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# Invariant measures and large deviation principles for stochastic Schrödinger delay lattice systems

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This paper is concerned with stochastic Schrödinger delay lattice systems with both locally Lipschitz drift and diffusion terms. Based on the uniform estimates and the equicontinuity of the segment of the solution in probability, we show the tightness of a family of probability distributions of the solution and its segment process, and hence the existence of invariant measures on $l^2 \times L^2((-\rho, 0); l^2)$ with $\rho > 0$. We also establish a large deviation principle for the solutions with small noise by the weak convergence method.

*Keywords:* Stochastic Schrödinger lattice system; time delay; invariant measure; tightness; large deviation principle; Laplace principle; weak convergence

2020 *Mathematics Subject Classification:* 37L40; 37L55; 60F10; 60H10

## 1. Introduction

Stochastic lattice systems can be used to model many practical systems with discrete character and random fluctuation. The long-time dynamics for stochastic lattice systems with or without delays have been investigated extensively in the literature. For stochastic lattice systems without delays, we refer the reader to [**2**, **3**, **9**, **20**] for pathwise random attractors and stability, [**31**, **32**, **34**, **36**] for weak mean random attractors and invariant measures. Since the current states of the practical systems often depend on their past history, stochastic lattice systems with delays have been investigated; see e.g., [**12**, **15**, **16**, **24**] for invariant measures and weak mean random attractors, and [**23**] for periodic measures. Recently, regime-switching was taken account into stochastic lattice systems, and invariant measures of such systems were studied in [**13**, **22**].

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In this paper we consider the following stochastic Schrödinger delay lattice system defined on the integer set $\mathbb{Z}$:

$$\begin{cases} \mathrm{d}u_n^\varepsilon(t) + i|u_n^\varepsilon(t)|^2 u_n^\varepsilon(t)\mathrm{d}t + \lambda u_n^\varepsilon(t)\mathrm{d}t - i(u_{n-1}^\varepsilon(t) - 2u_n^\varepsilon(t) + u_{n+1}^\varepsilon(t))\mathrm{d}t \\ = f_n(u_n^\varepsilon(t-\rho))\mathrm{d}t + g_n\mathrm{d}t \\ \quad + \sqrt{\varepsilon}\sum\limits_{k\in\mathbb{N}} (h_{k,n} + \sigma_{k,n}(u_n^\varepsilon(t-\rho)))\,\mathrm{d}W_k(t),\ t>0, \\ u_n^\varepsilon(0) = u_n^0,\ u_n^\varepsilon(s) = \xi_n(s),\ s\in(-\rho,0), \end{cases} \tag{1.1}$$

where $n \in \mathbb{Z}$; $\varepsilon \in (0, 1)$; $\lambda$ and $\rho$ are positive constants; $g = (g_n)_{n\in\mathbb{Z}}$ and $h_k = (h_{k,n})_{n\in\mathbb{Z}}$ are deterministic complex-valued sequences for each $k \in \mathbb{N}$; $f_n$ and $\sigma_{k,n}$ are locally Lipschitz continuous functions for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$, and $\{W_k\}_{k\in\mathbb{N}}$ are independent two-sided real-valued standard Wiener processes on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$.

The first goal of this paper is to investigate the existence of invariant measures of the stochastic Schrödinger delay lattice system (1.1) in $l^2 \times L^2((-\rho, 0); l^2)$. To that end, we need to establish the tightness of a family of distributions of the solution and its segment process of (1.1) in $l^2 \times L^2((-\rho, 0); l^2)$. Actually, such tightness can be obtained by proving the uniform tail-estimates, the uniform estimates of higher-order moments and the Hölder continuity of the solution as in [**15**]. Note that the derivation of uniform estimates of higher-order moments requires not only sophisticated calculations, but also strong dissipativeness assumptions on the nonlinear terms. In order to relax the strong dissipativeness restrictions and prove the existence of invariant measures under weaker conditions on the nonlinear terms, in the present paper, we will employ the equicontinuity of the segment of the solution in probability, instead of uniform estimates of higher-order moments, to establish the tightness of distributions of the solution and its segment process. The idea of equicontinuity in probability was used for proving the tightness of the segment of the solution in [**4**, **37**] for finite-dimensional stochastic ordinary differential equations and in [**14**] for fractional stochastic partial differential equations. In the present paper, we will use this method to deal with the infinite-dimensional lattice system (1.1).

The second goal of the paper is to investigate the large deviation principle (LDP) of the solutions of (1.1) on a finite interval $[0, T]$ with $T > 0$ by the weak convergence method. The weak convergence method is based on the variational representation of certain functionals of Brownian motion [**5**, **7**, **8**] as well as the equivalence of large deviation principles and Laplace principles. Compared with the classical discretization method as introduced in [**19**], the weak convergence method does not require any exponential-type probability estimates which are usually difficult to derive for infinite-dimensional models. The weak convergence method has been successfully applied to establish the LDP for many infinite-dimensional stochastic systems, see e.g. [**6**, **8**, **10**, **11**, **25**, **28**, **29**, **35**] for stochastic partial differential equations, and [**33**] for stochastic reaction-diffusion lattice systems without delay. We refer the reader to [**19**] and [**18**] for more details on the discretization method and the weak convergence method for LDPs, respectively.

Note that the LDPs of finite-dimensional stochastic delay differential equations have been studied by many authors, see e.g. [**1**, **21**, **27**] for constant delay and

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[26, 30] for general delay. However, to the best of our knowledge, there is no result available regarding the LDPs of infinite-dimensional delay lattice systems. We will close this gap and prove the LDP for the infinite-dimensional delay lattice system (1.1) in the last section of the paper. Compared with the finite-dimensional stochastic delay differential equations [21] the main difficulty to verify the conditions of the weak convergence for system (1.1) lies in the fact that bounded subsets of $\ell^2$ are not precompact. To deal with this issue, we adopt the idea of the finite-dimensional projection and the uniform tail-ends estimates to establish the precompactness of a family of solutions to the controlled system (4.5). The argument of the present paper can be extended to the path-dependent lattice systems driven by superlinear noise under certain conditions.

The paper is organized as follows. In Section 2, we discuss the assumptions on the nonlinear terms and present our main results. In the last two sections, we prove the existence of invariant measures and the LDP of (1.1), respectively.

For convenience, we will use $L^2(I; H)$ to denote the space of all square-integrable functions from an interval $I$ to a separable Hilbert space $H$ equipped with norm $\|\cdot\|_{L^2(I;H)}$. We also use $C(I; H)$ for the space of all continuous functions from $I$ to $H$ equipped with supremum norm $\|\cdot\|_{C(I;H)}$. As usual, we reserve $l^2$ for the space of all complex-valued square-summable sequences with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, respectively.

## 2. Assumptions and main results

In this section, we discuss the assumptions on the nonlinear terms in (1.1), and present the main results of the paper. First, we define the linear operators $A$, $B$, $B^* : l^2 \to l^2$ by:

$$(Au)_n = -u_{n-1} + 2u_n - u_{n+1}, \quad (Bu)_n = u_{n+1} - u_n, \quad (B^*u)_n = u_{n-1} - u_n,$$

for any $n \in \mathbb{Z}$ and $u = (u_n)_{n\in\mathbb{Z}} \in l^2$. Then we have

$$A = BB^* = B^*B, \quad (B^*u, v) = (u, Bv), \quad \forall\ u, v \in l^2.$$

Throughout the paper we make the following assumptions.

**(A1)** For any bounded subset $\mathcal{K}$ of $\mathbb{C}$, there exists a positive constant $L_{\mathcal{K}}$ such that

$$|f_n(z_1) - f_n(z_2)| \leqslant L_{\mathcal{K}}|z_1 - z_2|,$$

for any $z_1$, $z_2 \in \mathcal{K}$ and $n \in \mathbb{Z}$.

