BULL. AUSTRAL. MATH. SOC. 06A10, 08A40, 08B10
VOL. 47 (1993) [79-93]

POSETS, NEAR UNANIMITY FUNCTIONS AND ZIGZAGS

LÁSZLÓ ZÁDORI

With every poset we associate a class of coloured posets called zigzags. By means of zigzags we show that, if we delete a convex set from a finite lattice ordered set then the resulting poset has the strong selection property. We give the complete list of finite bounded irreducible posets admitting an n-ary near unanimity function, provided $n \leqslant 6$. We present some examples and classes of posets with full descriptions of their zigzags.

1. INTRODUCTION

For $n \geqslant 3$ an n-ary operation f on a set A is called a *near unanimity function*, briefly a *nuf*, if it obeys the identity $f\Big(x,\dots,x,\underset{i}{y},x,\dots,x\Big) = x$ for every $1 \leqslant i \leqslant n$. If $n = 3$, then f is called a *majority function*. We say that a poset $\mathbf{P}$ *admits* an n-ary operation g, if g is monotone on $\mathbf{P}$. In [**1, 2, 5, 11**] it is conjectured that a finite bounded poset $\mathbf{P}$ admits a near unanimity function, if there exists a finite generating set for the clone of all monotone operations on $\mathbf{P}$. This conjecture is important for the study of finite posets admitting a near unanimity function. In [**3**] Demetrovics, Hannák and Rónyai proved that every poset obtained from a finite lattice ordered set by deleting a convex subset admits a near unanimity function. In [**8**] Quackenbush, Rival and Rosenberg proved that every finite poset with the strong selection property admits a near unanimity function. We shall see that these results are easily proven by means of zigzags. In fact, we show that the above mentioned posets of Demetrovics et al. have the strong selection property. Due to a remark of Tardos in [**9**], zigzags have turned out to be a powerful tool in the exploration of finite posets admitting a near unanimity function. We list all the finite irreducible bounded posets admitting an n-ary near unanimity function with $n \leqslant 6$. Also, we describe some classes of finite posets with a full description of their zigzags. So we can easily decide whether or not these posets admit a near unanimity function.

The basic notions concerning zigzags are given in [**11**] and we review the results we need from this paper later in this section. We use boldface capital letters to denote a poset throughout the paper. A poset is called *bounded*, if it has a largest and a smallest

Received 23rd December, 1991.

element. For a poset $\mathbf{Q}$ let $\prec_{\mathbf{Q}}$ denote the covering relation determined by $\leqslant_{\mathbf{Q}}$. Let $\mathbf{P}$ be a poset and let T be a subset of $P \cup \prec_{\mathbf{P}}$ with $P \not\subseteq T$. We denote the poset $\left(P \setminus T, (\leqslant_{\mathbf{P}}|_{P\setminus T}) \setminus T\right)$ by $\mathbf{P} \setminus T$ and we say that T is *deleted* from $\mathbf{P}$. For two posets $\mathbf{P}$ and $\mathbf{Q}$, $\mathbf{P}+\mathbf{Q}$ denotes their ordinal sum. We say that a poset $\mathbf{Q}$ is *contained* in $\mathbf{P}$ if $\leqslant_{\mathbf{Q}} \subseteq \leqslant_{\mathbf{P}}$. If $\mathbf{Q}$ is contained in $\mathbf{P}$ we write $\mathbf{Q} \subseteq \mathbf{P}$. We say that $\mathbf{Q}$ is *properly contained* in $\mathbf{P}$ if $\mathbf{Q} \subseteq \mathbf{P}$ and $\mathbf{Q} \neq \mathbf{P}$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A pair $(\mathbf{Q}, f)$ is called a $\mathbf{P}$-*coloured poset* if f is a partially defined map from Q to P. If f can be extended to a fully defined monotone map $f' : \mathbf{Q} \to \mathbf{P}$ on Q then f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$-*extendible*; otherwise f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$-*nonextendible*. A $\mathbf{P}$-*zigzag* is a $\mathbf{P}$-nonextendible, $\mathbf{P}$-coloured poset $(\mathbf{H}, f)$, where H is finite and for every $\mathbf{K}$, properly contained in $\mathbf{H}$, the $\mathbf{P}$-coloured poset $(\mathbf{K}, f|_K)$ is $\mathbf{P}$-extendible. Roughly speaking, the $\mathbf{P}$-zigzags are the finite, minimal, nonextendible $\mathbf{P}$-coloured posets. When it is clear what $\mathbf{P}$ is we omit it in the terms such as $\mathbf{P}$-zigzags, $\mathbf{P}$-extendible, et cetera.

For two $\mathbf{P}$-coloured posets $(\mathbf{H}, f)$ and $(\mathbf{Q}, g)$ we say that $(\mathbf{H}, f)$ is *contained* in $(\mathbf{Q}, g)$ and we write $(\mathbf{H}, f) \subseteq (\mathbf{Q}, g)$ if $\mathbf{H} \subseteq \mathbf{Q}$ and $f = g|_H$. Observe that every finite nonextendible coloured poset contains a zigzag. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-coloured poset and let T be a subset of $H \cup \prec_{\mathbf{H}}$ with $H \not\subseteq T$. We denote the $\mathbf{P}$-coloured poset $\left(\mathbf{H} \setminus T, f|_{H\setminus T}\right)$ by $(\mathbf{H}, f) \setminus T$ and we say that T is *deleted* from $(\mathbf{H}, f)$.

For a $\mathbf{P}$-coloured poset $(\mathbf{H}, f)$ we define $C(\mathbf{H}, f) = \{h \in H : f(h) \text{ exists}\}$ and $N(\mathbf{H}, f) = H \setminus C(\mathbf{H}, f)$. We call the elements of $C(\mathbf{H}, f)$ *coloured elements* and the elements of $N(\mathbf{H}, f)$ *noncoloured elements*. If $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$ are nonempty we define the posets $\mathbf{C}(\mathbf{H}, f)$ and $\mathbf{N}(\mathbf{H}, f)$ by the restriction of $\leqslant_{\mathbf{H}}$ to $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$, respectively.

A figure of a $\mathbf{P}$-coloured poset $(\mathbf{H}, f)$ consists of the covering graph of $\mathbf{H}$ and an element of $\mathbf{H}$ is drawn as a small shaded circle or a small empty circle according to whether f is defined or not defined on the given point. Every shaded point is labelled by the value of f. For example, let $\mathbf{P}$ be the poset shown in Figure 1. Then $(\mathbf{H}, f)$ in Figure 1 is a $\mathbf{P}$-coloured poset. In fact, $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag.

