Canad. Math. Bull. Vol. **67** (2), 2024, pp. 289–303
http://dx.doi.org/10.4153/S0008439523000681


# Convexity of the radial sum of a star body and a ball

Shigehiro Sakata

*Abstract.* We investigate the convexity of the radial sum of two convex bodies containing the origin. Generally, the radial sum of two convex bodies containing the origin is not convex. We show that the radial sum of a star body (with respect to the origin) and any large centered ball is convex, which produces a pair of convex bodies containing the origin whose radial sum is convex.

We also investigate the convexity of the intersection body of a convex body containing the origin. Generally, the intersection body of a convex body containing the origin is not convex. Busemann's theorem states that the intersection body of any centered convex body is convex. We are interested in how to construct convex intersection bodies from convex bodies without any symmetry (especially, central symmetry). We show that the intersection body of the radial sum of a star body (with respect to the origin) and any large centered ball is convex, which produces a convex body with no symmetries whose intersection body is convex.

## 1 Introduction

The setting of this paper is in the Euclidean $n$-space $\mathbb{R}^n$ with $n \geq 2$. For two vectors $x_1$ and $x_2$ in $\mathbb{R}^n$, their *radial sum* is defined by

$$x_1 \widetilde{+} x_2 = \begin{cases} x_1 + x_2, & \exists\, (s_1, s_2) \in \mathbb{R}^2 \setminus \{(0,0)\} : s_1 x_1 + s_2 x_2 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Throughout this paper, we understand that every star-shaped subset of $\mathbb{R}^n$ is star-shaped with respect to the origin. For two star-shaped subsets $A_1$ and $A_2$ of $\mathbb{R}^n$, their *radial sum* is defined by

$$A_1 \widetilde{+} A_2 = \left\{ x_1 \widetilde{+} x_2 \,\middle|\, x_1 \in A_1,\ x_2 \in A_2 \right\}.$$

A subset $A$ of $\mathbb{R}^n$ is *centered* if $-x \in A$ for any $x \in A$. A subset $A$ of $\mathbb{R}^n$ is *centrally symmetric* if there exists a point $y \in \mathbb{R}^n$ such that $A - y = \{x - y \,|\, x \in A\}$ is centered. We denote by $rB^n$ the centered ball of radius $r$. Let $S^{n-1}$ be the boundary of $B^n = 1B^n$. For a star-shaped subset $A$ of $\mathbb{R}^n$, the *radial function* of $A$ is defined by

$$\rho_A(u) = \max\{\lambda \in [0, +\infty) \,|\, \lambda u \in A\},\ u \in S^{n-1}.$$

Received by the editors February 10, 2023; revised June 14, 2023; accepted August 30, 2023.
Published online on Cambridge Core September 4, 2023.
The author is partially supported by JSPS KAKENHI (Grant No. 17K14191) and JSPS Overseas Research Fellowships (Grant No. 201860263).
AMS subject classification: **52A39**, **52A40**, 41A58.
Keywords: Convexity of the radial sum of convex bodies, Busemann's theorem, convexity of intersection bodies.

A *body* in $\mathbb{R}^n$ is the closure of a bounded open subset of $\mathbb{R}^n$. A *star body* in $\mathbb{R}^n$ is a star-shaped body of $\mathbb{R}^n$ whose radial function is continuous.

It follows from the definition that the radial sum of two star-shaped subsets is star-shaped. For two star bodies $A_1$ and $A_2$ in $\mathbb{R}^n$, the relation $\rho_{A_1 \tilde{+} A_2} = \rho_{A_1} + \rho_{A_2}$ yields that $A_1 \tilde{+} A_2$ is a star body. However, the convex versions of these properties do not always hold. Namely, there exist two convex subsets/bodies containing the origin whose radial sum is not convex. Such two convex subsets are $\{(t, 0, 0, \dots, 0) \mid t \in [-1, 1]\}$ and $\{(0, t, 0, \dots, 0) \mid t \in [-1, 1]\}$. Such two convex bodies are $\{(t, 0, \dots, 0) + x \mid t \in [-2\sqrt{2}, 2\sqrt{2}],\ x \in B^n\}$ and $rB^n$ for any $r \in (0, -1 + \sqrt{6}/2)$ (see Example 2.2 for details).

In contrast to these examples, as a main investigation of this paper, we show the following two results:

(1) Let $\gamma \in [0, +\infty)$, and let $K$ be a convex body containing the origin whose radial function is of class $C^2$. If

$$(\rho_K(u_1) + \rho_K(u_2))\, \rho_K\left(\frac{u_1 + u_2}{|u_1 + u_2|}\right) - |u_1 + u_2|\, \rho_K(u_1)\, \rho_K(u_2) \geq \gamma \angle (u_1, u_2)^2$$

for any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$ with $u_1 + u_2 \neq 0$, then, for any "small enough" $f \in C^2(S^{n-1})$, the star body defined by $\rho_K + f$ is convex (see Proposition 3.1 for details).

(2) Let $A$ be a star body whose radial function is of class $C^2$. There exists a "large enough" $R \in (0, +\infty)$ such that, for any $r \in (0, +\infty)$, if $r \geq R$, then $A \tilde{+} rB^n$ is convex (see Theorem 3.3 and its corollaries for details).

Here, in the first assertion, if $\gamma = 0$, then the inequality for $\rho_K$ means that $K$ is convex (see Lemma 2.1). If $\gamma > 0$, then the inequality means that $K$ has a "stronger" convexity associated with $\gamma$, which is precisely explained right after Lemma 2.1. Any centered ball has this property. Thus, the first assertion yields that the radial sum of a "small perturbation" of any centered ball and a centered ball is convex.

The technical key point of the proofs is to approximate $C^2$-functions of two real variables by polynomials. As we know, Taylor's theorem yields that any $C^2$-function $\Phi$ of two real variables $\theta_1$ and $\theta_2$ has an approximation of the form

$$\Phi(\theta_1, \theta_2) \approx c_0 + c_1\theta_1 + c_2\theta_2 + c_{1,1}\theta_1^2 + c_{1,2}\theta_1\theta_2 + c_{2,2}\theta_2^2.$$

We improve this approximation for certain suitable $\Phi$ and obtain an approximation of the form

$$\Phi(\theta_1, \theta_2) \approx c_0 + c_1(\theta_1 - \theta_2) + c_2(\theta_1 - \theta_2)^2.$$

Using this approximation, as another main investigation of this paper, we give convex bodies with no symmetries whose intersection bodies are convex. We denote by $V_k$ the $k$-dimensional Lebesgue measure. For a unit vector $u$, we denote by $u^\perp$ the orthogonal complement of $u$. For a star body $A$ in $\mathbb{R}^n$, its *intersection body* is denoted by $IA$ and is the star body defined by

$$\rho_{IA}(u) = V_{n-1}(A \cap u^\perp),\ u \in S^{n-1}.$$

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

A star body $A$ is called an intersection body if there is a star body such that its intersection body is $A$. The notion of intersection body was introduced in [10] and played an important role in the solution of the Busemann–Petty problem.