**(A2)** For every $k \in \mathbb{N}$, $n \in \mathbb{Z}$ and every bounded subset $\mathcal{K}$ of $\mathbb{C}$, there exists a positive constant $L_{k,n,\mathcal{K}}$ such that for any $z_1$, $z_2 \in \mathcal{K}$,

$$|\sigma_{k,n}(z_1) - \sigma_{k,n}(z_2)| \leqslant L_{k,n,\mathcal{K}}|z_1 - z_2|,$$

where $L_{\mathcal{K}} = (L_{k,n,\mathcal{K}})_{k\in\mathbb{N},\, n\in\mathbb{Z}} \in l^2$.

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**(A3)** For any $n \in \mathbb{Z}$, there exist positive constants $\alpha_n$ and $\beta_0$ such that

$$|f_n(z)| \leqslant \beta_0 |z| + \alpha_n, \quad \forall\, z \in \mathbb{C},$$

where $\|\alpha\|^2 := \sum_{n\in\mathbb{Z}} |\alpha_n|^2 < \infty$.

**(A4)** For every $k \in \mathbb{N}$, $n \in \mathbb{Z}$, there exist positive constants $\delta_{k,n}$ and $\beta_k$ such that

$$|\sigma_{k,n}(z)| \leqslant \delta_{k,n} + \beta_k |z|, \quad \forall\, z \in \mathbb{C},$$

where $\|\delta\|^2 := \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{Z}} |\delta_{k,n}|^2 < \infty$, $\|\beta\|^2 := \sum_{k\in\mathbb{N}} |\beta_k|^2 < \infty$.

**(A5)**

$$\|g\|^2 := \sum_{n\in\mathbb{Z}} |g_n|^2 < \infty, \quad \|h\|^2 := \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{Z}} |h_{k,n}|^2 < \infty. \tag{2.1}$$

Consider operators $f$, $\sigma_k : l^2 \to l^2$ defined by

$$f(u) = (f_n(u_n))_{n\in\mathbb{Z}}, \quad \sigma_k(u) = (\sigma_{k,n}(u_n))_{n\in\mathbb{Z}}, \quad \forall\, u = (u_n)_{n\in\mathbb{Z}} \in l^2.$$

Then by assumptions **(A1)**–**(A4)**, we have:

(i) $f$ is well-defined, and

$$\|f(u)\|^2 \leqslant 2\beta_0^2 \|u\|^2 + 2\|\alpha\|^2, \quad \forall\, u \in l^2. \tag{2.2}$$

(ii) $f$ is locally Lipschitz continuous; that is, for every $R > 0$, there exists a positive constant $L_R^f$ such that for all $u$, $v \in l^2$ with $\|u\| \vee \|v\| \leqslant R$,

$$\|f(u) - f(v)\|^2 \leqslant L_R^f \|u - v\|^2. \tag{2.3}$$

(iii) $\sigma_k$ is well-defined and

$$\sum_{k\in\mathbb{N}} \|\sigma_k(u)\|^2 \leqslant 2\|\beta\|^2 \|u\|^2 + 2\|\delta\|^2, \quad \forall\, u \in l^2. \tag{2.4}$$

(iv) $\sigma_k$ is locally Lipschitz continuous; more precisely, for every $R > 0$, there exists a positive constant $L_R^\sigma$ such that for all $u$, $v \in l^2$ with $\|u\| \vee \|v\| \leqslant R$,

$$\sum_{k\in\mathbb{N}} \|\sigma_k(u) - \sigma_k(v)\|^2 \leqslant L_R^\sigma \|u - v\|^2. \tag{2.5}$$

With the above notation, problem (1.1) can be rewritten as the following form in $l^2$:

$$\begin{cases} \mathrm{d}u^\varepsilon(t) + i|u^\varepsilon(t)|^2 u^\varepsilon(t)\,\mathrm{d}t + \lambda u^\varepsilon(t)\,\mathrm{d}t + iAu^\varepsilon(t)\,\mathrm{d}t \\ = f(u^\varepsilon(t-\rho))\mathrm{d}t + g\,\mathrm{d}t + \sqrt{\varepsilon} \sum\limits_{k\in\mathbb{N}} (h_k + \sigma_k(u^\varepsilon(t-\rho)))\,\mathrm{d}W_k(t), \ t > 0, \\ u^\varepsilon(0) = u^0, \ u^\varepsilon(s) = \xi(s), \ s \in (-\rho, 0), \end{cases} \tag{2.6}$$

where $u^0 = (u_n^0)_{n\in\mathbb{Z}}$, $|u^\varepsilon(t)|^2 u^\varepsilon(t) = (|u_n^\varepsilon(t)|^2 u_n^\varepsilon(t))_{n\in\mathbb{Z}}$, $g = (g_n)_{n\in\mathbb{Z}}$, $h_k = (h_{k,n})_{n\in\mathbb{Z}}$ and $\xi = (\xi_n)_{n\in\mathbb{Z}}$.

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From now on, we denote the segment of $u^\varepsilon$ by $u^\varepsilon_t$ which is defined by

$$u^\varepsilon_t(s) = u^\varepsilon(t+s), \quad \forall\ s \in (-\rho, 0).$$

Under conditions **(A1)**–**(A5)**, for every $u^0 \in L^2(\Omega, \mathcal{F}_0; l^2)$ and $\xi \in L^2(\Omega, \mathcal{F}_0; L^2((-\rho,\, 0); l^2))$, system (2.6) admits a unique solution $u^\varepsilon$ (see [**15**, Theorem 2.2]) in the sense that $u^\varepsilon(t)$, $t \geqslant -\rho$, is an $l^2$-valued stochastic process such that

- $u^\varepsilon(t)$ for $t \geqslant 0$ is pathwise continuous and $\mathcal{F}_t$-adapted.
- $u^\varepsilon(0) = u^0$, $u^\varepsilon_0 = \xi$ and $u^\varepsilon \in L^2(\Omega; C([0,\, T]; l^2))$ for all $T > 0$.
- For $t \geqslant 0$, $\mathbb{P}$-almost surely,

$$u^\varepsilon(t) = u^0 + \int_0^t \left(-iAu^\varepsilon(s) - i|u^\varepsilon(s)|^2 u^\varepsilon(s) - \lambda u^\varepsilon(s) + f(u^\varepsilon(s-\rho)) + g\right) \mathrm{d}s \\ + \sqrt{\varepsilon} \sum_{k\in\mathbb{N}} \int_0^t \left(h_k + \sigma_k(u^\varepsilon(s-\rho))\right) \mathrm{d}W_k(s) \quad \text{in } l^2.$$

Moreover, one can verify that for every $T > 0$,

$$\mathbb{E}\left[\|u^\varepsilon\|^2_{C([0,T];l^2)}\right] \leqslant M_0 e^{M_0 T} \left(\mathbb{E}[\|u^0\|^2] + \int_{-\rho}^0 \mathbb{E}[\|\xi(s)\|^2]\,\mathrm{d}s + T\|g\|^2 + T\|h\|^2\right), \tag{2.7}$$

where $M_0$ is a positive constant independent of $u^0$, $\xi$ and $T$.

Based on the well-posedness of solutions of (2.6), we will prove the existence of invariant measures. For this purpose, we need an additional assumption as follows: **(H)** $\sqrt{2}\beta_0 + 2\|\beta\|^2 < \lambda$.

Theorem 2.1. *Suppose that **(A1)**–**(A5)** and **(H)** hold. Then* (2.6) *has an invariant measure on $l^2 \times L^2((-\rho,\, 0); l^2)$.*

Remark 2.2. Compared with [**15**, Theorem 4.1], the conditions on the nonlinear drift and the nonlinear diffusion terms are relaxed due to the fact that the uniform estimates of higher-order moments of solutions are not required in this paper.