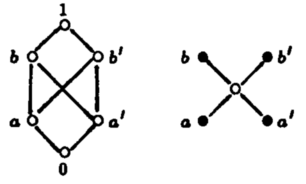

Figure 1. Poset $\mathbf{P}$ and a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$

A $\mathbf{P}$-coloured poset $(\mathbf{H}, f)$ is *monotone* if f is a monotone map on its domain, otherwise $(\mathbf{H}, f)$ is *nonmonotone*. For any poset $\mathbf{P}$ the $\mathbf{P}$-coloured two element chain

https://doi.org/10.1017/S0004972700012284 Published online by Cambridge University Press

in which the top is coloured by a and the bottom is coloured by b, where $b \not\leq a$ in $\mathbf{P}$, is a nonmonotone $\mathbf{P}$-zigzag and every nonmonotone $\mathbf{P}$-zigzag is of this form. In the proofs included in the paper we need some properties of zigzags which are established and proved in [11].

Claim 1.1. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-coloured poset, where H is finite. Then $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag if and only if $\mathbf{H}$ is connected, $(\mathbf{H}, f)$ is not $\mathbf{P}$-extendible and by deleting any covering pair of $(\mathbf{H}, f)$ the resulting coloured poset is $\mathbf{P}$-extendible.*

Claim 1.2. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. The subgraph spanned by $N(\mathbf{H}, f)$ in the covering graph of $\mathbf{H}$ is connected.*

Claim 1.3. *Let $(\mathbf{H}, f)$ be a monotone zigzag and let $a \in C(\mathbf{H}, f)$. For every $b \in H$ which satisfies $a \prec b$ or $b \prec a$ we have $b \in N(\mathbf{H}, f)$.*

A *monotone map* between two $\mathbf{P}$-coloured posets means a monotone map between the two base posets which maps each a-coloured element to an a-coloured element and each noncoloured element to a noncoloured element. We say that a $\mathbf{P}$-coloured poset $(\mathbf{H}, f)$ is a *monotone image* of a $\mathbf{P}$-coloured poset $(\mathbf{H}', f')$, if there exists a monotone map from $(\mathbf{H}', f')$ onto $(\mathbf{H}, f)$. A coloured poset in which every coloured element occurs in exactly one covering pair is called a *standard coloured poset.*

Claim 1.4. *For every $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ there exists a standard $\mathbf{P}$-zigzag $(\mathbf{H}', f')$ such that $(\mathbf{H}, f)$ is a monotone image of $(\mathbf{H}', f')$.*

Claim 1.5. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag and let a and b be two different elements of $C(\mathbf{H}, f)$. Let us suppose that there exists $c \in N(\mathbf{H}, f)$ with $c \prec a, b$. Then $f(a) \not\leq f(b)$.*

Claim 1.6. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag and let $a, b \in C(\mathbf{H}, f)$, where $a < b$. Then $f(a) \neq f(b)$.*

Claim 1.7. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. Let $a, b \in C(\mathbf{H}, f)$ be two different maximal elements of $\mathbf{H}$ for which $f(a) = f(b)$. Then there exists a zigzag $(\mathbf{H}', f')$ for which $N(\mathbf{H}', f') = N(\mathbf{H}, f)$ and there is an onto monotone map from $(\mathbf{H}, f)$ to $(\mathbf{H}', f')$ which identifies only a and b.*

Claim 1.8. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. Every monotone map $g : \mathbf{H} \to \mathbf{H}$ that is the identity map on $C(\mathbf{H}, f)$ has to be onto, that is, an automorphism of $\mathbf{H}$.*

Let $\mathbf{Q}$ be a finite poset. Then $a \in \mathbf{Q}$ is called *retractable* if there is a non-onto monotone map on $\mathbf{Q}$ that fixes each element different from a. An element $a \in \mathbf{Q}$ is called *irreducible* if there is a unique $b \in Q$ with $a \prec b$ or $b \prec a$. Observe that every irreducible element is retractable.

Claim 1.9. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. Then $N(\mathbf{H}, f)$ has no retractable element of $\mathbf{H}$.*

https://doi.org/10.1017/S0004972700012284 Published online by Cambridge University Press

Claim 1.10. *If $\mathbf{P} = \mathbf{Q}+\mathbf{1}$, then every maximal element of a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ is coloured.*

Claim 1.11. *For a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ the following hold.*

(1) *If $|N(\mathbf{H}, f)| = 0$, then $(\mathbf{H}, f)$ is a two element nonmonotone zigzag.*
(2) *If $|N(\mathbf{H}, f)| = 1$, then $(\mathbf{H}, f)$ is the first coloured poset shown in Figure 2, where m and n are nonnegative integers such that $m + n > 0$ and $n, m \neq 1$. Moreover, f is an order isomorphism on its domain.*
(3) *If $|N(\mathbf{H}, f)| = 2$, then $(\mathbf{H}, f)$ is the second coloured poset shown in Figure 2, where $k, l \geqslant 1$ and m and n are nonnegative integers for which $m, n \neq 1$. Moreover, any comparable pair in Range(f) not shown in the figure is of the form $d_i < c_j$, $c_j < b_s$ or $a_t < d_i$ for some $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$, $1 \leqslant s \leqslant m$ and $1 \leqslant t \leqslant n$.*

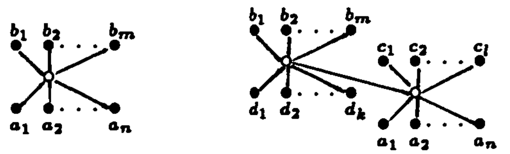

Figure 2. Monotone zigzags with one and two noncoloured elements

The next proposition, the same as [11, Proposition 2.3], characterises via zigzags the finite posets with the strong selection property. In this paper we take this proposition as a definition of a finite poset with the strong selection property. For the original definition see [7].

Proposition 1.12. *A finite poset $\mathbf{P}$ has the strong selection property if and only if every $\mathbf{P}$-zigzag has at most one noncoloured element.*

Tardos's remark in [9], see also [11, Remark 2.4], describes via zigzags the finite posets admitting an n-ary near unanimity function.

Remark 1.13. Let $n \geqslant 3$. A finite poset $\mathbf{P}$ admits an n-ary near unanimity function if and only if in every $\mathbf{P}$-zigzag the number of coloured elements is at most $n - 1$.

2. Finite posets with the strong selection property

In this section we show that every poset obtained from a finite lattice ordered set by deleting a convex subset has the strong selection property. Also, we show that the finite posets admitting a nuf with arity at most 6 have the strong selection property.

Let $\mathbf{L}$ be a finite lattice ordered set. We note that $\mathbf{L}$ has no monotone zigzags. A subset S of L is called a *convex subset* of $\mathbf{L}$ if $a, b \in S$, $c \in L$ and $a \leqslant_{\mathbf{L}} c \leqslant_{\mathbf{L}} b$ imply $c \in S$. There is a well known result [3] of Demetrovics, Hannák and Rónyai

https://doi.org/10.1017/S0004972700012284 Published online by Cambridge University Press

which states that for a finite lattice ordered set $\mathbf{L}$ the poset $\mathbf{P} = \mathbf{L} \setminus S$, where S is a proper, convex subset of $\mathbf{L}$, admits a nuf. We will show that $\mathbf{P}$, in fact, has the strong selection property, that is, by Proposition 1.12, every $\mathbf{P}$-zigzag has at most one noncoloured element. So by (1) and (2) of Claim 1.11 the number of $\mathbf{P}$-zigzags is finite. Thus by Remark 1.13 the poset $\mathbf{P}$ admits a nuf. We note that the preceding argument gives a proof of the result in [8] that a finite poset with the strong selection property admits a nuf. We need the following technical lemma about zigzags.