In [3], Busemann and Petty posed the following question:

> For any centered convex bodies $K$ and $L$ in $\mathbb{R}^n$, if $V_{n-1}(K \cap u^\perp) \leq V_{n-1}(L \cap u^\perp)$ for any $u \in S^{n-1}$, then is $V_n(K) \leq V_n(L)$ true?

This question is now referred to as the Busemann–Petty problem and appears often in the literature about geometric tomography (see, for example, [6]), in which information about a given geometric object is obtained from data concerning its sections and/or projections.

In the history of the Busemann–Petty problem, Lutwak's theorem [10, Theorem 10.1] is important as the first step toward the full solution. It states that, for any intersection body $K$ and any star body $L$, if $V_{n-1}(K \cap u^\perp) \leq V_{n-1}(L \cap u^\perp)$ for any $u \in S^{n-1}$, then $V_n(K) \leq V_n(L)$. In particular, this theorem yields that if $K$ is an intersection body, then the answer to the Busemann–Petty problem is affirmative. It was shown in [4, Theorem 3.1] with a regularity assumption and in [13, Theorem 2.22] without any regularity assumption that the Busemann–Petty problem has a positive answer in $\mathbb{R}^n$ if and only if each centered convex body in $\mathbb{R}^n$ is an intersection body. This equivalence gives negative answers to the Busemann–Petty problem: Gardner [4, Theorem 6.1] showed that if $n \geq 5$, then a cylinder in $\mathbb{R}^n$ is not an intersection body. Of course, this equivalence gives also positive answers: Gardner [5, Corollary 5.3] showed that every centered convex body in $\mathbb{R}^3$ is an intersection body; Zhang [14, Theorem 3] showed that every centered convex body in $\mathbb{R}^4$ is an intersection body (see also [7] for an analytic approach). We refer to [6, Chapter 8] for more information on the Busemann–Petty problem (see also [8, Section 17] and [12, Section 15]).

From the point of view of the Busemann–Petty problem, it is important to obtain a convex intersection body. Here, the term "a convex intersection body" means "an intersection body which is convex" and is not used in the sense of [11]. By definition, the intersection body of any star body is a centered star body. However, there exists a convex body containing the origin whose intersection body is not convex. Let us review some results on the convexity of intersection bodies. Gardner produced nonconvex intersection bodies in his textbook [6, Section 8.1]. Precisely, [6, Theorem 8.1.8] states that, for any convex body $K$, there exists a translate of $K$ such that it contains the origin in its interior and its intersection body is not convex. Also, [6, Theorem 8.1.9] remarks the following two examples:

(1) Let $K$ be an equilateral triangle whose centroid is at the origin. For any $y \in \mathbb{R}^2$, if the interior of $K + y$ contains the origin, then $I(K + y)$ is not convex.
(2) Let $K$ be a square whose centroid is at the origin. For any $y \in \mathbb{R}^2 \setminus \{0\}$, if the interior of $K + y$ contains the origin, then $I(K + y)$ is not convex.

In contrast to Gardner's indication, Busemann's theorem [2] yields that the intersection body of any centered convex body is convex (see also [6, Theorem 8.1.10 and Corollary 8.1.11]). As a generalization of Busemann's theorem, it was shown in [9, Theorem 3] that, for any $p \in (0,1]$, the intersection body of any centered $p$-convex body is $1/[(1/p - 1)(n - 1) + 1]$-convex. Here, we recall that a subset $K$ of

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

$\mathbb{R}^n$ is *p-convex* if, for any $(x_0, x_1, \lambda) \in K \times K \times [0,1]$, $(1-\lambda)^{1/p} x_0 + \lambda^{1/p} x_1 \in K$. We emphasize that central symmetry essentially works for Busemann's theorem and its generalization. In [1], the *local* convexity of intersection bodies of symmetric convex bodies of revolution was investigated. Namely, it is still open to concretely give convex bodies with *no symmetries* whose intersection bodies are *globally convex*.

We produce convex bodies with no symmetries whose intersection bodies are convex:

(1) Let $\gamma \in [0, +\infty)$, and let $A$ be a star body whose radial function is of class $C^2$. If

$$(\rho_{IA}(u_1) + \rho_{IA}(u_2))\,\rho_{IA}\left(\frac{u_1+u_2}{|u_1+u_2|}\right) - |u_1+u_2|\,\rho_{IA}(u_1)\,\rho_{IA}(u_2) \geq \gamma \angle (u_1, u_2)^2$$

for any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$ with $u_1 + u_2 \neq 0$, then, for any "small enough" $f \in C^2(S^{n-1})$, the intersection body of the star body defined by $\rho_A + f$ is convex (see Proposition 3.7 for details).

(2) Let $A$ be a star body whose radial function is of class $C^2$. There exists a "large enough" $\widetilde{R} \in (0, +\infty)$ such that, for any $r \in (0, +\infty)$, if $r \geq \widetilde{R}$, then the intersection body of $A \widetilde{+} rB^n$ is convex (see Theorem 3.9 and its corollaries for details).

Here, in the first assertion, any centered ball has the property since the intersection body of any centered ball is a centered ball. Thus, the first assertion yields that the intersection body of a "small perturbation" of any centered ball is convex.

## 2 Preliminaries

### 2.1 Notation and terminology

Let us prepare necessary notation and terminology in addition to those stated in the Introduction.

We denote by $\mathcal{S}^n$ the set of all star bodies (star-shaped bodies with respect to the origin whose radial functions are continuous) in $\mathbb{R}^n$. We denote by $\mathcal{K}^n$ the set of all convex bodies in $\mathbb{R}^n$, and let $\mathcal{K}^n_0 = \{K \in \mathcal{K}^n \mid 0 \in K\}$.

Let $\kappa_k = V_k(B^k)$. The symbol $\sigma_k$ denotes the $k$-dimensional spherical Lebesgue measure. We denote by $C^m(M)$ the set of all $C^m$-functions defined on a manifold $M$. Let $e_i$ be the $i$th unit vector of $\mathbb{R}^n$. Let $SO(n)$ denote the special orthogonal group of degree $n$.