Given $(u^0,\, \xi) \in l^2 \times L^2((-\rho,\, 0); l^2)$ and a positive constant $T$, we will prove the LDP for the family of solutions $\{u^\varepsilon\}$ of (2.6) on the finite time interval $[0,\, T]$ as $\varepsilon \to 0$, which is given below.

Theorem 2.3. *Suppose that **(A1)**–**(A5)** hold. Then the family of solutions $\{u^\varepsilon\}$ of system* (2.6) *on $[0,\, T]$, as $\varepsilon \to 0$, satisfies the large deviation principle on $C([0,\, T]; l^2)$ with the good rate function $I : C([0,\, T]; l^2) \to [0,\, \infty]$ defined by* (4.1).

Remark 2.4. We point out that theorem 2.1 and theorem 2.3 still hold with minor changes in the proofs if we replace the cubic term $i|u_n|^2 u_n$ in (1.1) by a more general nonlinear term $\pm iF(|u_n|)u_n$, where $F : [0,\, \infty] \to \mathbb{R}$ is continuous, $F(0) = 0$, and there exist $L_F > 0$ and $v \geqslant 0$ such that

$$|F(|z_1|)z_1 - F(|z_2|)z_2| \leqslant L_F(|z_1|^v + |z_2|^v)|z_1 - z_2|, \quad \forall\ z_1, z_2 \in \mathbb{C}.$$

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## 3. Invariant measures

In this section, we prove the existence of invariant measures of (2.6). To that end, we need to derive the uniform estimates of the solution as well as its segment process in the next subsection.

### 3.1. Uniform estimates

In this subsection, we firstly establish the uniform estimates of the solution of (2.6). Note that by **(H)**, there exist constants $\alpha_1 > 0$ and $\gamma > 0$ such that

$$\sqrt{2}\beta_0(1 + e^{\gamma\rho}) - 2\lambda + \alpha_1 + \gamma + 4\|\beta\|^2 e^{\gamma\rho} < 0. \tag{3.1}$$

LEMMA 3.1. *Suppose that **(A1)**–**(A5)** and **(H)** hold. Then for any $(u^0, \xi) \in L^2(\Omega; l^2) \times L^2(\Omega; L^2((-\rho, 0); l^2))$, the solution $u^\varepsilon$ of (2.6) satisfies that for all $t \geqslant 0$ and $0 < \varepsilon < 1$,*

$$\mathbb{E}[\|u^\varepsilon(t)\|^2] \leqslant M_1 \left(1 + \mathbb{E}[\|u^0\|^2] + \int_{-\rho}^0 \mathbb{E}[\|\xi(s)\|^2] \,\mathrm{d}s\right), \tag{3.2}$$

*where $M_1$ is a positive constant independent of $u^0$, $\xi$ and $\varepsilon$.*

*Proof.* Applying Itô's formula to (2.6), we obtain for all $t \geqslant 0$ and $0 < \varepsilon < 1$,

$$\begin{aligned} d(\|u^\varepsilon(t)\|^2) \leqslant & - 2\lambda\|u^\varepsilon(t)\|^2 \,\mathrm{d}t + 2\mathrm{Re}\ (u^\varepsilon(t), f(u^\varepsilon(t-\rho))) \,\mathrm{d}t \\ & + 2\mathrm{Re}\ (u^\varepsilon(t), g) \,\mathrm{d}t + \varepsilon \sum_{k\in\mathbb{N}} \|h_k + \sigma_k(u^\varepsilon(t-\rho))\|^2 \,\mathrm{d}t \\ & + 2\sqrt{\varepsilon}\mathrm{Re} \sum_{k\in\mathbb{N}} (u^\varepsilon(t), h_k + \sigma_k(u^\varepsilon(t-\rho))) \,\mathrm{d}W_k(t). \end{aligned} \tag{3.3}$$

Let $\gamma > 0$ be the positive constant satisfying (3.1). Then we get from (3.3) that

$$\begin{aligned} e^{\gamma t}\mathbb{E}[\|u^\varepsilon(t)\|^2] \leqslant & \mathbb{E}[\|u^0\|^2] + (\gamma - 2\lambda) \int_0^t e^{\gamma s}\mathbb{E}[\|u^\varepsilon(s)\|^2] \,\mathrm{d}s \\ & + 2\mathrm{Re}\ \int_0^t e^{\gamma s}\mathbb{E}\left[(u^\varepsilon(s), f(u^\varepsilon(s-\rho)))\right] \,\mathrm{d}s \\ & + 2\mathrm{Re}\ \int_0^t e^{\gamma s}\mathbb{E}\left[(u^\varepsilon(s), g)\right] \,\mathrm{d}s \\ & + \varepsilon \sum_{k\in\mathbb{N}} \int_0^t e^{\gamma s}\mathbb{E}\left[\|h_k + \sigma_k(u^\varepsilon(s-\rho))\|^2\right] \,\mathrm{d}s. \end{aligned} \tag{3.4}$$

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We now deal with the right-hand side of (3.4). For the third term on the right-hand side of (3.4), by Young's inequality and (2.2) we have

$$
\begin{aligned}
&2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[(u^\varepsilon(s), f(u^\varepsilon(s-\rho)))\right] \,\mathrm{d}s \\
&\quad\leqslant 2\int_0^t e^{\gamma s}\mathbb{E}\left[\|u^\varepsilon(s)\|\|f(u^\varepsilon(s-\rho))\|\right] \,\mathrm{d}s \\
&\quad\leqslant \sqrt{2}\beta_0\int_0^t e^{\gamma s}\mathbb{E}\left[\|u^\varepsilon(s)\|^2\right] \,\mathrm{d}s + \frac{1}{\sqrt{2}\beta_0}\int_0^t e^{\gamma s}\mathbb{E}\left[\|f(u^\varepsilon(s-\rho))\|^2\right] \,\mathrm{d}s \\
&\quad\leqslant \sqrt{2}\beta_0\int_0^t e^{\gamma s}\mathbb{E}\left[\|u^\varepsilon(s)\|^2\right] \mathrm{d}s + \frac{\sqrt{2}\|\alpha\|^2}{\gamma\beta_0}(e^{\gamma t}-1) \\
&\qquad + \sqrt{2}\beta_0 e^{\gamma\rho}\int_{-\rho}^{t-\rho} e^{\gamma s}\mathbb{E}\left[\|u^\varepsilon(s)\|^2\right] \mathrm{d}s \\
&\quad\leqslant \sqrt{2}\beta_0(1+e^{\gamma\rho})\int_0^t e^{\gamma s}\mathbb{E}\left[\|u^\varepsilon(s)\|^2\right] \mathrm{d}s \\
&\qquad + \sqrt{2}\beta_0 e^{\gamma\rho}\int_{-\rho}^0 \mathbb{E}\left[\|\xi(s)\|^2\right] \mathrm{d}s + \frac{\sqrt{2}\|\alpha\|^2}{\gamma\beta_0}e^{\gamma t}.
\end{aligned} \tag{3.5}
$$

Let $\alpha_1 > 0$ be a constant satisfying (3.1). By Young's inequality we get

$$
2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[(u^\varepsilon(s), g)\right] \,\mathrm{d}s \leqslant \alpha_1\int_0^t e^{\gamma s}\mathbb{E}\left[\|u^\varepsilon(s)\|^2\right] \,\mathrm{d}s + \frac{\|g\|^2}{\alpha_1\gamma}e^{\gamma t}. \tag{3.6}
$$