Lemma 2.1. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag with $|N(\mathbf{H}, f)| \geqslant 2$. Then for every $a \in N(\mathbf{H}, f)$, $a \notin \{b, c\} \subseteq H$ with $b \prec_{\mathbf{H}} c$ there exists a monotone $\mathbf{P}$-zigzag $(\mathbf{H}', f')$ such that $\mathbf{H}' \subseteq \mathbf{H}$, $a, b, c \in \mathbf{H}'$, $f' \mid_{H' \setminus \{a\}} = f \mid_{H' \setminus \{a\}}$ and $f'(a)$ is defined in such a way that $f(d) \leqslant f'(a) \leqslant f(e)$ for every $d, e \in C(\mathbf{H}, f)$ with $d <_{\mathbf{H}} a <_{\mathbf{H}} e$.*

Proof: Since $(\mathbf{H}, f)$ is a $\mathbf{P}$-zigzag and $|N(\mathbf{H}, f)| \geqslant 2$ there exist $p \in P$ and a monotone partial map f'' from $\mathbf{H}$ to $\mathbf{P}$ given by $f'' \mid_{C(\mathbf{H}, f)} = f$ and $f''(a) = p$. For every such p we select one $\mathbf{P}$-zigzag contained in $(\mathbf{H}, f'')$ and we denote it by $(\mathbf{H}_p, f_p)$. Clearly, $a \in H_p$. Let us suppose there is no $p \in P$ such that $(b, c) \in \leqslant_{\mathbf{H}_p}$. Let $\mathbf{H}_{b,c} = \mathbf{H} \setminus \{(b, c)\}$. Then $\mathbf{H}_{b,c} \subseteq \mathbf{H}$ and $(\mathbf{H}_{b,c}, f)$ is not $\mathbf{P}$-extendible because every monotone extension of $(\mathbf{H}_{b,c}, f)$ to a contains a zigzag $(\mathbf{H}_p, f_p)$ for some p. But this contradicts that $(\mathbf{H}, f)$ is a zigzag. Thus there is a $p_0 \in P$ such that $(b, c) \in \leqslant_{\mathbf{H}_{p_0}}$. Taking $(\mathbf{H}', f') = (\mathbf{H}_{p_0}, f_{p_0})$ we get the claim. ∎

Proposition 2.2. *Let $\mathbf{L}$ be a finite lattice ordered set. Let S be a proper, convex subset of $\mathbf{L}$. Then $\mathbf{P} = \mathbf{L} \setminus S$ has the strong selection property.*

Proof: By Proposition 1.12 it suffices to show that for every monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ we have $|N(\mathbf{H}, f)| = 1$. Let us suppose this is not true. Then there exists a $\mathbf{P}$-zigzag $(\mathbf{H}, f)$ such that $|N(\mathbf{H}, f)|$ is minimal with respect to $|N(\mathbf{H}, f)| \geqslant 2$. Let $a \neq b \in N(\mathbf{H}, f)$. By Lemma 2.1 there are two $\mathbf{P}$-zigzags $(\mathbf{H}_1, f_1)$ and $(\mathbf{H}_2, f_2)$ such that $\mathbf{H}_1, \mathbf{H}_2 \subseteq \mathbf{H}$, $\{a, b\} \subseteq \mathbf{H}_1, \mathbf{H}_2$ and $f_1 \mid_{H_1 \setminus \{a\}} = f \mid_{H_1 \setminus \{a\}}$, $f_2 \mid_{H_2 \setminus \{b\}} = f \mid_{H_2 \setminus \{b\}}$. Moreover $f_1(a)$ and $f_2(b)$ are defined in such a way that $f(c) \leqslant f_1(a) \leqslant f(d)$ for every $c, d \in C(\mathbf{H}, f)$ with $c < a < d$ and $f(c) \leqslant f_2(b) \leqslant f(d)$ for every $c, d \in C(\mathbf{H}, f)$ with $c < b < d$. The minimality of $|N(\mathbf{H}, f)| \geqslant 2$ implies that $|N(\mathbf{H}_1, f_1)| = |N(\mathbf{H}_2, f_2)| = 1$. Hence (2) in Claim 1.11 applies to $(\mathbf{H}_1, f_1)$ and $(\mathbf{H}_2, f_2)$. So by $a, b \in H_1$, the elements a and b have to be comparable in $\mathbf{H}$. Let us say $a < b$. Let $\wedge$ and $\vee$ be the join and meet operation of $\mathbf{L}$. We define $t = \wedge\{f(p) :\ p \in C(\mathbf{H}_1, f_1),\ b < p\}$ and $u = \vee\{f(p) :\ p \in C(\mathbf{H}_2, f_2),\ p < a\}$. Now, clearly, $f_1(a) \leqslant t$ and by the above inequalities for $f_1(a)$, $u \leqslant f_1(a)$. Observe that t and u have to be in S otherwise $(\mathbf{H}_1, f_1)$ or $(\mathbf{H}_2, f_2)$ would be extendible $\mathbf{P}$-coloured posets. Since S is convex we get that $f_1(a) \in S$, which contradicts $f_1(a) \in P$. ∎

We remark that not every finite poset $\mathbf{P}$ with the strong selection property can be obtained from a finite lattice ordered set in the above way. Let $\mathbf{P}' = \mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$, the poset in Figure 1, and let $\mathbf{P} = \mathbf{P}' \times \mathbf{P}'$. Then $\mathbf{P}$ has the strong selection property since $\mathbf{P}'$ has it by Proposition 2.2 and if $\mathbf{H}$ and $\mathbf{K}$ have the strong selection property then $\mathbf{H} \times \mathbf{K}$ has it too, see [7]. In [3] it is shown that $\mathbf{P}$ cannot be obtained from a finite lattice ordered set by deleting a convex subset.

With the help of the next proposition we get a procedure how to construct all zigzags of a finite poset. Although this procedure is not efficient it can be used to determine all the zigzags of a small poset such as the one in Figure 7.

