_{j=1}^{k} P_j(z)\big)\big(m(\Omega_k)\big)^{1/2}$ and $|Q_k(z)|^2 = \frac{d\rho_k}{dz}$.

Now fix $n \in \mathbb{Z}$ and let k be so large that the first return time of any $x \in \Omega_k$ back to Ω_k (under T or T^{-1}) is bigger than $|n|$; equivalently, let k be so large that $h_k > |n|$. If $r, s \in N_k$ then $s + n - r$ can never exceed or equal the second return time of any $x \in \Omega_k$ back to Ω_k (under T or T^{-1}). Moreover there are at most n^2 pairs (r, s) with $(r, s) \in N_k$ such that $s + n - r$ equals the first return time of an $x \in \Omega_k$ back to Ω_k. For suppose $n > 0$ and $T^{s+n-r}\Omega_k \cap \Omega_k \neq \emptyset$ and $s + n - r \neq 0$. Then $r = n + s - u$ where u is the first return time of some $x \in \Omega_k$ back to Ω_k. Since each n, r, s is less than h_k, $h_k \leq u$ and $r \geq 0$, we see that $0 \leq r < n$ and $s + n - r = u \geq h_k$, so $s \geq h_k - (n - r)$. Thus there can be at most n^2 pairs with $T^{s+n-r}\Omega_k \cap \Omega_k \neq \emptyset$ and $s + n - r \neq 0$. (But note that for each fixed u, there are at most n pairs with this property.) A similar argument holds for $n < 0$. So, if $T^{s+n-r}\Omega_k \cap \Omega_k \neq \emptyset$, then we must have $s + n - r = 0$ except for at most n^2 pairs (r, s),

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$r, s \in N_k$. Now

$$\begin{aligned}(U^n f_0, f_0) &= \sum_{r,s\in N_k} (U^{n-r} f_k, U^{-s} f_k)\, m(\Omega_k)\\ &= \sum_{r,s\in N_k} (U^{s+n-r}\mathbf{1}_{\Omega_k}, \mathbf{1}_{\Omega_k})\\ &= \sum_{r,s\in A} (U^{s+n-r}\mathbf{1}_{\Omega_k}, \mathbf{1}_{\Omega_k}) + \sum_{r,s\in B} (U^{s+n-r}\mathbf{1}_{\Omega_k}, \mathbf{1}_{\Omega_k})\end{aligned}$$

where A is the set of pairs $(r, s) \in N_k$ with $s+n-r = 0$ and B is the set of pairs $(r, s) \in N_k$ with $s + n - r$ equal to the first return time of an $x \in \Omega_k$ back to Ω_k (under T or T^{-1}). Now

$$\sum_{r,s\in B} (U^{s+n-r}\mathbf{1}_{\Omega_k}, \mathbf{1}_{\Omega_k}) \leq n^2 m(\Omega_k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,$$

whence

$$(U^n f_0, f_0) = \lim_{k\to\infty} L_k . m(\Omega_k)$$

where L_k is the number of pairs (r, s) with $s + n - r = 0$, $r, s \in N_k$. (By breaking up Ω_k into the disjoint sets corresponding to each first return time u we could even replace the term $n^2 m(\Omega_k)$ by $|n| m(\Omega_k)$.) On the other hand, it is easy to see that

$$\begin{aligned}\int_{S^1} z^n d\rho_k &= \int_{S^1} z^n m(\Omega_k) \Big| \sum_{s\in N_k} z^{-s} \Big|^2 dz\\ &= m(\Omega_k). L_k,\end{aligned}$$

so that $\hat{\rho}_k(n) \longrightarrow \hat{\sigma}_0(n)$ as $k \longrightarrow \infty$, for each $n \in \mathbb{Z}$, and so σ_0 is the weak limit of ρ_k.