For the last term on the right-hand side of (3.4), by (2.4) we obtain

$$
\begin{aligned}
&\varepsilon\sum_{k\in\mathbb{N}}\int_0^t e^{\gamma s}\mathbb{E}[\|h_k+\sigma_k(u^\varepsilon(s-\rho))\|^2]\,\mathrm{d}s \\
&\quad\leqslant 2\sum_{k\in\mathbb{N}}\int_0^t e^{\gamma s}\mathbb{E}[\|h_k\|^2]\,\mathrm{d}s + 2\sum_{k\in\mathbb{N}}\int_0^t e^{\gamma s}\mathbb{E}[\|\sigma_k(u^\varepsilon(s-\rho))\|^2]\,\mathrm{d}s \\
&\quad\leqslant \frac{2\|h\|^2}{\gamma}e^{\gamma t} + \frac{4\|\delta\|^2}{\gamma}e^{\gamma t} + 4\|\beta\|^2\int_0^t e^{\gamma s}\mathbb{E}\left[\|u^\varepsilon(s-\rho)\|^2\right]\mathrm{d}s \\
&\quad\leqslant \frac{2\|h\|^2}{\gamma}e^{\gamma t} + \frac{4\|\delta\|^2}{\gamma}e^{\gamma t} + 4\|\beta\|^2 e^{\gamma\rho}\int_0^t e^{\gamma s}\mathbb{E}\left[\|u^\varepsilon(s)\|^2\right]\mathrm{d}s \\
&\qquad + 4\|\beta\|^2 e^{\gamma\rho}\int_{-\rho}^0 \mathbb{E}\left[\|\xi(s)\|^2\right]\mathrm{d}s.
\end{aligned} \tag{3.7}
$$

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It follows from (3.4)–(3.7) that for all $t \geqslant 0$,

$$
\begin{aligned}
&e^{\gamma t}\mathbb{E}[\|u^{\varepsilon}(t)\|^2] \\
&\quad\leqslant \mathbb{E}[\|u^0\|^2] + \left[\gamma - 2\lambda + \sqrt{2}\beta_0(1+e^{\gamma\rho}) + \alpha_1 + 4\|\beta\|^2 e^{\gamma\rho}\right]\int_0^t e^{\gamma s}\mathbb{E}[\|u^{\varepsilon}(s)\|^2]\mathrm{d}s \\
&\qquad + (\sqrt{2}\beta_0 + 4\|\beta\|^2)e^{\gamma\rho}\int_{-\rho}^0 \mathbb{E}\left[\|\xi(s)\|^2\right]\,\mathrm{d}s + \frac{\sqrt{2}\|\alpha\|^2}{\gamma\beta_0}e^{\gamma t} \\
&\qquad + \frac{\|g\|^2}{\alpha_1\gamma}e^{\gamma t} + \frac{2\|h\|^2}{\gamma}e^{\gamma t} + \frac{4\|\delta\|^2}{\gamma}e^{\gamma t},
\end{aligned}
$$

which along with (3.1) indicates that for all $t \geqslant 0$,

$$
\begin{aligned}
\mathbb{E}[\|u^{\varepsilon}(t)\|^2] &\leqslant e^{-\gamma t}\mathbb{E}[\|u^0\|^2] + (\sqrt{2}\beta_0 + 4\|\beta\|^2)e^{-\gamma(t-\rho)}\int_{-\rho}^0 \mathbb{E}\left[\|\xi(s)\|^2\right]\,\mathrm{d}s \\
&\quad + \frac{\sqrt{2}\|\alpha\|^2}{\gamma\beta_0} + \frac{\|g\|^2}{\alpha_1\gamma} + \frac{2\|h\|^2}{\gamma} + \frac{4\|\delta\|^2}{\gamma}.
\end{aligned}
$$

This implies (3.2), and thus completes the proof. □

Next we give the uniform estimates of the solution in probability.

LEMMA 3.2. *Suppose that* ***(A1)***–***(A5)*** *and* ***(H)*** *hold. If* $(u^0, \xi) \in L^2(\Omega; l^2) \times L^2(\Omega; L^2((-\rho, 0); l^2))$ *satisfies that* $\mathbb{E}[\|u^0\|^2] \vee \int_{-\rho}^0 \mathbb{E}[\|\xi(s)\|^2]\,\mathrm{d}s \leqslant R$ *for some* $R > 0$, *then for any* $T > 0$ *and* $\varepsilon' > 0$, *there exists a positive constant* $M_2 = M_2(\varepsilon', T, R)$, *independent of* $\varepsilon \in (0, 1)$, *such that*

$$
\mathbb{P}\left(\left\{\sup_{s\in[t,t+T]}\|u^{\varepsilon}(s)\| \leqslant m\right\}\right) \geqslant 1-\varepsilon', \quad \forall\ t \geqslant 0,\ m \geqslant M_2.
$$

*Proof.* For any $t \geqslant \rho$ and $m \in \mathbb{N}$, let

$$
\tau_m^t = \inf\{s \geqslant t : \|u^{\varepsilon}(s)\| > m\},
$$

and we set $\tau_m^t = \infty$ if $\{s \geqslant t : \|u^{\varepsilon}(s)\| > m\} = \emptyset$.

For any $T > 0$, applying Itô's formula to (2.6), we have

$$
\begin{aligned}
&\mathbb{E}[\|u^{\varepsilon}((t+T)\wedge\tau_m^t)\|^2] \\
&\quad\leqslant \mathbb{E}[\|u^{\varepsilon}(t)\|^2] - 2\lambda\int_t^{(t+T)\wedge\tau_m^t}\mathbb{E}[\|u^{\varepsilon}(s)\|^2]\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\mathrm{Re} \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}\left[(u^\varepsilon(s), f(u^\varepsilon(s-\rho)))\right] \mathrm{d}s \\
&+ 2\mathrm{Re} \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}\left[(u^\varepsilon(s), g)\right] \mathrm{d}s \\
&+ \varepsilon \sum_{k\in\mathbb{N}} \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}[\|h_k + \sigma_k(u^\varepsilon(s-\rho))\|^2] \,\mathrm{d}s. 
\end{aligned} \tag{3.8}
$$

For the third term on the right-hand side of (3.8), by (2.2) we obtain

$$
\begin{aligned}
&2\mathrm{Re} \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}\left[(u^\varepsilon(s), f(u^\varepsilon(s-\rho)))\right] \,\mathrm{d}s \\
&\quad \leqslant 2\beta_0^2 \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}[\|u^\varepsilon(s-\rho)\|^2] \,\mathrm{d}s + 2\|\alpha\|^2 T + \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}[\|u^\varepsilon(s)\|^2] \,\mathrm{d}s.
\end{aligned} \tag{3.9}
$$

For the fourth term on the right-hand side of (3.8), by Young's inequality we have

$$
2\mathrm{Re} \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}\big[\big(u^\varepsilon(s), g\big)\big] \,\mathrm{d}s \leqslant \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}[\|u^\varepsilon(s)\|^2] \,\mathrm{d}s + \|g\|^2\, T. \tag{3.10}
$$

For the last term on the right-hand side of (3.8), by (2.4) we get

$$
\begin{aligned}
&\varepsilon \sum_{k\in\mathbb{N}} \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}[\|h_k + \sigma_k(u^\varepsilon(s-\rho))\|^2] \,\mathrm{d}s \\
&\quad \leqslant 2\|h\|^2\, T + 4\|\delta\|^2\, T + 4\|\beta\|^2 \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}[\|u^\varepsilon(s-\rho)\|^2] \,\mathrm{d}s.
\end{aligned} \tag{3.11}
$$