PROPOSITION 2.3. *Let $\mathbf{P}$ be a finite poset. Let $(\mathbf{H}, f)$ be a monotone $\mathbf{P}$-zigzag. Then for every $h \in N(\mathbf{H}, f)$ there exist $\mathbf{P}$-coloured posets $(\mathbf{H}_i, f_i) \subseteq (\mathbf{H}, f)$, $i \in I$, for which $h \in N(\mathbf{H}_i, f_i)$, and there exist $p_i \in P$, $i \in I$, such that if h is coloured by p_i in $(\mathbf{H}_i, f_i)$ the resulting coloured poset is a $\mathbf{P}$-zigzag and if h is coloured by p_j in $(\mathbf{H}_i, f_i)$, $j \in I \setminus \{i\}$, the resulting coloured poset is $\mathbf{P}$-extendible. Moreover, for every $p \in P$ there exists $i \in I$ such that, if h is coloured by p in $(\mathbf{H}_i, f_i)$ the resulting coloured poset is not $\mathbf{P}$-extendible.*

PROOF: As in Lemma 14, if we colour h by an element of $\mathbf{P}$ in $(\mathbf{H}, f)$ the resulting coloured poset is still nonextendible so it contains some $\mathbf{P}$-zigzags which must contain h. Let $X = \{(\mathbf{G}_t, g_t) : t \in T\}$ be the set of all zigzags which can be obtained in this way. We assign a subset $S_t \subseteq P$ to every $(\mathbf{G}_t, g_t)$, $t \in T$, so that S_t contains $g_t(h)$ and all the elements of $\mathbf{P}$ by which recolouring h in $(\mathbf{G}_t, g_t)$, the resulting coloured poset is nonextendible. We select a subset I of T as follows.

(1) I is a minimal set with respect to $\cup_{i\in I} S_i = P$.

(2) I has the maximal cardinality with respect to (1).

(3) I satisfies $S_t \cup \left(\cup_{j \in I\setminus\{i\}} S_j\right) \neq P$ for every S_t that is a proper subset of S_i, where $t \in T$ and $i \in I$.

For every $i \in I$ we select $(\mathbf{G}_t, g_t)$ such that $S_t = S_i$ and $\mathbf{G}_t$ is minimal with respect to the containment of posets. For simplicity, we can assume $i = t$. Then we define $\mathbf{H}_i = \mathbf{G}_i$ and $f_i = g_i \,|_{G_i \setminus \{h\}}$ for $i \in I$.

Let $T_i = S_i \setminus \left(\cup_{j \in I\setminus\{i\}} S_j\right)$ for $i \in I$. These sets are nonempty by (1). We claim that every T_i, $i \in I$, contains an element p_i such that, if h is coloured by p_i in $(\mathbf{H}_i, f_i)$, then we get a zigzag. Let us suppose this is not true. So there exists an i such that for every $p \in T_i$, if h in $(\mathbf{H}_i, f_i)$ is coloured by p the resulting coloured poset is not extendible but also not a zigzag. Hence it properly contains some zigzags. Let these zigzags be $(\mathbf{G}_v, g_v)$, where $v \in V \subseteq T$. Observe that $S_v \subseteq S_i$ and $\cup_{v\in V} S_v \cup \left(\cup_{j \in I\setminus\{i\}} S_j\right) = P$. Let us take a subset V_0 of V minimal with respect to $\cup_{v\in V_0} S_v \cup \left(\cup_{j \in I\setminus\{i\}} S_j\right) = P$. Clearly, $V_0 \cup (I \setminus \{i\})$ is minimal in the sense of (1). Since $|V_0| \geqslant 1$ and I satisfies (2), $V_0 = \{v_0\}$ for some v_0. Then by (3), $S_i = S_{v_0}$. But $\mathbf{G}_{v_0}$ is properly contained in $\mathbf{H}_i$ which contradicts the fact that $\mathbf{H}_i$ is minimal. Now, $(\mathbf{H}_i, f_i)$ and p_i, $i \in I$, clearly satisfy the claims of the lemma. $\square$

By the use of Proposition 2.3 we can construct all **P**-zigzags of a finite poset **P**. Let us suppose we have determined all **P**-zigzags with fewer than m noncoloured elements. Then any **P**-zigzag $(\mathbf{H}, f)$ with m noncoloured elements can be obtained as a monotone image of a **P**-zigzag $(\mathbf{H}', f')$. The zigzag $(\mathbf{H}', f')$ is obtained from the **P**-zigzags $(\mathbf{G}_i, g_i)$, $i \in I$, with at most $m-1$ noncoloured elements, by deleting the colour of an $h_i \in C(\mathbf{G}_i, g_i)$ in each $(\mathbf{G}_i, g_i)$, $i \in I$, and sticking together the resulting coloured posets at h_i, $i \in I$. By Proposition 2.3 we can choose $(\mathbf{G}_i, g_i)$ and h_i, $i \in I$ in such a way that the coloured poset $(\mathbf{H}', f')$ indeed is a zigzag and $(\mathbf{H}, f)$ is a monotone image of $(\mathbf{H}', f')$.

The following definitions can be found in [4]. A *representation* of a poset **P** is a family $(\mathbf{P}_i \mid i \in I)$ of posets such that $\mathbf{P}_i$ is a retract of **P** for each $i \in I$, and **P** is a retract of $\prod_{i \in I} \mathbf{P}_i$. A poset **P** is *irreducible* if for every representation $(\mathbf{P}_i \mid i \in I)$ of **P**, **P** is a retract of $\mathbf{P}_i$ for some $i \in I$. If **P** is not irreducible then it is called *reducible*. For example, the two element antichain, fences and crowns are known to be irreducible posets, see [4].

Remark 1.13 gives a characterisation of finite posets admitting an n-ary near unanimity function. We would like to find a somewhat more constructive description, similar to the result of Quackenbush, Rival and Rosenberg in [8], which states that every finite poset admitting a majority function is a retract of a finite product of fences and the two element antichain. Observe, the building elements here, that is, the fences and the two element antichain, are irreducible posets admitting a majority function. In general, we can expect a similar characterisation of finite posets admitting an n-ary near unanimity function as the next proposition states.

PROPOSITION 2.4. *The class of finite posets admitting an n-ary near unanimity operation coincides with the class of retracts of finite products of irreducible posets that admit an n-ary near unanimity operation.*

PROOF: The retract and the product of posets preserve the existence of an n-ary near unanimity operation. On the other hand, every finite poset has a finite representation by irreducibles, see [4]. These facts imply the claim. $\square$

We note that a similar claim is true for finite bounded posets admitting an n-ary near unanimity function.

In the following proposition we list all the finite irreducible bounded posets that admit a 6-ary near unanimity function.

PROPOSITION 2.5. *The list of the finite irreducible bounded posets admitting a 6-nuf is the following:* $\mathbf{1}$, $\mathbf{1}+\mathbf{1}$ *with 3-nufs,* $\mathbf{S}_2+\mathbf{S}_2^d$ *with a 5-nuf,* $\mathbf{S}_2+\mathbf{S}_3^d$, $\mathbf{S}_3+\mathbf{S}_2^d$ *with 6-nufs, where* $\mathbf{S}_n$ *is the poset given by the Boolean lattice of n atoms without its*

https://doi.org/10.1017/S0004972700012284 Published online by Cambridge University Press

top element and $\mathbf{S}_n^d$ *is the dual of* $\mathbf{S}_n$.