For a star-shaped subset $A$ of $\mathbb{R}^n$, the *extended radial function* of $A$ is denoted by the same symbol as the usual radial function and is defined by

$$\rho_A(x) = \begin{cases} |x|\rho_A\left(\dfrac{x}{|x|}\right), & x \in \mathbb{R}^n \setminus \{0\}, \\ \infty, & x = 0. \end{cases}$$

For a star-shaped subset $A$ of $\mathbb{R}^n$ and a function $f$ defined on $S^{n-1}$ such that $\rho_A + f \geq 0$ on $S^{n-1}$, let $A_f$ be the star-shaped subset whose radial function is $\rho_A + f$. In particular, for a star-shaped subset $A$ of $\mathbb{R}^n$ and a nonnegative constant $r$, $A_r = A \widetilde{+} rB^n$.

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

For a function $f$ defined on $S^{n-1}$ and a nonnegative integer $i$, let $f^i$ be the function $S^{n-1} \ni u \mapsto f(u)^i$. For two functions $f$ and $g$ defined on $S^{n-1}$, let $f+g$ and $fg$ be the functions $S^{n-1} \ni u \mapsto f(u)+g(u)$ and $S^{n-1} \ni u \mapsto f(u)g(u)$, respectively. For a continuous function $f$ defined on $S^{n-1}$, we denote by $\mathcal{R}[f]$ the *spherical Radon transform* of $f$, that is,

$$\mathcal{R}[f](u) = \int_{S^{n-1}\cap u^\perp} f(v)\, \mathrm{d}\sigma_{n-2}(v),\ u \in S^{n-1}.$$

For two continuous functions $f$ and $g$ defined on $S^{n-1}$, put $\langle f, g\rangle = \mathcal{R}[fg]$.

For two unit vectors $u_1$ and $u_2$, we use the following notation:

$$\angle\,(u_1, u_2) = \arccos u_1 \cdot u_2,\ u_3 = \begin{cases} \dfrac{u_1+u_2}{|u_1+u_2|}, & u_1+u_2 \neq 0, \\ 0, & u_1+u_2 = 0. \end{cases}$$

For two functions $f$ and $g$ defined on $S^{n-1}$, we define $\Delta[f,g] : S^{n-1}\times S^{n-1} \to (-\infty, \infty]$ as:

(i) If $u_1+u_2 \neq 0$, then

$$\begin{aligned}\Delta[f,g]\,(u_1,u_2) = (f\,(u_1)+f\,(u_2))\,g\,(u_3) + (g\,(u_1)+g\,(u_2))\,f\,(u_3) \\ -|u_1+u_2|\,(f\,(u_1)\,g\,(u_2)+f\,(u_2)\,g\,(u_1)).\end{aligned}$$

(ii) If $u_1+u_2 = 0$, then $\Delta[f,g](u_1,u_2) = \infty$.

For a function $f$ defined on $S^{n-1}$, put

$$\Gamma[f]\,(u_1,u_2) = \frac{\Delta[f,f]\,(u_1,u_2)}{2},\ (u_1,u_2) \in S^{n-1}\times S^{n-1}.$$

We denote by $\|\cdot\|_\infty$ the sup-norm for bounded functions. For a function $\varphi$ of $k$ real variables $\theta_1,\dots,\theta_k$ with bounded partial derivatives, we write

$$\varphi' = \left(\frac{\partial\varphi}{\partial\theta_1},\dots,\frac{\partial\varphi}{\partial\theta_k}\right),\ \|\varphi'\|_\infty = \max\left\{\left\|\frac{\partial\varphi}{\partial\theta_i}\right\|_\infty \,\middle|\, i \in \{1,\dots,k\}\right\},$$

$$\varphi'' = \left(\frac{\partial^2\varphi}{\partial\theta_i\partial\theta_j}\right),\ \|\varphi''\|_\infty = \max\left\{\left\|\frac{\partial^2\varphi}{\partial\theta_i\partial\theta_j}\right\|_\infty \,\middle|\, i,j \in \{1,\dots,k\}\right\}.$$

Put

$$\phi\,(\theta_1,\dots,\theta_{n-1}) = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1\cos\theta_2 \\ \vdots \\ \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\ \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1} \end{pmatrix},\ (\theta_1,\dots,\theta_{n-1}) \in \mathbb{R}^{n-1}.$$

We define $\Phi_0 : C^2(S^{n-1})\times C^2(S^{n-1}) \to [0,+\infty)$ as:

(I) On the off-diagonal set,

  (i) if $f(u_1)g(u_2)+f(u_2)g(u_1) < 0$ for some $(u_1,u_2) \in S^{n-1}\times S^{n-1}$, then

$$\begin{aligned}\Phi_0(f,g) = 2\,\|(f\circ\phi)'\|_\infty\,\|(g\circ\phi)'\|_\infty + \|f\circ\phi\|_\infty\,\|(g\circ\phi)''\|_\infty \\ +\|f\circ\phi\|_\infty\|g\circ\phi\|_\infty;\end{aligned}$$

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

(ii) if $f(u_1)g(u_2) + f(u_2)g(u_1) \geq 0$ for every $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, then

$$\Phi_0(f, g) = 2\,\|(f \circ \phi)'\|_\infty \,\|(g \circ \phi)'\|_\infty + \|f \circ \phi\|_\infty \,\|(g \circ \phi)''\|_\infty .$$

(II) On the diagonal set,

(i) if $\Gamma[f](u_1, u_2) < 0$ for some $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, and if $f(u_1)$ $f(u_2) < 0$ for some $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, then

$$\Phi_0(f, f) = 2\,\|(f \circ \phi)'\|_\infty^2 + \|f \circ \phi\|_\infty \,\|(f \circ \phi)''\|_\infty + \|f \circ \phi\|_\infty^2;$$

(ii) if $\Gamma[f](u_1, u_2) < 0$ for some $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, and if $f(u_1)$ $f(u_2) \geq 0$ for every $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, then

$$\Phi_0(f, f) = 2\,\|(f \circ \phi)'\|_\infty^2 + \|f \circ \phi\|_\infty \,\|(f \circ \phi)''\|_\infty ;$$

(iii) if $\Gamma[f](u_1, u_2) \geq 0$ for every $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, then $\Phi_0(f, f) = 0$.

We define $\Phi_1 : C^2(S^{n-1}) \to [0, +\infty)$ as:

(i) If $f(u) < 0$ for some $u \in S^{n-1}$, then $\Phi_1(f) = 2\|f \circ \phi\|_\infty + \|(f \circ \phi)''\|_\infty$.
(ii) If $f(u) \geq 0$ for every $u \in S^{n-1}$, then $\Phi_1(f) = \|(f \circ \phi)''\|_\infty$.

## 2.2 Examples of nonconvex radial sums of convex bodies

The following lemma is useful in investigating the convexity of a star body.