From (3.8)–(3.11) and lemma 3.1, it follows that there exists a positive constant $C_{T,R}$ depending only on $R$ and $T$ such that for all $t \geqslant \rho$,

$$
\begin{aligned}
&\mathbb{E}[\|u^\varepsilon((t+T)\wedge\tau_m^t)\|^2] \\
&\quad \leqslant \mathbb{E}[\|u^\varepsilon(t)\|^2] + (2-2\lambda) \int_t^{(t+T)\wedge\tau_m^t} \mathbb{E}[\|u^\varepsilon(s)\|^2] \,\mathrm{d}s \\
&\qquad + (\|g\|^2 + 4\|\delta\|^2 + 2\|\alpha\|^2 + 2\|h\|^2)T + (2\beta_0^2 + 4\|\beta\|^2) \int_{t-\rho}^{t+T-\rho} \mathbb{E}[\|u^\varepsilon(s)\|^2] \,\mathrm{d}s \\
&\quad \leqslant \left[1 + (2 - 2\lambda + 2\beta_0^2 + 4\|\beta\|^2)T\right] M_1 \left(1 + \mathbb{E}[\|u^0\|^2] + \int_{-\rho}^0 \mathbb{E}[\|\xi(s)\|^2] \,\mathrm{d}s\right) \\
&\qquad + (\|g\|^2 + 4\|\delta\|^2 + 2\|\alpha\|^2 + 2\|h\|^2)T \\
&\quad \leqslant C_{T,R}.
\end{aligned} \tag{3.12}
$$

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Recalling the definition of $\tau_m^t$, we obtain by (3.12) that for all $t \geqslant \rho$,

$$m^2\mathbb{P}(\{\tau_m^t < t+T\}) \leqslant \mathbb{E}\left[\|u^\varepsilon(\tau_m^t)\|^2 I_{\{\tau_m^t<t+T\}}\right] \\ \leqslant \mathbb{E}[\|u^\varepsilon((t+T)\wedge\tau_m^t)\|^2] \leqslant C_{T,R}.$$

Then, we have

$$\mathbb{P}(\{\tau_m^t < t+T\}) \leqslant \frac{C_{T,R}}{m^2}. \tag{3.13}$$

By (3.13) we find that for every $\varepsilon' > 0$, $T > 0$ and $R > 0$, there exists $m_1 = m_1(\varepsilon', T, R) > 0$ such that for $m \geqslant m_1$,

$$\mathbb{P}(\{\tau_m^t < t+T\}) \leqslant \frac{\varepsilon'}{2}, \quad \forall\, t \geqslant \rho,$$

which implies that

$$\mathbb{P}\left(\left\{\sup_{s\in[t,t+T]}\|u^\varepsilon(s)\| > m\right\}\right) \leqslant \frac{\varepsilon'}{2}, \quad \forall\, t \geqslant \rho,\ m \geqslant m_1. \tag{3.14}$$

On the other hand, making use of (2.7) and the Chebyshev inequality, we obtain that there exists $m_2 = m_2(\varepsilon', R) > 0$ such that

$$\mathbb{P}\left(\left\{\sup_{0\leqslant s\leqslant\rho}\|u^\varepsilon(s)\| > m\right\}\right) \leqslant \frac{\mathbb{E}\left[\sup_{0\leqslant s\leqslant\rho}\|u^\varepsilon(s)\|^2\right]}{m^2} \leqslant \frac{\varepsilon'}{2}, \quad \forall\, m \geqslant m_2,$$

which along with (3.14) implies that there exists $M_2 = M_2(\varepsilon', T, R) > 0$ such that

$$\mathbb{P}\left(\left\{\sup_{s\in[t,t+T]}\|u^\varepsilon(s)\| > m\right\}\right) \leqslant \varepsilon', \quad \forall\, t \geqslant 0,\ m \geqslant M_2,$$

as desired. □

By lemma 3.2, we have the uniform estimates of the segment of the solution in probability as follows.

Remark 3.3. If $T = 2\rho$ in lemma 3.2, then we obtain

$$\mathbb{P}\left(\left\{\sup_{s\in[t,t+\rho]}\|u_s^\varepsilon\|_{C([-\rho,0];l^2)} \leqslant m\right\}\right) \geqslant 1-\varepsilon', \quad \forall\, t \geqslant \rho,\ m \geqslant M_2.$$

Moreover, if $(u^0, \xi) \in L^2(\Omega; l^2) \times L^2(\Omega; C([-\rho, 0]; l^2))$, from the proof of lemma 3.2, we can proceed to obtain that for any $T > 0$ and $\varepsilon' > 0$, there exists a positive constant $M_2 = M_2(\varepsilon', T, R)$, independent of $\varepsilon \in (0, 1)$, such that

$$\mathbb{P}\left(\left\{\sup_{s\in[t,t+T]}\|u_s^\varepsilon\|_{C([-\rho,0];l^2)} \leqslant m\right\}\right) \geqslant 1-\varepsilon', \quad \forall\, t \geqslant 0,\ m \geqslant M_2,$$

when $\mathbb{E}[\|u^0\|^2] \vee \mathbb{E}[\sup_{s\in[-\rho,0]}\|\xi(s)\|^2] \leqslant R$ for some $R > 0$.

LEMMA 3.4. *Suppose **(A1)**–**(A5)** and **(H)** hold. If* $(u^0, \xi)\in L^2(\Omega; l^2)\times L^2(\Omega; L^2((-\rho, 0); l^2))$ *satisfies that* $\mathbb{E}[\|u^0\|^2]\vee\int_{-\rho}^0\mathbb{E}[\|\xi(s)\|^2]\,\mathrm{d}s\leqslant R$ *for some* $R>0$*, then for any* $\varepsilon'>0$ *and* $\delta_1>0$*, there exists* $\eta=\eta(\varepsilon', \delta_1, R)\in(0, \rho)$*, independent of* $\varepsilon\in(0, 1)$*, such that*

$$\mathbb{P}\left(\left\{\sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|<\eta}\|u_t^\varepsilon(t_1)-u_t^\varepsilon(t_2)\|\geqslant\delta_1\right\}\right)\leqslant\varepsilon',\quad\forall\, t\geqslant\rho.$$

*Proof.* For any $\varepsilon'>0$, it follows from remark 3.3 that there exists $m_3=m_3(\varepsilon', R)>0$ such that for any $t\geqslant\rho$,

$$\mathbb{P}\left(\left\{\sup_{t\leqslant s\leqslant t+\rho}\|u_s^\varepsilon\|_{C([-\rho,0];l^2)}\leqslant m_3\right\}\right)\geqslant 1-\frac{\varepsilon'}{2}. \tag{3.15}$$

For each $r\geqslant\rho$, define a stopping time $\tau_r$ by

$$\tau_r=\inf\{s\geqslant r:\|u_s^\varepsilon\|_{C([-\rho,0];l^2)}>m_3\},$$

and we set $\tau_r=\infty$ if $\{s\geqslant r:\|u_s^\varepsilon\|_{C([-\rho,0];l^2)}>m_3\}=\emptyset$. By (3.15) we know that

$$\mathbb{P}\{\tau_r<r+\rho\}\leqslant\frac{\varepsilon'}{2},\quad\forall\, r\geqslant\rho. \tag{3.16}$$

By (2.6) we have for any $\rho\leqslant r\leqslant t$,

$$\begin{aligned}\|u^\varepsilon(t)-u^\varepsilon(r)\|&\leqslant(\lambda+4)\int_r^t\|u^\varepsilon(s)\|\,\mathrm{d}s+\int_r^t\|u^\varepsilon(s)\|^3\,\mathrm{d}s+\int_r^t\|f(u^\varepsilon(s-\rho))\|\,\mathrm{d}s\\&\quad+\|g\||t-r|+\sqrt{\varepsilon}\left\|\sum_{k\in\mathbb{N}}\int_r^t(h_k+\sigma_k(u^\varepsilon(s-\rho)))\,\mathrm{d}W_k(s)\right\|,\end{aligned}$$

and hence for any $r\geqslant\rho$, $0<\eta<\rho$ and $p>1$, we get

$$\begin{aligned}&\sup_{t\in[r,r+\eta]}\|u^\varepsilon(t\wedge\tau_r)-u^\varepsilon(r)\|^{2p}\\&\leqslant 5^{2p-1}(\lambda+4)^{2p}\sup_{t\in[r,r+\eta]}\left(\int_r^{t\wedge\tau_r}\|u^\varepsilon(s)\|\,\mathrm{d}s\right)^{2p}\\&\quad+5^{2p-1}\sup_{t\in[r,r+\eta]}\left(\int_r^{t\wedge\tau_r}\|u^\varepsilon(s)\|^3\,\mathrm{d}s\right)^{2p}\\&\quad+5^{2p-1}\sup_{t\in[r,r+\eta]}\left(\int_r^{t\wedge\tau_r}\|f(u^\varepsilon(s-\rho))\|\,\mathrm{d}s\right)^{2p}+5^{2p-1}\|g\|^{2p}\eta^{2p}\\&\quad+5^{2p-1}\varepsilon^p\left[\sup_{t\in[r,r+\eta]}\left\|\sum_{k\in\mathbb{N}}\int_r^{t\wedge\tau_r}(h_k+\sigma_k(u^\varepsilon(s-\rho)))\,\mathrm{d}W_k(s)\right\|^{2p}\right].\end{aligned} \tag{3.17}$$