PROOF: Let $\mathbf{P}$ be a finite irreducible bounded poset with a 6-nuf. We show that for every monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$, $|N(\mathbf{H}, f)| = 1$. Because of Claim 1.4 it suffices to show this for a standard zigzag. So let us assume $(\mathbf{H}, f)$ is a standard zigzag and $|N(\mathbf{H}, f)| \geqslant 2$. By Remark 1.13 we have $|C(\mathbf{H}, f)| \leqslant 5$. By claims 1.9 and 1.10 every maximal element of $\mathbf{N}(\mathbf{H}, f)$ is covered by at least two coloured elements of $(\mathbf{H}, f)$ and every minimal element of $\mathbf{N}(\mathbf{H}, f)$ covers at least two coloured elements of $(\mathbf{H}, f)$. These facts imply that $N(\mathbf{H}, f)$ has one maximal and one minimal element. Since $|N(\mathbf{H}, f)| \geqslant 2$ there is a noncoloured element which covers the bottom element of $\mathbf{N}(\mathbf{H}, f)$. This noncoloured element must also cover a coloured element, for otherwise it would be irreducible, which contradicts Claim 1.9. Dually, there is a noncoloured element covered by the top element of $\mathbf{N}(\mathbf{H}, f)$ which is also covered by a coloured element. But then $|C(\mathbf{H}, f)| \geqslant 6$ which is a contradiction.

So $|N(\mathbf{H}, f)| = 1$ and by Proposition 1.12, $\mathbf{P}$ has the strong selection property. Now, we can invoke a result of Nevermann and Wille in [7] which gives a complete list of the finite irreducible posets having the strong selection property. By using this result and Remark 1.13 we get the list of posets mentioned in the claim. $\square$

We note that there are finite irreducible bounded posets admitting a 7-nuf that do not have the strong selection property. For example, the bounded posets in Figure 5 and Figure 7 in Section 3 admit 7-nufs by Remark 1.13 and are easily shown to be irreducible by [**10**, Proposition 6.6]. Both posets have a zigzag with two noncoloured elements. So, by Proposition 1.12, they do not have the strong selection property. We do not know if there are infinitely many finite irreducible bounded posets which admit a 7-nuf.

3. Examples of zigzags of special posets

First we describe the zigzags of finite posets admitting a majority function. Then in Theorem 3.3 we present a construction of posets from smaller posets and show that for this construction it is easy to describe all zigzags if we know the zigzags of the smaller posets. For example, by this construction we get locked fences, defined in [**5**], and Tardos's eight element poset in [**9**] from fences and the two element antichain, respectively. The description of the zigzags of locked fences yields an easy proof that each locked fence admits a near unanimity function, which result is mentioned in [**5**].

In a connected poset $\mathbf{Q}$ we define the *up distance* from a to b to be the least positive integer n such that there is a subset $\{a_0, \dots, a_n\} \subseteq Q$ with $a = a_0$, $b = a_n$ and $a_0 \leqslant a_1 \geqslant a_2 \leqslant \dots$. We define the *down distance* from a to b dually. Let $\uparrow(a, b)$ and $\downarrow(a, b)$ denote the up and down distance from a to b, respectively. We note that by the definition $\uparrow(a, a) = \downarrow(a, a) = 1$.

PROPOSITION 3.1. *Let* **P** *be a finite connected poset admitting a majority function. Then every* **P***-zigzag* $(\mathbf{H}, f)$ *is a* **P***-coloured fence satisfying the following properties. If* a *and* b *denote the endpoints of the coloured fence, then* a *and* b *are the only coloured points and at least one of the inequalities* $\uparrow(a,b) < \uparrow(f(a), f(b))$ *and* $\downarrow(a,b) < \downarrow(f(a), f(b))$ *holds.*

PROOF: Certainly, a **P**-coloured fence of the above form is a **P**-zigzag for any poset **P**. Let us suppose now that **P** is a finite poset admitting a majority function and $(\mathbf{H}, f)$ is a **P**-zigzag. We want to prove that $(\mathbf{H}, f)$ is of the above form. By Remark 1.13 we have $|C(\mathbf{H}, f)| = 2$. Let $C(\mathbf{H}, f) = \{a, b\}$. Then $f(a)$ and $f(b)$ must be different otherwise $(\mathbf{H}, f)$ would be extendible. If at least one of $\uparrow(a,b) < \uparrow(f(a), f(b))$ and $\downarrow(a,b) < \downarrow(f(a), f(b))$ holds then there is a fence **F** between a and b in **H** such that $(\mathbf{F}, f|_F)$ is a zigzag of the above form. By the minimality of $(\mathbf{H}, f)$, $(\mathbf{H}, f) = (\mathbf{F}, f|_F)$. So we have the claim.

Thus we assume $(\mathbf{H}, f)$ satisfies both $\uparrow(a,b) \geqslant \uparrow(f(a), f(b))$ and $\downarrow(a,b) \geqslant \downarrow(f(a), f(b))$. In this case we show that $(\mathbf{H}, f)$ is extendible, thereby obtaining a contradiction. Without loss of generality, we assume that a shortest path between $f(a)$ and $f(b)$ in **P** is given by $f(a) = a_1 < a_2 > \ldots a_n = f(b)$, $n \geqslant 2$. Since the number of coloured elements is two in every **P**-zigzag, $(\mathbf{H}, f)$ must be a standard zigzag. This implies that a is maximal or minimal. If a is minimal, then by $\uparrow(a,b) \geqslant \uparrow(f(a), f(b))$ the sets $B_i = \{c \in H : \downarrow(a,c) = i\}$, $1 \leqslant i \leqslant n-1$, and $B_n = H \setminus \cup_{i=1}^{n-1} B_i$ are nonempty. Moreover, $a \in B_1$ and $b \in B_n$. Let $f' : H \to P$ be the function that takes on the value a_i for every element of B_i, $1 \leqslant i \leqslant n$. Now, f' is clearly a monotone function from **H** to **P** and f' extends f to **H**. If a is maximal then the only problem that can occur in the preceding argument is when $b \in B_{n-1}$. Because $\downarrow(a,b) \geqslant \downarrow(f(a), f(b))$, then there exists a shortest fence given by $f(a) = b_1 > b_2 < \ldots b_n = f(b)$ in **P**. Hence we have the dual of the case when a was minimal. Thus, we can extend f to **H**. ∎

We define the *middle element* of three elements of a fence as the one which is on the path connecting the other two. It was noticed, see for example [8], that for every fence the ternary function which assigns the middle element to each 3-tuple is a monotone majority function. So by Proposition 3.1 we easily get all the monotone zigzags of fences. For example, the four element fence has the five monotone zigzags shown in Figure 3.

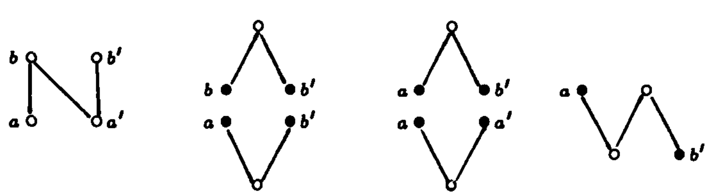

Figure 3. The four element fence and its monotone zigzags

PROPOSITION 3.2. *Let* $\mathbf{P}$ *be an antichain. Then the* $\mathbf{P}$*-zigzags are the coloured fences which are coloured on the two endpoints by different elements of* $\mathbf{P}$.