***Lemma 2.1*** [6, Lemma 5.1.4] *Let $A \in \mathcal{S}^n$. The following conditions are equivalent:*

(i) *$A$ is convex.*
(ii) *For any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, we have $\Gamma[\rho_A](u_1, u_2) \geq 0$.*

For the condition (ii), we remark that if $u_1 + u_2 = 0$, then $\Gamma[\rho_A](u_1, u_2) = \infty$. Thus, the condition (ii) essentially works for the case where $u_1 + u_2 \neq 0$. In order to understand the philosophy of Lemma 2.1, assume $u_1 + u_2 \neq 0$ and $\rho_A(u_1)\rho_A(u_2)\rho_A(u_3) \neq 0$. Let us compute the unique point of the intersection

$$\{ s\rho_A(u_3)\,u_3 \mid s \in [0, +\infty) \} \cap \{ (1-t)\rho_A(u_1)\,u_1 + t\rho_A(u_2)\,u_2 \mid t \in [0,1] \} .$$

There exists a pair $(s, t) \in [0, +\infty) \times [0, 1]$ such that

$$s\rho_A(u_3)\,u_3 = (1-t)\rho_A(u_1)\,u_1 + t\rho_A(u_2)\,u_2,$$

and we get

$$s = \frac{\rho_A(u_1)\,\rho_A(u_2)}{\rho_A(u_1) + \rho_A(u_2)}\,\frac{|u_1 + u_2|}{\rho_A(u_3)}.$$

Since $\rho_A(u_1)u_1$, $\rho_A(u_2)u_2$, and $\rho_A(u_3)u_3$ are on the boundary of $A$, $A$ is convex if and only if $s \leq 1$ which is equivalent to $\Gamma[\rho_A](u_1, u_2) \geq 0$. This finishes the proof of Lemma 2.1.

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

From this observation, if there is a positive function $\varepsilon$ with $\varepsilon(0^+) = 0$ such that, for any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$,

$$\frac{\rho_A(u_1)\rho_A(u_2)}{\rho_A(u_1) + \rho_A(u_2)} \frac{|u_1 + u_2|}{\rho_A(u_3)} \leq 1 - \varepsilon(\angle(u_1, u_2))$$

holds, then $A$ has a "stronger" convexity associated with $\varepsilon$. This observation will be used in Propositions 3.1 and 3.7 with $\varepsilon(\theta) = \theta^2$ (up to constant multiple).

Using Lemma 2.1, let us concretely give a pair $(K, r) \in \mathcal{K}_0^n \times (0, +\infty)$ such that $K \widetilde{+} rB^n$ is not convex.

***Example 2.2*** Let $\ell \in (0, +\infty)$. Put

$$K = \{te_1 + x \mid t \in [-\ell, \ell],\ x \in B^n\},\ \alpha(\ell) = \arctan\frac{1}{\ell},$$

$$r_*(\ell) = -\frac{2\sin^3\left(\frac{\alpha(\ell)}{2} + \frac{\pi}{4}\right) + \sin^2\left(\frac{\alpha(\ell)}{2} + \frac{\pi}{4}\right) - \sin\left(\frac{\alpha(\ell)}{2} + \frac{\pi}{4}\right) - 1}{\sin\alpha(\ell)\sin\left(\frac{\alpha(\ell)}{2} + \frac{\pi}{4}\right)},$$

$$s_* = \frac{\left(44 + 3\sqrt{177}\right)^{1/3} + \left(44 - 3\sqrt{177}\right)^{1/3} - 1}{6},\ \ell_* = \frac{2s_*\sqrt{1 - s_*^2}}{2s_*^2 - 1}.$$

(1) $s_*$ is the unique real root of $2s^3 + s^2 - s - 1 = 0$, and $0.829 < s_* < 0.830$.
(2) $\ell_*$ is the unique root of $\sin(\alpha(\ell)/2 + \pi/4) = s_*$, and $2.450 < \ell_* < 2.476$.
(3) If $\ell > \ell_*$, then $r_*(\ell) > 0$.
(4) If $\ell > \ell_*$, then, for any $r \in (0, r_*(\ell))$, $K \widetilde{+} rB^n$ is not convex.

**Proof** (4) Let us check that the condition (ii) in Lemma 2.1 does not hold. Put

$$u_1 = \begin{pmatrix} \cos\alpha(\ell) \\ \sin\alpha(\ell) \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ u_2 = e_2.$$

Then, we have

$$|u_1 + u_2| = 2\cos\left(\frac{\alpha(\ell)}{2} - \frac{\pi}{4}\right),\ u_3 = \begin{pmatrix} \cos\left(\frac{\alpha(\ell)}{2} + \frac{\pi}{4}\right) \\ \sin\left(\frac{\alpha(\ell)}{2} + \frac{\pi}{4}\right) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\rho_K(u_1) = \sqrt{\ell^2 + 1},\ \rho_K(u_2) = 1,\ \rho_K(u_3) = \frac{1}{\sin\left(\frac{\alpha(\ell)}{2} + \frac{\pi}{4}\right)},$$

$$r_*(\ell) = -\frac{2\rho_K(u_3) + (1 - |u_1 + u_2|)(\rho_K(u_1) + \rho_K(u_2))}{2 - |u_1 + u_2|}.$$

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

Noting $\Gamma[\rho_K](u_1, u_2) = 0$, we obtain

$$\Gamma\left[\rho_K + r\right](u_1, u_2) = (2 - |u_1 + u_2|)\, r\,(r - r_*(\ell)),$$

which is negative for any $r \in (0, r_*(\ell))$. ∎

***Example 2.3*** Let $A \in \mathcal{S}^2$. By definition, we have

$$IA = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A.$$

Combining this and Example 2.2 with $n = 2$, we have a star body of the form $K \widetilde{\mp} rB^2$ whose intersection body is not convex.

## 2.3 Technical lemmas and remarks

***Lemma 2.4*** *Let $I \subset \mathbb{R}$ be an open interval, and let $\varphi$ and $\psi \in C^2(I)$. For any $(\theta_1, \theta_2) \in I \times I$, we have*

$$\left|(\varphi(\theta_1) + \varphi(\theta_2))\,\psi\left(\frac{\theta_1+\theta_2}{2}\right) - \varphi(\theta_1)\,\psi(\theta_2) - \varphi(\theta_2)\,\psi(\theta_1)\right|$$
$$\leq \left(2\,\|\varphi'\|_\infty\,\|\psi'\|_\infty + \|\varphi\|_\infty\,\|\psi''\|_\infty\right)\frac{(\theta_1-\theta_2)^2}{4}.$$