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By (2.2), (2.4), the Hölder inequality and the Burkholder–Davis–Gundy (BDG) inequality, it follows from (3.17) that for any $r \geqslant \rho$, $0 < \eta < \min\{1, \rho\}$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[r,r+\eta]} \|u^\varepsilon(t\wedge\tau_r) - u^\varepsilon(r)\|^{2p}\right] \\
&\quad \leqslant 5^{2p-1}(\lambda+4)^{2p} m_3^{2p}\eta^{2p} + 5^{2p-1} m_3^{6p}\eta^{2p} + 5^{2p-1}(2\|\alpha\|^2 + 2\beta_0^2 m_3^2)^p \eta^{2p} \\
&\qquad + 5^{2p-1}\|g\|^{2p}\eta^{2p} + 5^{2p-1} C_p (4\|\delta\|^2 + 4\|\beta\|^2 m_3^2 + 2\|h\|^2)^p \eta^p \\
&\quad \leqslant C_0 \eta^p (1+\eta^p) \leqslant 2C_0\eta^p,
\end{aligned} \tag{3.18}$$

where $C_p$ is the coefficient of the BDG inequality and $C_0$ is a positive constant independent of $\eta$, $r$ and $\varepsilon \in (0, 1)$. From (3.16) and (3.18), we can derive that for any $\delta_1 > 0$ and $t \geqslant 2\rho$,

$$\begin{aligned}
&\mathbb{P}\left(\left\{\sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|\leqslant\eta} \|u_t^\varepsilon(t_1) - u_t^\varepsilon(t_2)\| \geqslant \delta_1\right\}\right) \\
&\quad \leqslant \mathbb{P}(\{\tau_{t-\rho} < t\}) + \mathbb{P}\left(\left\{\tau_{t-\rho} \geqslant t, \sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|\leqslant\eta} \|u_t^\varepsilon(t_1) - u_t^\varepsilon(t_2)\| \geqslant \delta_1\right\}\right) \\
&\quad = \mathbb{P}(\{\tau_{t-\rho} < t\}) + \mathbb{P}\left(\left\{\tau_{t-\rho} \geqslant t, \sup_{t_1\in[-\rho,0],t_2\in[t_1,(t_1+\eta)\wedge 0]} \|u^\varepsilon(t+t_1) - u^\varepsilon(t+t_2)\| \geqslant \delta_1\right\}\right) \\
&\quad \leqslant \frac{\varepsilon'}{2} + \mathbb{P}\left(\left\{\tau_{t-\rho} \geqslant t, \max_{0\leqslant k\leqslant\left[\frac{\rho}{\eta}\right]} \sup_{s\in[t-(k+1)\eta\wedge\rho,t-k\eta]} \|u^\varepsilon(s) - u^\varepsilon(t-(k+1)\eta\wedge\rho)\| \geqslant \frac{\delta_1}{3}\right\}\right) \\
&\quad \leqslant \frac{\varepsilon'}{2} + \sum_{k=0}^{[\rho/\eta]} \mathbb{P}\left(\left\{\tau_{t-\rho} \geqslant t, \sup_{s\in[t-(k+1)\eta\wedge\rho,t-k\eta]} \|u^\varepsilon(s) - u^\varepsilon(t-(k+1)\eta\wedge\rho)\| \geqslant \frac{\delta_1}{3}\right\}\right) \\
&\quad \leqslant \frac{\varepsilon'}{2} + \sum_{k=0}^{[\rho/\eta]} \mathbb{P}\left(\left\{\tau_{t-(k+1)\eta\wedge\rho} \geqslant t, \sup_{s\in[t-(k+1)\eta\wedge\rho,t-k\eta]} \|u^\varepsilon(s) - u^\varepsilon(t-(k+1)\eta\wedge\rho)\| \geqslant \frac{\delta_1}{3}\right\}\right) \\
&\quad \leqslant \frac{\varepsilon'}{2} + \sum_{k=0}^{[\rho/\eta]} \mathbb{P}\left(\left\{\sup_{s\in[t-(k+1)\eta\wedge\rho,t-k\eta]} \|u^\varepsilon(s\wedge\tau_{t-(k+1)\eta\wedge\rho}) - u^\varepsilon(t-(k+1)\eta\wedge\rho)\| \geqslant \frac{\delta_1}{3}\right\}\right) \\
&\quad \leqslant \frac{\varepsilon'}{2} + \left(\left[\frac{\rho}{\eta}\right]+1\right)\frac{3^{2p}2C_0\eta^p}{\delta_1^{2p}}.
\end{aligned} \tag{3.19}$$

Let

$$\eta_1 = \left(\frac{\varepsilon'\delta_1^{2p}}{4(1+\rho)C_0 3^{2p}}\right)^{1/p-1} \wedge 1 \wedge \rho.$$

Then by (3.19) we obtain that for any $t \geqslant 2\rho$,

$$\mathbb{P}\left(\left\{\sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|\leqslant\eta_1} \|u_t^\varepsilon(t_1) - u_t^\varepsilon(t_2)\| \geqslant \delta_1\right\}\right) \leqslant \varepsilon'. \tag{3.20}$$

On the other hand, since $u^\varepsilon \in L^2(\Omega; C([0, 2\rho]; l^2))$, we find that there exists a constant $\eta_2 = \eta_2(\delta_1, \varepsilon') > 0$ such that for any $\rho \leqslant t \leqslant 2\rho$,

$$\mathbb{P}\left(\left\{\sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|\leqslant\eta_2} \|u_t^\varepsilon(t_1) - u_t^\varepsilon(t_2)\| \geqslant \delta_1\right\}\right) \leqslant \varepsilon',$$

which along with (3.20) yields that

$$\mathbb{P}\left(\left\{\sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|\leqslant\eta} \|u_t^\varepsilon(t_1) - u_t^\varepsilon(t_2)\| \geqslant \delta_1\right\}\right) \leqslant \varepsilon', \quad \forall\, t \geqslant \rho,$$

where $\eta = \eta_1 \wedge \eta_2$, as desired. □

REMARK 3.5. If $(u^0, \xi) \in L^2(\Omega; l^2) \times L^2(\Omega; C([-\rho, 0]; l^2))$, from the proof of lemma 3.4, we can further obtain that for any $\varepsilon' > 0$ and $\delta_1 > 0$, there exists $\eta = \eta(\varepsilon', \delta_1, R) \in (0, \rho)$, independent of $\varepsilon \in (0, 1)$, such that

$$\mathbb{P}\left(\left\{\sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|<\eta} \|u_t^\varepsilon(t_1) - u_t^\varepsilon(t_2)\| \geqslant \delta_1\right\}\right) \leqslant \varepsilon', \quad \forall\, t \geqslant 0,$$

when $\mathbb{E}[\|u^0\|^2] \vee \mathbb{E}[\sup_{s\in[-\rho,0]} \|\xi(s)\|^2] \leqslant R$ for some $R > 0$.