3. **Examples.** (a) Consider Chacon's transformation (see Friedman [3]), where at each stage we divide the stack into three equal parts and place a single spacer on the top of the middle column. We have

$$\begin{aligned}h_1 &= 3 + 1\\ h_2 &= 3(3 + 1) + 1\\ &\vdots\\ h_n &= 3^n + 3^{n-1} + \cdots + 1 = \frac{3^{n+1} - 1}{2}.\end{aligned}$$

The ρ_k's take the form

$$d\rho_k = \left(\frac{1}{3^k} \prod_{j=1}^{k} |(1 + z^{-h_j} + z^{-(2h_j+1)})|^2 \right) dz$$

or

$$d\rho_k = \left(\prod_{j=1}^{k} \left(1 + \frac{2}{3} \operatorname{Re}(z^{-h_j} + z^{-(h_j+1)} + z^{-(2h_j+1)}) \right) \right) dz.$$

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Chacon showed that this transformation is weakly but not strongly mixing (see [3]). Further, it has singular spectrum; this was first shown by Baxter, ([1], Section 2.3). Ivo Klemes has shown us how to use the methods of this paper to prove the singularity of the spectrum. One shows that $z^{h_k}\,d\sigma_0$ tends weakly to $\frac{1}{2}(1+z)\,d\sigma_0$, which implies that $(1+z)\,d\sigma_0$ and hence $d\sigma_0$ itself are both singular.

(b) The Staircase transformation. Here at the k-th stage, we divide the $(k-1)$-st stack into k equal columns and put j spacers above the j-th column, $1 \le j \le k$, (hence the name 'staircase') and then stack. Note that at the first stage we do not divide Ω_0 at all, but only add a spacer equal to the length of Ω_0. We have

$$h_1 = 2$$
$$h_2 = 2 \times 2 + 1 + 2 = 7$$
$$\vdots$$
$$h_k = kh_{k-1} + \frac{k(k+1)}{2}$$
$$d\rho_k = \Big(\prod_{j=1}^{k} |P_j(z)|^2 \cdot m(\Omega_k)\Big)\,dz$$

where

$$P_j(z) = 1 + z^{-(h_{j-1}+1)} + z^{-(2h_{j-1}+1+2)} + \cdots + z^{-[(j-1)h_{j-1}+\frac{(j-1)j}{2}]}.$$

The staircase transformation is known to be weakly mixing, since it mixes along the sequence $\{h_k\}$ (see Friedman [4]). However, it is an open problem whether it is mixing, and it is not known if its spectrum has absolutely continuous component (with respect to Lebesgue measure). It seems plausible that for any analytic trigonometric polynomial $p(z)$, $\|p(U)\mathbf{1}_{\Omega_1} - \mathbf{1}_{\Omega_2}\| \ge \alpha\|\mathbf{1}_{\Omega_1}\|$ for some $\alpha > 0$. If this can be established, then clearly the staircase transformation has a Lebesgue component in its spectrum. It is to be noted that with $p(z) = z^{-n}$, $n \le 10$, $\|p(U)\mathbf{1}_{\Omega_1} - \mathbf{1}_{\Omega_2}\| \ge \frac{1}{2}\|\mathbf{1}_{\Omega_1}\|$. On the other hand, the considerations of Remark 6 in the next section suggest that the spectrum of the staircase transformation may well be singular.

In case T is known to be weakly mixing (as in the case of Chacon's transformation) or if P_k's have no zeros on S^1 or if $m(X) = \infty$, then the weak limit of the ρ_k's is the maximal spectral type.

4. **Remarks.** 1. In case $m(X)$ is finite, σ_0 has non-trivial mass at $z = 1$, so that $\sum_{n\in\mathbb{Z}} |\hat{\sigma}_0(n)|^2 = \infty$. It is interesting to note that this fact, *viz.* $\sum_{n\in\mathbb{Z}} |\hat{\sigma}_0(n)|^2 = \infty$, always holds, whether $m(X)$ is finite or not. Indeed, since the coefficients of powers of z in the formal expansion (without grouping terms) of the infinite product $\prod_{k=1}^{\infty} \frac{1}{m_k}|P_k(z)|^2$ are all positive and since $\hat{\sigma}_0(n) =$ sum of the coefficients of z^n in this formal expansion, we see that $\sum_{n=-\infty}^{\infty} |\hat{\sigma}(n)|^2 \ge$ sum of the squares of the coefficients of powers of z in the formal expansion of the infinite product. This second sum of squares in turn is bigger than $\sum_{k=1}^{\infty} \frac{m_k(m_k-1)}{m_k^2}$, a sum which is ∞. Moreover if $(m_k)_{k=1}^{\infty}$ is bounded over a subsequence,

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then $\left(\hat{\sigma}_0(n)\right)_{n=1}^{\infty}$ is bounded away from zero over a suitable subsequence; hence in such a case σ_0 can not be absolutely continuous with respect to Lebesgue measure.