**Proof** By the fundamental theorem of calculus, we have

$$\begin{aligned}
&(\varphi(\theta_1) + \varphi(\theta_2))\,\psi\left(\frac{\theta_1+\theta_2}{2}\right) - \varphi(\theta_1)\,\psi(\theta_2) - \varphi(\theta_2)\,\psi(\theta_1)\\
&= \varphi(\theta_1)\left(\psi\left(\frac{\theta_1+\theta_2}{2}\right) - \psi(\theta_2)\right) + \varphi(\theta_2)\left(\psi\left(\frac{\theta_1+\theta_2}{2}\right) - \psi(\theta_1)\right)\\
&= \frac{\theta_1-\theta_2}{2}\int_0^1\left(\varphi(\theta_1)\,\psi'\left(s\frac{\theta_1+\theta_2}{2} + (1-s)\theta_2\right)\right.\\
&\qquad \left.-\varphi(\theta_2)\,\psi'\left(s\frac{\theta_1+\theta_2}{2} + (1-s)\theta_1\right)\right)\mathrm{d}s.
\end{aligned}$$

Using the fundamental theorem of calculus again, the integrand is

$$\begin{aligned}
&(\theta_1-\theta_2)\left[\int_0^1 \varphi'(t\theta_1 + (1-t)\theta_2)\right.\\
&\qquad\times\psi'\left(t\left(s\frac{\theta_1+\theta_2}{2} + (1-s)\theta_2\right) + (1-t)\left(s\frac{\theta_1+\theta_2}{2} + (1-s)\theta_1\right)\right)\mathrm{d}t\\
&\quad-(1-s)\int_0^1 \varphi(t\theta_1 + (1-t)\theta_2)\\
&\qquad\left.\times\psi''\left(t\left(s\frac{\theta_1+\theta_2}{2} + (1-s)\theta_2\right) + (1-t)\left(s\frac{\theta_1+\theta_2}{2} + (1-s)\theta_1\right)\right)\mathrm{d}t\right].
\end{aligned}$$

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

Estimating

$$\left|\int_0^1 \left(\int_0^1 \varphi'\psi'\,\mathrm{d}t - (1-s)\int_0^1 \varphi\psi''\,\mathrm{d}t\right)\mathrm{d}s\right| \le \|\varphi'\|_\infty \|\psi'\|_\infty + \frac{\|\varphi\|_\infty \|\psi''\|_\infty}{2},$$

we obtain the conclusion. ∎

***Corollary 2.5*** *Let $I \subset \mathbb{R}$ be an open interval, and let $\varphi$ and $\psi \in C^2(I)$. For any $(\theta_1, \theta_2) \in I \times I$, we have*

$$\left|(\varphi(\theta_1) + \varphi(\theta_2))\,\psi\left(\frac{\theta_1+\theta_2}{2}\right) - \cos\frac{\theta_1-\theta_2}{2}\left(\varphi(\theta_1)\psi(\theta_2) + \varphi(\theta_2)\psi(\theta_1)\right)\right|$$
$$\le \left(2\|\varphi'\|_\infty \|\psi'\|_\infty + \|\varphi\|_\infty \|\psi''\|_\infty + \|\varphi\|_\infty \|\psi\|_\infty\right)\frac{(\theta_1-\theta_2)^2}{4}.$$

**Proof** By Taylor's theorem with integral remainder, we have

$$\cos\theta = 1 + \int_0^\theta (\xi - \theta)\cos\xi\,\mathrm{d}\xi.$$

Thus, the left-hand side is

$$\left|(\varphi(\theta_1) + \varphi(\theta_2))\,\psi\left(\frac{\theta_1+\theta_2}{2}\right) - \varphi(\theta_1)\psi(\theta_2) - \varphi(\theta_2)\psi(\theta_1)\right.$$
$$\left. - \left(\varphi(\theta_1)\psi(\theta_2) + \varphi(\theta_2)\psi(\theta_1)\right)\int_0^{\frac{\theta_1-\theta_2}{2}}\left(\xi - \frac{\theta_1-\theta_2}{2}\right)\cos\xi\,\mathrm{d}\xi\right|.$$

Estimating

$$\left|\left(\varphi(\theta_1)\psi(\theta_2) + \varphi(\theta_2)\psi(\theta_1)\right)\int_0^{\frac{\theta_1-\theta_2}{2}}\left(\xi - \frac{\theta_1-\theta_2}{2}\right)\cos\xi\,\mathrm{d}\xi\right|$$
$$\le 2\|\varphi\|_\infty \|\psi\|_\infty \int_0^{\frac{\theta_1-\theta_2}{2}}\left(\frac{\theta_1-\theta_2}{2} - \xi\right)\mathrm{d}\xi$$
$$= \|\varphi\|_\infty \|\psi\|_\infty \frac{(\theta_1-\theta_2)^2}{4},$$

Lemma 2.4 completes the proof. ∎

***Lemma 2.6*** *Let $f$ and $g \in C^2(S^{n-1})$. For any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, we have*

$$(f(u_1) + f(u_2))\,g(u_3) - \cos\frac{\angle(u_1,u_2)}{2}\left(f(u_1)g(u_2) + f(u_2)g(u_1)\right)$$
$$\ge -\Phi_0(f,g)\frac{\angle(u_1,u_2)^2}{4}.$$

**Proof** If $f = g$, then the left-hand side is $\Gamma[f](u_1, u_2)$. Thus, in the case (II-iii) of the definition of $\Phi_0$, the proof is completed. Let us consider the other cases.

There exists a triple $(q, \theta_1, \theta_2) \in SO(n) \times \mathbb{R} \times \mathbb{R}$ such that $u_j = q\phi(\theta_j, 0, \dots, 0)$. Then, we have $u_3 = q\phi((\theta_1+\theta_2)/2, 0, \dots, 0)$. Put $\varphi_q(\theta) = f(q\phi(\theta, 0, \dots, 0))$ and $\psi_q(\theta) = g(q\phi(\theta, 0, \dots, 0))$. In the case (I-i) or (II-i), Corollary 2.5 with $(\varphi, \psi) =$

$(\varphi_q, \psi_q)$ completes the proof. In the case (I-ii) or (II-ii), the inequality $\cos \le 1$ and Lemma 2.4 with $(\varphi, \psi) = (\varphi_q, \psi_q)$ complete the proof. ■

***Corollary 2.7*** *Let $f$ and $g \in C^2(S^{n-1})$. For any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, we have*

$$\Delta[f, g](u_1, u_2) \ge -(\Phi_0(f, g) + \Phi_0(g, f)) \frac{\angle(u_1, u_2)^2}{4}.$$

***Corollary 2.8*** *Let $f \in C^2(S^{n-1})$. For any $(u_1, u_2) \in S^{n-1}$, we have*

$$\Gamma[f](u_1, u_2) \ge -\Phi_0(f, f) \frac{\angle(u_1, u_2)^2}{4}.$$

***Lemma 2.9*** *Let $I \subset \mathbb{R}$ be an open interval, and let $\varphi \in C^2(I)$. For any $(\theta_1, \theta_2) \in I \times I$, we have*

$$\left| 2\varphi\left(\frac{\theta_1 + \theta_2}{2}\right) - \varphi(\theta_1) - \varphi(\theta_2) \right| \le \|\varphi''\|_\infty \frac{(\theta_1 - \theta_2)^2}{4}.$$