LEMMA 3.6. *Suppose that* ***(A1)***–***(A5)*** *and* ***(H)*** *hold. Then for every compact subset $E$ of $L^2(\Omega; l^2) \times L^2(\Omega; L^2((-\rho, 0); l^2))$ and $\varepsilon' > 0$, there exists a positive integer $N_1 = N_1(\varepsilon', E)$ such that for all $m \geqslant N_1$, $\varepsilon \in (0, 1)$ and $t \geqslant 0$, the solution $u^\varepsilon(t)$ of (2.6) with $(u^0, \xi) \in E$ satisfies*

$$\sum_{|n|\geqslant m} \mathbb{E}[|u_n^\varepsilon(t)|^2] \leqslant \varepsilon'.$$

*Proof.* Hereafter, we denote by $C$ a generic positive constant independent of $E$, $T$ and $\varepsilon'$. Consider a smooth function $\theta : \mathbb{R} \to [0, 1]$ satisfying

$$\theta(s) = 0 \text{ for } |s| \leqslant 1; \quad \text{and} \quad \theta(s) = 1 \text{ for } |s| \geqslant 2. \tag{3.21}$$

Fixed $m \in \mathbb{N}$, denote by $\theta_m = (\theta(n/m))_{n\in\mathbb{Z}}$ and $\theta_m u = (\theta(n/m)u_n)_{n\in\mathbb{Z}}$ for $u = (u_n)_{n\in\mathbb{Z}} \in l^2$. Then by (2.6) we have

$$\begin{aligned} &d(\theta_m u^\varepsilon(t)) + \left(i\theta_m A u^\varepsilon(t) + i\theta_m |u^\varepsilon(t)|^2 u^\varepsilon(t) + \lambda\theta_m u^\varepsilon(t)\right) \mathrm{d}t \\ &\quad = \theta_m f(u^\varepsilon(t-\rho))\, \mathrm{d}t + \theta_m g\, \mathrm{d}t \\ &\qquad + \sqrt{\varepsilon} \sum_{k\in\mathbb{N}} (\theta_m h_k + \theta_m \sigma_k(u^\varepsilon(t-\rho)))\, \mathrm{d}W_k(t). \end{aligned} \tag{3.22}$$

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Similar to (3.4), by (3.22) we get that for all $t \geqslant 0$,

$$
\begin{aligned}
e^{\gamma t}\mathbb{E}[\|\theta_m u^\varepsilon(t)\|^2] =& \mathbb{E}[\|\theta_m u^0\|^2] + (\gamma - 2\lambda)\int_0^t e^{\gamma s}\mathbb{E}[\|\theta_m u^\varepsilon(s)\|^2]\,\mathrm{d}s \\
& - 2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[\left(\theta_m^2 u^\varepsilon(s), iAu^\varepsilon(s)\right)\right]\,\mathrm{d}s \\
& - 2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[\left(\theta_m^2 u^\varepsilon(s), i|u^\varepsilon(s)|^2 u^\varepsilon(s)\right)\right]\,\mathrm{d}s \\
& + 2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[\left(\theta_m u^\varepsilon(s), \theta_m f(u^\varepsilon(s-\rho))\right)\right]\,\mathrm{d}s \\
& + 2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[\left(\theta_m u^\varepsilon(s), \theta_m g\right)\right]\,\mathrm{d}s \\
& + \varepsilon\sum_{k\in\mathbb{N}}\int_0^t e^{\gamma s}\mathbb{E}\left[\|\theta_m h_k + \theta_m\sigma_k(u^\varepsilon(s-\rho))\|^2\right]\,\mathrm{d}s. \qquad (3.23)
\end{aligned}
$$

By the argument of (4.4)–(4.6) in [**31**], we have

$$
\begin{aligned}
& - 2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[\left(\theta_m^2 u^\varepsilon(s), iAu^\varepsilon(s)\right)\right]\,\mathrm{d}s \\
& = -2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[\left(B(\theta_m^2 u^\varepsilon(s)), iBu^\varepsilon(s)\right)\right]\,\mathrm{d}s \\
& \leqslant \frac{C}{m}\int_0^t e^{\gamma s}\mathbb{E}[\|u^\varepsilon(s)\|^2]\,\mathrm{d}s. \qquad (3.24)
\end{aligned}
$$

For the fifth term on the right-hand side of (3.23), by assumption **(A3)** we obtain

$$
\begin{aligned}
& 2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[\left(\theta_m u^\varepsilon(s), \theta_m f(u^\varepsilon(s-\rho))\right)\right]\,\mathrm{d}s \\
& \leqslant \sqrt{2}\beta_0\int_0^t e^{\gamma s}\mathbb{E}\left[\|\theta_m u^\varepsilon(s)\|^2\right]\,\mathrm{d}s + \frac{1}{\sqrt{2}\beta_0}\int_0^t e^{\gamma s}\mathbb{E}[\|\theta_m f(u^\varepsilon(s-\rho))\|^2]\,\mathrm{d}s \\
& \leqslant \sqrt{2}\beta_0(1+e^{\gamma\rho})\int_0^t e^{\gamma s}\mathbb{E}\left[\|\theta_m u^\varepsilon(s)\|^2\right]\,\mathrm{d}s + \frac{\sqrt{2}}{\beta_0\gamma}e^{\gamma t}\sum_{|n|\geqslant m}|\alpha_n|^2 \\
& \quad + \sqrt{2}\beta_0 e^{\gamma\rho}\int_{-\rho}^0 \mathbb{E}\left[\|\theta_m\xi(s)\|^2\right]\,\mathrm{d}s. \qquad (3.25)
\end{aligned}
$$

By Young's inequality we have

$$
\begin{aligned}
2\mathrm{Re}\int_0^t e^{\gamma s}\mathbb{E}\left[\left(\theta_m u^\varepsilon(s), \theta_m g\right)\right]\,\mathrm{d}s \leqslant & \alpha_1\int_0^t e^{\gamma s}\mathbb{E}\left[\|\theta_m u^\varepsilon(s)\|^2\right]\,\mathrm{d}s \\
& + \frac{1}{\alpha_1\gamma}e^{\gamma t}\sum_{|n|\geqslant m}|g_n|^2. \qquad (3.26)
\end{aligned}
$$

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For the last term on the right-hand side of (3.23), by assumption **(A4)** we get

$$
\begin{aligned}
&\varepsilon \sum_{k\in\mathbb{N}} \int_0^t e^{\gamma s}\mathbb{E}\left[\|\theta_m h_k + \theta_m \sigma_k(u^\varepsilon(s-\rho))\|^2\right] \mathrm{d}s \\
&\quad \leqslant 2\sum_{k\in\mathbb{N}}\sum_{|n|\geqslant m} |h_{k,n}|^2 \frac{e^{\gamma t}}{\gamma} + 2\sum_{k\in\mathbb{N}} \int_0^t e^{\gamma s}\mathbb{E}\left[\|\theta_m \sigma_k(u^\varepsilon(s-\rho))\|^2\right] \mathrm{d}s \\
&\quad \leqslant 2\sum_{k\in\mathbb{N}}\sum_{|n|\geqslant m} |h_{k,n}|^2 \frac{e^{\gamma t}}{\gamma} + 4\sum_{k\in\mathbb{N}}\sum_{|n|\geqslant m} |\delta_{k,n}|^2 \frac{e^{\gamma t}}{\gamma} + 4\|\beta\|^2 e^{\gamma\rho} \int_{-\rho}^0 \mathbb{E}\left[\|\theta_m \xi(s)\|^2\right] \mathrm{d}s \\
&\qquad + 4\|\beta\|^2 e^{\gamma\rho} \int_0^t e^{\gamma s}\mathbb{E}\left[\|\theta_m u^\varepsilon(s)\|^2\right] \mathrm{d}s. \qquad (3.27)
\end{aligned}
$$