PROOF: Clearly, every coloured fence of the above type is a $\mathbf{P}$-zigzag. On the other hand, every zigzag has to contain at least two coloured points. Since every $\mathbf{P}$-zigzag is connected every $\mathbf{P}$-zigzag contains a copy of a coloured fence mentioned in the claim. Hence, by the minimality of zigzags we have the claim. □

We remark that the notion of 2-hole defined in [6] concides with the one of zigzag in the case of finite posets admitting a majority function. Moreover, 2.4, 3.2, the proof of 3.1, Proposition 10 and Corollary 11 in [10] yield a proof of a variant of [7, Theorem 1] that every finite poset admitting a majority function is a retract of a finite product of fences and the two element antichain.

By the use of the following theorem we can easily describe the zigzags of certain posets constructed from fences and antichains.

THEOREM 3.3. *Let* $\mathbf{P}$ *be a finite poset and let* $\mathbf{A}$ *be a finite poset with the strong selection property. Let* $\mathbf{P}' = \mathbf{P}+\mathbf{A}$. *Then every monotone* $\mathbf{P}'$*-zigzag is one of the following form: a* $\mathbf{P}$*-zigzag in which every maximal element is coloured, an* $\mathbf{A}$*-zigzag in which every minimal element is coloured or it can be obtained from a monotone* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$, *such that above each noncoloured maximal element of* $(\mathbf{H}, f)$ *we place the coloured elements of some* $\mathbf{A}$*-zigzag having a noncoloured minimal element. Moreover, every* $\mathbf{P}'$*-coloured poset of this form will be a* $\mathbf{P}'$*-zigzag.*

PROOF: First, we prove the last claim. Let $(\mathbf{H}', f')$ be a $\mathbf{P}'$-coloured poset of the above form. If $(\mathbf{H}', f')$ is one of the first two forms, then the claim is obvious. So let us suppose that $(\mathbf{H}', f')$ is of the third form. Observe that $\mathbf{H}'$ is connected since $\mathbf{H}$ is. The coloured poset $(\mathbf{H}', f')$ is not $\mathbf{P}'$-extendible since any $\mathbf{P}'$ extension of f' restricted to H must lie in P and so it would be a $\mathbf{P}$-extension of f to $\mathbf{H}$. So by Claim 1.1 we only have to show that by deleting any covering pair in $(\mathbf{H}', f')$, the resulting coloured poset is $\mathbf{P}'$-extendible. The only problem occurs if we delete a covering pair (h, h') not in $(\mathbf{H}, f)$. Since h and the coloured elements covering it form an $\mathbf{A}$-zigzag f' can be extended to h in $(\mathbf{H}', f') \setminus \{(h, h')\}$ by an element from $\mathbf{A}$. If we colour the other elements of $N(\mathbf{H}', f')$ by a $\mathbf{P}$-extension of $(\mathbf{H}, f) \setminus \{h\}$ restricted to $N(\mathbf{H}', f')$, we get a $\mathbf{P}'$-extension of $(\mathbf{H}', f') \setminus \{(h, h')\}$.

Let $(\mathbf{H}', f')$ be a monotone $\mathbf{P}'$-zigzag. By an induction on $|N(\mathbf{H}', f')|$ we show that $(\mathbf{H}', f')$ is of the above form. If $|N(\mathbf{H}', f')| = 1$ the claim is obvious from Claim1.11. Let $|N(\mathbf{H}', f')| \geqslant 2$. Let h be an arbitrary maximal element of $\mathrm{N}(\mathbf{H}', f')$. Let $h_1, h_2, \dots, h_l$ denote the upper covers of h in $(\mathbf{H}', f')$. So $h_1, h_2, \dots, h_l \in C(\mathbf{H}', f')$. First we show that f' does not have a monotone extension f'' to h defined by $f''(h) = a$ for some $a \in A$. Let us suppose the contrary. We assume that a is a maximal element in $\mathbf{A}$

for which there exists a monotone extension f'' of f' to h with $f''(h) = a$. Since the coloured poset $(\mathbf{H}', f'')$ is nonextendible it contains a monotone $\mathbf{P}'$-zigzag $(\mathbf{Q}, g)$. By the minimality of $(\mathbf{H}', f')$ we get $h \in Q$. Let h' be a lower cover of h in $\mathbf{Q}$. So $h' \in N(\mathbf{Q}, g)$. Let h' be covered by the elements $h = s_1, s_2, \ldots, s_m$ in $(\mathbf{Q}, g)$. By the induction hypothesis each s_i, $i = 1, 2, \ldots, m$, must be coloured from A so that the first coloured poset in Figure 4 is an A-zigzag. By Claim 1.1 it is easy to check that second coloured poset in Figure 4 is an A-zigzag, too. But this contradicts the fact that A has the strong selection property.

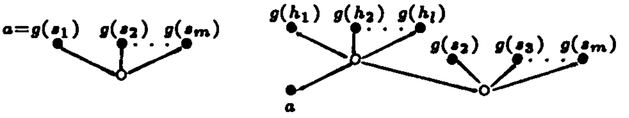

Figure 4. Two A-coloured posets

Now, let us suppose that $(\mathbf{H}', f')$ has an element h' that is not maximal and coloured from $\mathbf{A}$. Since $(\mathbf{H}', f')$ is monotone we must have a maximal element $h > h'$ in $\mathrm{N}(\mathbf{H}', f')$. Because $|N(\mathbf{H}', f')| \geqslant 2$, f' is extendible to h by $f''(h) = a$ for some $a \in A$. This is impossible as we saw in the preceding paragraph. So every A-coloured element in $(\mathbf{H}', f')$ is maximal. Let now h' be an element covered by a A-coloured element in $(\mathbf{H}', f')$. Let us suppose that there is an element $h \in N(\mathbf{H}', f')$ with $h' \prec h$. By the minimality of $(\mathbf{H}', f')$ there must be an element $a \in P'$ such that, if we colour h by a in $(\mathbf{H}', f')$ then the resulting coloured poset is monotone and it contains a monotone zigzag in which $h' \prec h$. By the induction hypothesis, Claim 1.5 and Claim 1.9, $a \in A$. This contradicts the claim in the preceding paragraph. So any A-coloured element covers only maximal elements of $\mathrm{N}(\mathbf{H}', f')$ in $(\mathbf{H}', f')$. Also, by the claim in the preceding paragraph every maximal element of $(\mathbf{H}', f')$ must be coloured. If every maximal element of $(\mathbf{H}', f')$ is coloured from $\mathbf{P}$ then $(\mathbf{H}', f')$ is a P-zigzag and it is of the desired form. For otherwise let h be a maximal element of $\mathrm{N}(\mathbf{H}', f')$ such that there is an upper cover of h coloured from A. Then by Claim 1.5 every upper cover of h must be coloured from A. By the claim in the preceding paragraph the coloured subposet determined by h and its upper covers in $(\mathbf{H}', f')$ is an A-zigzag. By deleting the A-coloured elements from $(\mathbf{H}', f')$ we get a connected P-coloured poset $(\mathbf{H}, f)$. Since $(\mathbf{H}', f')$ is $\mathbf{P}'$-nonextendible $(\mathbf{H}, f)$ is P-nonextendible. By deleting a covering pair in $(\mathbf{H}, f)$ we get a P-extendible coloured poset since, if we delete the same edge in $(\mathbf{H}', f')$ then any monotone extension of f' restricted to $N(\mathbf{H}', f')$ must have its range in P. So by Claim 1.1, $(\mathbf{H}, f)$ is a P-zigzag. Thus $(\mathbf{H}', f')$ is of the desired form. ∎