2. If $\left(\hat{\sigma}_0(n)\right)_{n=1}^{\infty}$ decreases to zero as $n \longrightarrow \infty$ (this is possible only in case $m(X) = \infty$), then it follows from Zygmund ([9], Chapter V, p. 184, Theorem 1.8) that the cosine series $1 + 2\sum_{n=1}^{\infty} \hat{\sigma}(n) \cos nx$ converges uniformly outside every neighbourhood of 0 to a continuous function f which has improper Riemann integral on $[0, 2\pi]$. Further, the series $1+2\sum_{n=1}^{\infty} \hat{\sigma}(n) \cos nx$ is the Fourier-Riemann series of f. However, the series also converges to the limit of the Cesàro sums of the series. The limit of the Cesàro sums is the Radon-Nikodym derivative of σ_0 with respect to the Lebesgue measure, whence $f \geq 0$, and hence summable in view of being (improperly) Riemann integrable. Thus σ_0 is absolutely continuous whenever $\left(\hat{\sigma}_0(n)\right)_{n=1}^{\infty}$ decreases to zero as $n \longrightarrow \infty$. In case $m(X) < \infty$ and $\hat{\sigma}_0(n)$ decreases to $1/m(X)$ as $n \longrightarrow \infty$, then the spectrum of U_T on $L_0^2(X, \mathcal{B}, m) = \{g : \int_X g\, dm = 0, g \in L^2(X, \mathcal{B}, m)\}$ is absolutely continuous with respect to Lebesgue measure.

J. P. Thouvenot conjectures that there exists a measure preserving T such that U_T has simple absolutely continuous spectrum which is not Lebesgue on $L_0^2(X, \mathcal{B}, m)$. The present remark makes this seem plausible since a cosine series $1+2\sum_{k=1}^{\infty} a_k \cos kx$ with a_k decreasing may vanish on a closed nowhere dense set of positive measure and at the same time be a generalized Riesz product of the kind appearing in this note. On the contrary, there are also good reasons for believing that all rank one transformations have singular spectrum; see Remarks 5 and 6 below!

3. The infinite product $\prod_{\ell=1}^{\infty}\left(\frac{1}{m_{j_\ell}}|P_{j_\ell}|^2\right)$ taken over a subsequence $j_1 < j_2 < j_3 \cdots$ also represents the maximal spectral type (up to discrete measure) of some rank one transformation. If we represent this maximal spectral type by σ_1, then it is easy to see that for each $n, \hat{\sigma}_0(n) \geq \hat{\sigma}_1(n)$. Although we can choose $j_1 < j_2 < j_3 \cdots$ in such a way that the formal expansion of the product $\prod_{\ell=1}^{\infty} P_{j_\ell}(z)$ is sufficiently lacunary, it is not clear that one can do this for the product $\prod_{\ell=1}^{\infty} \frac{1}{m_{j_\ell}}|P_{j_\ell}|^2$. If one could do so then one could choose the subsequence so that σ_1 was singular to Lebesgue measure. Ivo Klemes has shown us that the methods used in Bourgain's preprint (see Remark 6) would then imply that σ_0 was itself singular to Lebesgue measure.

4. One can consider non-singular T obtained by cutting and stacking. This means that at the k-th stage we divide the stack obtained at the $(k-1)$-st stage in the ratios

$$p_{0,k}, p_{1,k}, \ldots, p_{m_k-1,k},\ p_{ik} > 0, \quad \sum_{i=0}^{m_k-1} p_{ik} = 1.$$