**Proof** Lemma 2.4 with $(\varphi, \psi) = (1, \varphi)$ completes the proof. ■

***Corollary 2.10*** *Let $I \subset \mathbb{R}$ be an open interval, and let $\varphi \in C^2(I)$. For any $(\theta_1, \theta_2) \in I \times I$, we have*

$$\left| 2\varphi\left(\frac{\theta_1 + \theta_2}{2}\right) + \left(1 - 2\cos\frac{\theta_1 - \theta_2}{2}\right)(\varphi(\theta_1) + \varphi(\theta_2)) \right|$$
$$\le (2\|\varphi\|_\infty + \|\varphi''\|_\infty) \frac{(\theta_1 - \theta_2)^2}{4}.$$

**Proof** Applying Taylor's theorem with integral remainder to cos, in the same manner as in Corollary 2.5, Lemma 2.9 completes the proof. ■

***Remark 2.11*** Let $f$ be a function defined on $S^{n-1}$, and let $c \in \mathbb{R}$. For any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, we have

$$\begin{aligned}
\Delta[f, c](u_1, u_2) &= \Delta[f, 1](u_1, u_2)\, c \\
&= \Delta[cf, 1](u_1, u_2) \\
&= (2f(u_3) + (1 - |u_1 + u_2|)(f(u_1) + f(u_2)))\, c \\
&= \left(2f(u_3) + \left(1 - 2\cos\frac{\angle(u_1, u_2)}{2}\right)(f(u_1) + f(u_2))\right) c.
\end{aligned}$$

***Lemma 2.12*** *Let $f \in C^2(S^{n-1})$, and let $c \in \mathbb{R}$. For any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, we have*

$$\Delta[f, c](u_1, u_2) \ge -\Phi_1(f)|c| \frac{\angle(u_1, u_2)^2}{4}.$$

**Proof** Fix an arbitrary $(u_1, u_2) \in S^{n-1} \times S^{n-1}$. There exists a triple $(q, \theta_1, \theta_2) \in SO(n) \times \mathbb{R} \times \mathbb{R}$ such that $u_j = q\phi(\theta_j, 0, \dots, 0)$. Then, we have $u_3 = q\phi((\theta_1 + \theta_2)/2, 0, \dots, 0)$. Put $\varphi_q(\theta) = f(q\phi(\theta, 0, \dots, 0))$. If $cf(u) < 0$ for some $u \in S^{n-1}$, then Remark 2.11 and Corollary 2.10 with $\varphi = \varphi_q$ complete the proof. If $cf(u) \ge 0$ for

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

any $u \in S^{n-1}$, then the triangle inequality $|u_1 + u_2| \le 2$ and Lemma 2.9 with $\varphi = \varphi_q$ complete the proof. ∎

***Remark 2.13*** Let $f$ and $g$ be functions defined on $S^{n-1}$. We have $\Gamma[f+g] = \Gamma[f] + \Delta[f,g] + \Gamma[g]$.

***Remark 2.14*** By Taylor's theorem with integral remainder, for any $\theta \in \mathbb{R}$, we have

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} = \theta^2 + \frac{1}{12}\int_0^{2\theta} (\xi - 2\theta)^3 \cos\xi \, d\xi \ge \theta^2 - \frac{\theta^4}{3}.$$

***Remark 2.15*** Let $c \in \mathbb{R}$. By Remark 2.14, for any $(u_1,u_2) \in S^{n-1} \times S^{n-1}$, we have

$$\Gamma[c](u_1,u_2) = 4c^2 \sin^2 \frac{\angle(u_1,u_2)}{4} \ge \left(1 - \frac{\pi^2}{48}\right) c^2 \frac{\angle(u_1,u_2)^2}{4}.$$

***Corollary 2.16*** *Let $f \in C^2(S^{n-1})$, and let $c \in \mathbb{R}$. For any $(u_1,u_2) \in S^{n-1} \times S^{n-1}$, we have*

$$\Gamma[f+c](u_1,u_2) \ge \left(\left(1 - \frac{\pi^2}{48}\right) c^2 - \Phi_1(f)|c| - \Phi_0(f,f)\right) \frac{\angle(u_1,u_2)^2}{4}.$$

**Proof** By Remarks 2.13 with $g = c$ and 2.15, we have

$$\Gamma[f+c](u_1,u_2) = \Gamma[f](u_1,u_2) + \Delta[f,c](u_1,u_2) + 4c^2 \sin^2 \frac{\angle(u_1,u_2)}{4}.$$

Applying Corollary 2.8, Lemma 2.12, and the inequality in Remark 2.15 to the first, second, and third terms, respectively, we obtain the conclusion. ∎

# 3 Main results

## 3.1 Construction of convex radial sums

We keep the notation from the Introduction and Section 2.1.

***Proposition 3.1*** *Let $\gamma \in [0,+\infty)$, and let $K \in \mathcal{K}_0^n$ be such that $\rho_K \in C^2(S^{n-1})$. Assume that, for any $(u_1,u_2) \in S^{n-1} \times S^{n-1}$, the inequality $\Gamma[\rho_K](u_1,u_2) \ge \gamma\angle(u_1,u_2)^2$ holds. Let $f \in C^2(S^{n-1})$ be such that $\rho_K + f \ge 0$. If the inequality*

$$4\gamma \ge \Phi_0(f,f) + \Phi_0(f,\rho_K) + \Phi_0(\rho_K,f)$$

*holds, then $K_f$ is convex.*

**Proof** Let us check the condition (ii) in Lemma 2.1. Let $(u_1,u_2) \in S^{n-1} \times S^{n-1}$ be such that $u_1 + u_2 \ne 0$. By Remark 2.13 with $g = \rho_K$ and the assumption, we have

$$\Gamma[\rho_K + f] = \Gamma[f + \rho_K] \ge \Gamma[f](u_1,u_2) + \Delta[f,\rho_K](u_1,u_2) + \gamma\angle(u_1,u_2)^2.$$

Corollaries 2.7 with $g = \rho_K$ and 2.8 complete the proof. ∎

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

***Remark 3.2*** Proposition 3.1 essentially works for the case where $\gamma > 0$ (see Example 2.2).

***Theorem 3.3*** *Let $A \in \mathcal{S}^n$ be such that $\rho_A \in C^2(S^{n-1})$. Let $\gamma \in [0, +\infty)$, and put*

$$R_A(\gamma) = \frac{24}{48 - \pi^2}\left(\Phi_1(\rho_A) + \sqrt{\Phi_1(\rho_A)^2 + \frac{48 - \pi^2}{12}\left(\Phi_0(\rho_A, \rho_A) + 4\gamma\right)}\right).$$