We can use the above theorem, its dual and the previous propositions to obtain the zigzags of certain posets like the famous poset $\mathbf{1{+}2{+}2{+}2{+}1}$, or $\mathbf{1{+}2{+}P{+}2{+}1}$, where $\mathbf{P}$ is a fence. The latter posets are called *locked fences*. In [5] McKenzie leaves it to the

https://doi.org/10.1017/S0004972700012284 Published online by Cambridge University Press

reader that locked fences admit near unanimity functions. In the case of a four element fence a 7-nuf is given in [8]. In general, it follows from Proposition 3.1, Theorem 3.3 and Remark 1.13 that a locked fence with $|P| = n \geqslant 2$ admits a $(2n-1)$-nuf and $2n-1$ is the smallest possible arity.

To illustrate the claims in this section we present some examples. In Figure 5 we have two posets with all of their monotone zigzags. These posets are obtained from the four element fence in such a way that Theorem 3.3 and its dual apply. Since they have finitely many zigzags they admit a near unanimity function.

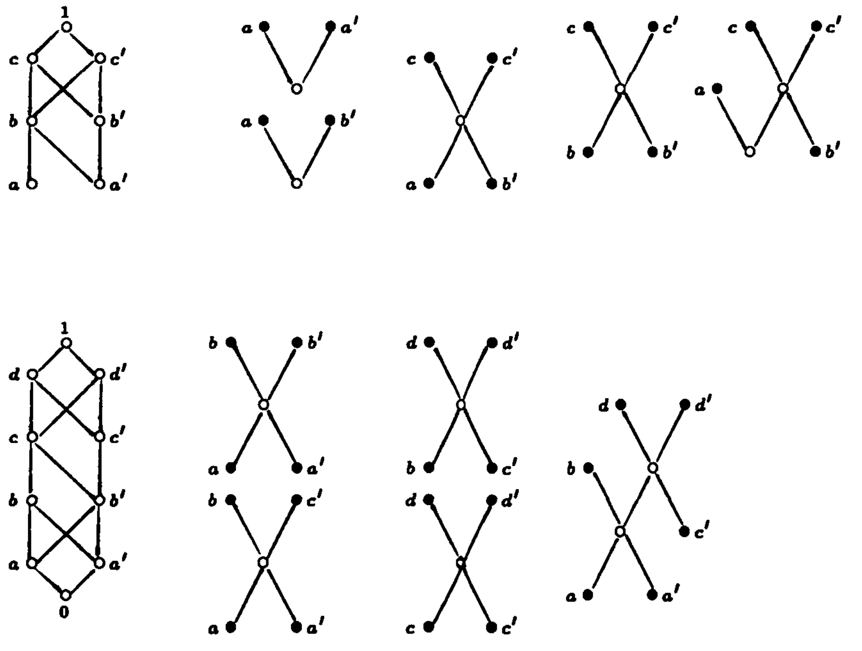

Figure 5. Posets admitting a nuf and their monotone zigzags

In Figure 6 we have two posets constructed from the two element antichain. All of their zigzags can be determined by applying Theorem 3.3 and its dual. They have infinitely many zigzags. In fact, they admit no near unanimity function.

The poset $\mathbf{P}$ in Figure 7 is different. We need Lemma 2.1 and Proposition 2.3 to prove that the list of the monotone zigzags in Figure 7 is complete. By the argument below Figure 7 we show that $\{x,y\} = \{a,a'\}$ and $\{v,z\} = \{d,d'\}$. There are sixteen possibilities for $(\{p,q\},\{r,s\})$. If $\{p,q\} = \{a,a'\}$, then $\{r,s\}$ is $\{b,b'\}$, $\{b,c\}$, $\{c',c\}$ or $\{c',b'\}$. Dually, if $\{r,s\} = \{d,d'\}$, then $\{p,q\}$ is $\{c,c''\}$, $\{c,b\}$, $\{b'',b\}$ or $\{b'',c''\}$. If $(p,r) = (b'',c')$, then (q,s) is (b,d), (b,d'), (a,d), (a,d'), (a,c), (a',d), (a',d') or (a',c). For (u,t) there are four possibilities, (u,t) equals (c,b), (c,c'), (b'',b) or (b'',c').

https://doi.org/10.1017/S0004972700012284 Published online by Cambridge University Press