The spacers are added in the usual manner, by which we mean that the sizes of the spacers added on top of the j-th column are all the same and equal to that of the top piece of the j-th column. The extension of T to the spacers is done linearly as usual. Note that at the k-th stage the resultant measure m is defined only for the algebra generated by the levels of the k-th stack. The resulting T after all the stages of construction are complete is a non-singular ergodic transformation for which $\frac{dm_T}{dm}$ is constant on all but the top layer of every stack. On $L^2(X, \mathcal{B}, m)$ we can now define

$$(Vf)(x) = A(x) \cdot \left(\frac{dm_T}{dm}\right)^{1/2}(x) f(Tx), \quad f \in L^2(X, \mathcal{B}, m)$$

where A is a function of absolute value one which is constant on all but the top layer of every stack and $m_T = m \circ T$. It can be shown that the maximal spectral type of V (up to a discrete measure) is given by the weak limit of the measures ρ_k defined as follows:

$$d\rho_k = \left(\prod_{j=1}^{k} |P_j(z)|^2\right) \cdot m(\Omega_k) \cdot dz$$

where

$$P_j(z) = 1 + a_{1,j}\left(\frac{p_{1j}}{p_{0j}}\right)^{1/2} z^{-R_{1,j}} + \cdots + a_{m_j-1,j}\left(\frac{p_{m_j-1,j}}{p_{0j}}\right)^{1/2} z^{-R_{m_j-1,j}}$$

$m(\Omega_k) = p_{01} \cdot p_{02} \cdots p_{0k}$ and where $a_{1,j}, a_{2,j}, \ldots, a_{m_j-1,j}$ are constants of absolute value one determined by A. Suppose $A = 1$ so that all a_{ij}'s are equal to 1. Then it can be seen after a calculation that $\sum_{n\in\mathbb{Z}} |\hat{\sigma}_0(n)|^2 \geq \sum_{j=1}^{\infty} p_{0j}(1 - p_{0j})$, where $\hat{\sigma}_0(n) = (V^n \mathbf{1}_{\Omega_0}, \mathbf{1}_{\Omega_0})$, $n \in \mathbb{Z}$. If $\sum_{n\in\mathbb{Z}} |\hat{\sigma}_0(n)|^2$ is finite, then $\sum_{j=1}^{\infty} p_{0j}(1 - p_{0j})$ is finite so that by ergodicity of T the measure on Ω_0 is discrete, whence the measure $(X, \mathcal{B}, m)$ is discrete.

5. For the classical Riesz product $\prod_{j=1}^{\infty}\left(1 + 2c_k(z^{n_k} + z^{-n_k})\right)$, $|c_k| < 1/2$, $n_{k+1} > 3n_k$, it is known that if $\sum_{k=1}^{\infty} |c_k|^2 = \infty$, then the product represents a singular measure, (Zygmund [9]). Could a similar result be true for the general Riesz product appearing in this note where no lacunarity condition is satisfied, but the convergence of the Riesz product is known from the considerations of the theorem of Section 2?

6. Classical Riesz products and some variants of them satisfy quasi-invariance and purity laws. (See Brown [2], Kilmer and Saeki [6], Parreau [7], J. Peyrière [8].) Similar results for the type of products appearing in this note may help resolve spectral questions associated with rank one transformations, such as whether a rank one transformation always has spectrum singular to Lebesgue measure. In this connection we mention that in a recent preprint entitled *On the spectral type of Ornstein's class one transformations*, J. Bourgain has suggested that the following conjecture on trigonometric polynomials may imply the singularity of the spectrum of any rank one transformation. For $n > 1$, let $\beta_n = \sup_{k_1<k_2<\cdots<k_n} \frac{1}{\sqrt{n}} \| \sum_{j=1}^{n} e^{2\pi i k_j \theta} \|_{L^1}$; then it is conjectured that $\sup_n \beta_n < 1$. Ivo Klemes has suggested that arguments similar to those used in Bourgain's preprint may show that the spectrum of the staircase transformation is singular. Klemes has also pointed out that if, in the notation of our Section 2 above, the trigonometric polynomials $Q_k(z)$ tend to zero in (Lebesgue) measure, then the spectrum of the corresponding transformation T is singular. Of course, it is still an open question whether a measure preserving transformation can have simple Lebesgue spectrum, or even whether a transformation with simple spectrum can have a Lebesgue component.

7. We have recently shown that z is an eigenvalue of U_T if and only if the infinite product $\prod_{j=1}^{\infty} \frac{1}{m_j^2} |P_j(z)|^2$ converges. This will appear in another paper.

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