*If $r \geq R_A(\gamma)$, then, for any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, we have*

$$\Gamma[\rho_A + r](u_1, u_2) \geq \gamma \angle (u_1, u_2)^2.$$

**Proof** This follows from Corollary 2.16 with $(f, c) = (\rho_A, r)$. ∎

***Corollary 3.4*** *Let $A$ and $R_A$ be as in Theorem 3.3. If $r \geq R_A(0)$, then $A \widetilde{+} rB^n$ is convex.*

Combining Theorem 3.3 and Proposition 3.1, for any $A \in \mathcal{S}^n$ with $\rho_A \in C^2(S^{n-1})$, there exists a "large enough" $F \in C^2(S^{n-1})$ such that $A_F$ is convex. Precisely, we obtain the following corollary.

***Corollary 3.5*** *Let $A$, $\gamma$, and $R_A$ be as in Theorem 3.3. Let $(r, f) \in [R_A(\gamma), +\infty) \times C^2(S^{n-1})$ be such that $\rho_A + r + f \geq 0$. If the inequality*

$$4\gamma \geq \Phi_0(f, f) + \Phi_0(f, \rho_A + r) + \Phi_0(\rho_A + r, f)$$

*holds, then $A_{r+f}$ is convex.*

**Proof** Put $K = A \widetilde{+} rB^n$. $\rho_K = \rho_A + r$ is of class $C^2$. Theorem 3.3 guarantees that, for any $(u_1, u_2) \in S^{n-1} \times S^{n-1}$, the inequality $\Gamma[\rho_K](u_1, u_2) \geq \gamma \angle (u_1, u_2)^2$ holds. By Proposition 3.1, $A_{r+f} = K_f$ is convex. ∎

### 3.2 Construction of convex intersection bodies

We keep the notation from the Introduction and Section 2.1.

***Remark 3.6*** Let $A \in \mathcal{S}^n$. Let $f \in C^0(S^{n-1})$ be such that $\rho_A + f \geq 0$. For any $u \in S^{n-1}$, we have

$$\begin{aligned}
V_{n-1}\left(A_f \cap u^\perp\right) &= \frac{1}{n-1}\mathcal{R}\left[(\rho_A + f)^{n-1}\right](u) \\
&= \frac{1}{n-1}\sum_{i=0}^{n-1}\binom{n-1}{i}\left\langle \rho_A^i, f^{n-1-i}\right\rangle(u) \\
&= V_{n-1}(A \cap u^\perp) + \frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\left\langle \rho_A^i, f^{n-1-i}\right\rangle(u) \\
&= \frac{1}{n-1}\mathcal{R}\left[f^{n-1}\right](u) + \frac{1}{n-1}\sum_{i=1}^{n-1}\binom{n-1}{i}\left\langle \rho_A^i, f^{n-1-i}\right\rangle(u).
\end{aligned}$$

https://doi.org/10.4153/S0008439523000681 Published online by Cambridge University Press

In particular, if $f$ is a constant $c$, then

$$\begin{aligned} V_{n-1}\left(A_c \cap u^{\perp}\right) &= V_{n-1}\left(A \cap u^{\perp}\right) + \frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right](u)c^{n-1-i} \\ &= \kappa_{n-1}c^{n-1} + \frac{1}{n-1}\sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right](u)c^{n-1-i}. \end{aligned}$$

***Proposition 3.7*** *Let $\gamma \in [0,+\infty)$, and let $A \in \mathcal{S}^n$ be such that $\rho_A \in C^2(S^{n-1})$. Assume that, for any $(u_1,u_2) \in S^{n-1} \times S^{n-1}$, the inequality $\Gamma[\rho_{IA}](u_1,u_2) \geq \gamma \angle (u_1,u_2)^2$ holds. Let $f \in C^2(S^{n-1})$ be such that $\rho_A + f \geq 0$. If the inequality*

$$\begin{aligned} 4\gamma(n-1)^2 \geq\ & \Phi_0\left(\mathcal{R}\left[\rho_A^{n-1}\right], \sum_{i=0}^{n-2}\left\langle \rho_A^i, f^{n-1-i}\right\rangle\right) \\ &+ \Phi_0\left(\sum_{i=0}^{n-2}\left\langle \rho_A^i, f^{n-1-i}\right\rangle, \mathcal{R}\left[\rho_A^{n-1}\right]\right) \\ &+ \Phi_0\left(\sum_{i=0}^{n-2}\left\langle \rho_A^i, f^{n-1-i}\right\rangle, \sum_{i=0}^{n-2}\left\langle \rho_A^i, f^{n-1-i}\right\rangle\right) \end{aligned}$$

*holds, then $IA_f$ is convex.*

**Proof** Let $(u_1,u_2) \in S^{n-1} \times S^{n-1}$ be such that $u_1 + u_2 \neq 0$. Let us check the condition (ii) in Lemma 2.1. By Remarks 3.6 and 2.13, we have

$$\begin{aligned} \Gamma\left[\rho_{IA_f}\right] = \Gamma\left[\rho_{IA}\right] &+ \Delta\left[\frac{1}{n-1}\mathcal{R}\left[\rho_A^{n-1}\right], \frac{1}{n-1}\sum_{i=0}^{n-2}\left\langle \rho_A^i, f^{n-1-i}\right\rangle\right] \\ &+ \Gamma\left[\frac{1}{n-1}\sum_{i=0}^{n-2}\left\langle \rho_A^i, f^{n-1-i}\right\rangle\right]. \end{aligned}$$

By the assumption, we have $\Gamma[\rho_{IA}](u_1,u_2) \geq \gamma\angle(u_1,u_2)^2$. Hence, Corollaries 2.7 and 2.8 complete the proof. ∎