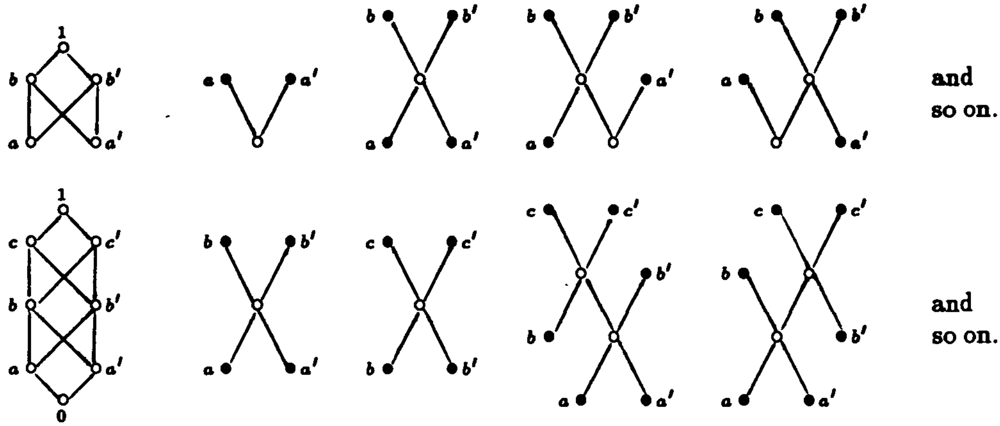

Figure 6. Posets admitting no nuf and their monotone zigzags

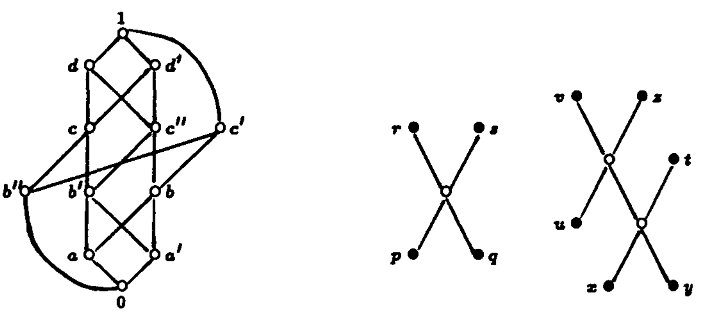

Figure 7. A poset with its monotone zigzags

First of all, it is easy to get all zigzags with exactly one noncoloured point by using (2) of Claim 1.11. To obtain the zigzags with exactly two noncoloured elements we use Lemma 2.1 and (3) of Claim 1.11. Let $(\mathbf{H}, f)$ be a zigzag with exactly two noncoloured elements. We apply Lemma 2.1 to $(\mathbf{H}, f)$ and its two noncoloured elements. Then $(\mathbf{H}, f)$ must contain a coloured poset $(\mathbf{Q}, g)$ with two noncoloured elements of the form in Figure 7, where there is a monotone extension of g to each noncoloured point such that the resulting coloured poset contains a zigzag with one noncoloured element of the form in Figure 7. So by using the list of zigzags with exactly one noncoloured element we have that $\{r, s\} = \{d, d'\}$, $\{p, q\} = \{b, b''\}$ or $\{p, q\} = \{b, c\}$ for one of the zigzags and $\{p, q\} = \{a, a'\}$, $\{r, s\} = \{c, c'\}$ or $\{r, s\} = \{c, b\}$ for the other zigzag. From these we get the four zigzags with exactly two noncoloured elements as listed above.

We show that there are no other monotone zigzags. Let us suppose there exists a zigzag with more than two noncoloured elements. Then let $(\mathbf{H}, f)$ be a zigzag such that $|N(\mathbf{H}, f)|$ is minimal with $|N(\mathbf{H}, f)| > 2$. By applying Proposition 2.3 for a maximal noncoloured point h in $N(\mathbf{H}, f)$, there exist an index set I and $h \in (\mathbf{H}_i, f_i) \subseteq (\mathbf{H}, f)$ and $p_i \in P$, $i \in I$, with the properties given in 2.3. Let $(\mathbf{G}_i, g_i)$ be the zigzag obtained

https://doi.org/10.1017/S0004972700012284 Published online by Cambridge University Press

by colouring h by p_i in $(\mathbf{H}_i, f_i)$, $i \in I$. First we examine the case, when there is a $j \in I$ such that $(\mathbf{G}_j, g_j)$ has two noncoloured points. Since h is maximal in $(\mathbf{G}_j, g_j)$, by looking at the above list of the zigzags with two noncoloured elements, p_j must be one of d, d', c' and b. The first three choices are not possible because, by claims 1.9 and 1.10, h is covered by at least two coloured elements in $(\mathbf{H}, f)$. By Claim 1.5 these elements must be coloured by incomparable elements that, by 2.3, are greater than p_j, which is impossible. So p_j must be b. Hence, by 2.3, only $(\mathbf{G}_j, g_j)$ can have two noncoloured elements.

Let $(\mathbf{G}_i, g_i)$ be a zigzag with one noncoloured element. In $(\mathbf{H}_i, f_i)$ colouring h by b we get an extendible coloured poset by 2.3. Now, from the above list of zigzags with one noncoloured element $(\mathbf{G}_i, g_i)$ must be one of the four zigzags with $\{p, q\} = \{a, a'\}$. Similarly, a nonmonotone zigzag $(\mathbf{G}_i, g_i)$ in which h is maximal, the minimal element is coloured by one of b, a and a'. A nonmonotone zigzag $(\mathbf{G}_i, g_i)$ in which h is minimal, the maximal element must be coloured by d or d'. Here b and c' are not possible otherwise $(\mathbf{H}, f)$ would contain a zigzag with exactly two noncoloured elements. Also c'' is not possible since h is covered by two elements coloured by incomparable elements in $(\mathbf{H}, f)$. So we have determined the list of all possible $(\mathbf{H}_i, f_i)$. Observe that any of these $(\mathbf{H}_i, f_i)$ is extendible by a map that colours h by c''. This contradicts 2.3.

So we can assume that each $(\mathbf{H}_i, f_i)$ contains at most one noncoloured element. In each $(\mathbf{G}_i, g_i)$, h is maximal and it is covered by at least two coloured elements in $(\mathbf{H}, f)$ with incomparable colours. So, by 2.3 and the above list of zigzags with exactly one noncoloured element, the colour of h must be one of b, b' and c. All three cannot occur together, since, by 2.3, these elements have to be incomparable. So we have b, c or b, b' as possible colours of h in two of the $(\mathbf{G}_i, g_i)$. But then in a nonmonotone $(\mathbf{G}_i, g_i)$, where h is minimal, the maximal element must be coloured by d or d'. There does not exist a nonmonotone $(\mathbf{G}_i, g_i)$, where h is maximal since the colour of its minimal element would be smaller than b, c or b, b'. So it would be a or a'. But both a and a' occur as the colour of a minimal element in one of the $(\mathbf{G}_i, g_i)$ with one noncoloured element. Then by sticking together the possible, and at the same time necessary, $(\mathbf{H}_i, f_i)$ at h we get a zigzag $(\mathbf{H}', f')$ with exactly three noncoloured elements.

Let us apply Proposition 2.3 to a minimal noncoloured element in $(\mathbf{H}', f')$. Observe that one of the resulting $(\mathbf{G}_i, g_i)$ must have two noncoloured elements. By the dual of the first part of this proof this is impossible. So there is no zigzag with more than two noncoloured elements.

References

[1] B. A. Davey, 'Monotone clones and congruence modularity', *Order* 6 (1990), 389–400.

[2] B. A. Davey, R. W. Quackenbush and D. Schweigert, 'Monotone clones and the varieties they determine', *Order* **7** (1990), 145–167.

[3] J. Demetrovics, L. Hannák and L. Rónyai, 'Near unanimity functions of partial orders', in *Proc. 14* (ISMVL, Manitoba, 1984), pp. 52–56.

[4] D. Duffus and I. Rival, 'A structure theory for ordered sets', *Discrete Math.* **35** (1981), 53 –118.

[5] R. McKenzie, 'Algebraic properties of monotone clones: residual smallness and congruence distributivity', *Bull. Austral. Math. Soc.* **41** (1990), 283–300.

[6] P. Nevermann and I. Rival, 'Holes in ordered sets', *Graphs Combin.* **1** (1985), 339–350.

[7] P. Nevermann and R. Wille, 'The strong selection property and ordered sets of finite length', *Algebra Universalis* **18** (1984), 18–28.

[8] R. Quackenbush, I. Rival and I.G. Rosenberg, 'Clones, order varieties, near unanimity functions and holes', *Order* **7** (1990), 239–248.

[9] G. Tardos, 'A maximal clone of monotone operations which is not finitely generated', *Order* **3** (1986), 211–218.

[10] L. Zadori, 'Order varieties generated by finite posets', *Order* **8** (1992), 341–348.

[11] L. Zadori, 'Monotone Jónsson operations and near unanimity functions', (submitted).

József Attila Tudom-anyegyetem
Bolyai Intézet
Aradi vértanúk tere 1
H-6720 Szeged
Hungary