***Remark 3.8*** Let $A \in \mathcal{S}^n$ be such that $\rho_A \in C^2(S^{n-1})$, and let $\gamma \in [0,+\infty)$. The function of $r \in [0,+\infty)$,

$$\begin{aligned} &\left(1 - \frac{\pi^2}{48}\right)\kappa_{n-1}^2 r^{2(n-1)} - \frac{\kappa_{n-1}}{n-1}\Phi_1\left(\sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{2(n-1)-i}\right) \\ &- \frac{1}{(n-1)^2}\Phi_0\left(\sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{n-1-i}, \sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{n-1-i}\right) - 4\gamma, \end{aligned}$$

has at least one real root, and the set of the real roots is bounded. In fact, we have the following asymptotic behaviors as $r$ goes to infinity:

$$\Phi_1\left(\sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{2(n-1)-i}\right) \leq \sum_{i=1}^{n-1}\binom{n-1}{i}\left\|\left(\mathcal{R}\left[\rho_A^i\right]\circ\phi\right)''\right\|_\infty r^{2(n-1)-i} = O\left(r^{2n-3}\right),$$

$$\Phi_0\left(\sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{n-1-i}, \sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{n-1-i}\right)$$

$$\leq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\binom{n-1}{i}\binom{n-1}{j}\left(2\left\|\left(\mathcal{R}\left[\rho_A^i\right]\circ\phi\right)'\right\|_\infty\left\|\left(\mathcal{R}\left[\rho_A^j\right]\circ\phi\right)'\right\|_\infty\right.$$
$$\left.+\left\|\mathcal{R}\left[\rho_A^i\right]\circ\phi\right\|_\infty\left\|\left(\mathcal{R}\left[\rho_A^j\right]\circ\phi\right)''\right\|_\infty\right)r^{2(n-1)-i-j}$$
$$= O\left(r^{2n-4}\right).$$

***Theorem 3.9*** *Let $A \in \mathcal{S}^n$ be such that $\rho_A \in C^2(S^{n-1})$, and let $\gamma \in [0,+\infty)$. Let $\widetilde{R}_A(\gamma)$ be the maximum real root of the function of $r$,*

$$\left(1-\frac{\pi^2}{48}\right)\kappa_{n-1}^2 r^{2(n-1)} - \frac{\kappa_{n-1}}{n-1}\Phi_1\left(\sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{2(n-1)-i}\right)$$
$$-\frac{1}{(n-1)^2}\Phi_0\left(\sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{n-1-i}, \sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{n-1-i}\right) - 4\gamma.$$

*If $r \geq \widetilde{R}_A(\gamma)$, then, for any $(u_1,u_2) \in S^{n-1}\times S^{n-1}$, we have*

$$\Gamma\left[\rho_{I(A\mp rB^n)}\right](u_1,u_2) \geq \gamma\angle\left(u_1,u_2\right)^2.$$

**Proof** Remark 3.6 with $c = r$ and Corollary 2.16 with

$$(f,c) = \left(\frac{1}{n-1}\sum_{i=1}^{n-1}\binom{n-1}{i}\mathcal{R}\left[\rho_A^i\right]r^{n-1-i}, \kappa_{n-1}r^{n-1}\right)$$

complete the proof. ∎

***Corollary 3.10*** *Let $A$ and $\widetilde{R}_A$ be as in Theorem 3.9. If $r \geq \widetilde{R}_A(0)$, then $I(A\mp rB^n)$ is convex.*

Combining Theorem 3.9 and Proposition 3.7, for any $A \in \mathcal{S}^n$ with $\rho_A \in C^2(S^{n-1})$, there exists a "large enough" $F \in C^2(S^{n-1})$ such that $IA_F$ is convex. Precisely, we obtain the following corollary.

***Corollary 3.11*** *Let $A$, $\gamma$, and $\widetilde{R}_A$ be as in Theorem 3.9. Let $(r,f) \in [\widetilde{R}_A(\gamma),+\infty)\times C^2(S^{n-1})$ be such that $\rho_A + r + f \geq 0$. If the inequality*

$$4\gamma(n-1)^2 \geq \Phi_0\left(\mathcal{R}\left[(\rho_A+r)^{n-1}\right], \sum_{i=0}^{n-2}\left\langle(\rho_A+r)^i, f^{n-1-i}\right\rangle\right)$$
$$+\Phi_0\left(\sum_{i=0}^{n-2}\left\langle(\rho_A+r)^i, f^{n-1-i}\right\rangle, \mathcal{R}\left[(\rho_A+r)^{n-1}\right]\right)$$
$$+\Phi_0\left(\sum_{i=0}^{n-2}\left\langle(\rho_A+r)^i, f^{n-1-i}\right\rangle, \sum_{i=0}^{n-2}\left\langle(\rho_A+r)^i, f^{n-1-i}\right\rangle\right)$$

*holds, then $IA_{r+f}$ is convex.*

**Acknowledgment** The author would like to express his deep gratitude to the anonymous reviewer(s) for careful reading of this paper and helpful suggestions.

## References


[1] M. A. Alfonseca and J. Kim, *On the local convexity of intersection bodies of revolution*. Can. J. Math. **67**(2015), no. 1, 3–27.
[2] H. Busemann, *A theorem of convex bodies of the Brunn–Minkowski type*. Proc. Natl. Acad. Sci. USA **35**(1949), 27–31.
[3] H. Busemann and C. M. Petty, *Problems on convex bodies*. Math. Scand. **4**(1956), 88–94.
[4] R. J. Gardner, *Intersection bodies and the Busemann–Petty problem*. Trans. Amer. Math. Soc. **342**(1994), no. 1, 435–445.
[5] R. J. Gardner, *A positive answer to the Busemann–Petty problem in three dimensions*. Ann. Math. **140**(1994), no. 2, 435–447.
[6] R. J. Gardner, *Geometric tomography*. 2nd ed., Encyclopedia of Mathematics and Its Applications, 58, Cambridge University Press, Cambridge, 2006.
[7] R. J. Gardner, A. Koldobsky, and T. Schlumprecht, *An analytic solution to the Busemann–Petty problem on sections of convex bodies*. Ann. Math. **149**(1999), no. 2, 691–703.
[8] G. Hansen, I. Herburt, H. Martini, and M. Moszyńska, *Starshaped sets*. Aequat. Math. **94**(2020), 1001–1092.
[9] J. Kim, V. Yaskin, and A. Zvavitch, *The geometry of $p$-convex intersection bodies*. Adv. Math. **226**(2011), no. 6, 5320–5337.
[10] E. Lutwak, *Intersection bodies and dual mixed volumes*. Adv. Math. **71**(1988), 232–261.
[11] M. Meyer and S. Reisner, *The convex intersection body of a convex body*. Glasg. Math. J. **53**(2011), 523–534.
[12] M. Moszyńska, *Selected topics in convex geometry*, Birkhäuser, Boston, 2006.
[13] G. Y. Zhang, *Centered bodies and dual mixed volumes*. Trans. Amer. Math. Soc. **345**(1994), no. 2, 777–801.
[14] G. Y. Zhang, *A positive solution to the Busemann–Petty problem in* $\mathbb{R}^4$. Ann. Math. **149**(1999), no. 2, 535–543.


*Department of Applied Mathematics, Fukuoka University, 8-19-1 Nanakuma, Jonan, Fukuoka 814-0180, Japan*
*e-mail*: ssakata@fukuoka-u.ac.jp