It follows from (3.23)–(3.27) that for all $t \geqslant 0$,

$$
\begin{aligned}
&\mathbb{E}[\|\theta_m u^\varepsilon(t)\|^2] \leqslant \mathbb{E}[\|\theta_m u^0\|^2] e^{-\gamma t} \\
&\quad + \left[\gamma - 2\lambda + \sqrt{2}\beta_0(1+e^{\gamma\rho}) + \alpha_1 + 4\|\beta\|^2 e^{\gamma\rho}\right] \int_0^t e^{\gamma(s-t)}\mathbb{E}\left[\|\theta_m u^\varepsilon(s)\|^2\right] \mathrm{d}s \\
&\quad + \frac{2}{\gamma}\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}} |h_{k,n}|^2 + \frac{4}{\gamma}\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}} |\delta_{k,n}|^2 + \frac{1}{\alpha_1\gamma}\sum_{|n|\geqslant m} |g_n|^2 + \frac{\sqrt{2}}{\beta_0\gamma}\sum_{|n|\geqslant m} |\alpha_n|^2 \\
&\quad + \frac{C}{m}\int_0^t e^{\gamma(s-t)}\mathbb{E}[\|u^\varepsilon(s)\|^2] \mathrm{d}s + (\sqrt{2}\beta_0 + 4\|\beta\|^2) e^{\gamma(\rho-t)} \int_{-\rho}^0 \mathbb{E}\left[\|\theta_m \xi(s)\|^2\right] \mathrm{d}s.
\end{aligned} \qquad (3.28)
$$

Since $\|h\|^2 \vee \|\delta\|^2 \vee \|\alpha\|^2 \vee \|g\|^2 < \infty$, we infer that there exists $m_4 = m_4(\varepsilon') \geqslant 0$ such that for all $m \geqslant m_4$,

$$
\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}} |h_{k,n}|^2 \vee \sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}} |\delta_{k,n}|^2 \vee \sum_{|n|\geqslant m} |g_n|^2 \vee \sum_{|n|\geqslant m} |\alpha_n|^2 \leqslant \varepsilon'. \qquad (3.29)
$$

For any $\varepsilon' > 0$, since $E$ is compact in $L^2(\Omega; l^2) \times L^2(\Omega; L^2((-\rho, 0); l^2))$, then it has a finite open cover of balls with radius $\sqrt{\varepsilon'}/2$, which is denoted by $\left\{B\left((u^j, \xi^j), \sqrt{\varepsilon'}/2\right)\right\}_{j=1}^l$. Since $(u^j, \xi^j) \in L^2(\Omega; l^2) \times L^2(\Omega; L^2((-\rho, 0); l^2))$ for $j = 1, 2, \cdots, l$, there exists $m_5 = m_5(\varepsilon', E) \geqslant m_4$ such that for all $m \geqslant m_5$ and $j = 1, 2, \cdots, l$,

$$
\sum_{|n|\geqslant m}\left(\mathbb{E}[|u_n^j|^2] + \int_{-\rho}^0 \mathbb{E}[|\xi_n^j(s)|^2] \mathrm{d}s\right) \leqslant \frac{\varepsilon'}{4},
$$

which implies for all $m \geqslant m_5$ and $(u^0, \xi) \in E$,

$$
\sum_{|n|\geqslant m}\left(\mathbb{E}[|u_n^0(s)|^2] + \int_{-\rho}^0 \mathbb{E}[|\xi_n(s)|^2] \mathrm{d}s\right) \leqslant \varepsilon'.
$$

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We therefore obtain that for all $m \geqslant m_5$ and $(u^0, \xi) \in E$,

$$\int_{-\rho}^{0} \mathbb{E}\left[\|\theta_m \xi(s)\|^2\right] \mathrm{d}s \leqslant \int_{-\rho}^{0} \sum_{|n|\geqslant m} \mathbb{E}\left[|\xi_n(s)|^2\right] \mathrm{d}s \leqslant \varepsilon', \tag{3.30}$$

and

$$\mathbb{E}[\|\theta_m u^0\|^2] \leqslant \sum_{|n|\geqslant m} \mathbb{E}[|u_n^0|^2] \leqslant \varepsilon'. \tag{3.31}$$

On the other hand, by lemma 3.1 we know that there exists $m_6 = m_6(\varepsilon', E) \geqslant m_5$ such that for all $m \geqslant m_6$ and $t \geqslant 0$,

$$\frac{C}{m} \int_0^t e^{\gamma(s-t)} \mathbb{E}[\|u^\varepsilon(s)\|^2] \, \mathrm{d}s \leqslant \frac{C_1}{m} \int_0^t e^{\gamma(s-t)} \, \mathrm{d}s \leqslant \varepsilon', \tag{3.32}$$

where $C_1 > 0$ is a constant depending only on $E$. Substituting (3.29)-(3.32) into (3.28), we obtain for all $m \geqslant m_6$ and $t \geqslant 0$,

$$\mathbb{E}[\|\theta_m u^\varepsilon(t)\|^2] \leqslant 2\varepsilon' + \frac{2\varepsilon'}{\gamma} + \frac{4\varepsilon'}{\gamma} + \frac{\varepsilon'}{\alpha_1 \gamma} + \frac{\sqrt{2}\varepsilon'}{\beta_0 \gamma} + (\sqrt{2}\beta_0 + 4\|\beta\|^2) e^{\gamma\rho} \varepsilon' \leqslant C\varepsilon',$$

which implies that for all $m \geqslant m_6$ and $t \geqslant 0$,

$$\sum_{|n|\geqslant 2m} \mathbb{E}[|u_n^\varepsilon(t)|^2] \leqslant \mathbb{E}[\|\theta_m u^\varepsilon(t)\|^2] \leqslant C\varepsilon',$$

as desired. □

LEMMA 3.7. *Suppose that **(A1)**–**(A5)** and **(H)** hold. Then for every compact subset $E$ of $L^2(\Omega; l^2) \times L^2(\Omega; L^2((-\rho, 0); l^2))$, the solution $u^\varepsilon(t)$ of (2.6) with $(u^0, \xi) \in E$ satisfies*

$$\limsup_{m\to\infty} \sup_{(u^0,\xi)\in E} \sup_{t\geqslant 0} \mathbb{E}[\sup_{t\leqslant s\leqslant t+T} \sum_{|n|\geqslant m} |u_n^\varepsilon(s)|^2] = 0.$$

*Proof.* Let $\theta$ be the smooth cut-off function as given by (3.21). It follows from lemma 3.6 that for every $\varepsilon' > 0$, there exists $N_1 = N_1(\varepsilon', E) > 0$ such that for any $t \geqslant 0$ and $m \geqslant N_1$,

$$\mathbb{E}[\|\theta_m u^\varepsilon(t)\|^2] \leqslant \varepsilon'. \tag{3.33}$$

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Applying Itô's formula to (3.22), we obtain for all $t \geqslant 0$, $r \in [t, t+T]$, $\varepsilon \in (0, 1)$ and $m \geqslant N_1$,

$$
\begin{aligned}
\|\theta_m u^\varepsilon(r)\|^2 =& \|\theta_m u^\varepsilon(t)\|^2 - 2\mathrm{Re}\int_t^r \left(iAu^\varepsilon(s), \theta_m^2 u^\varepsilon(s)\right)\,\mathrm{d}s \\
& - 2\mathrm{Re}\int_t^r \left(i|u^\varepsilon(s)|^2 u^\varepsilon(s), \theta_m^2 u^\varepsilon(s)\right)\,\mathrm{d}s - 2\lambda\int_t^r \|\theta_m u^\varepsilon(s)\|^2\,\mathrm{d}s \\
& + 2\mathrm{Re}\int_t^r (\theta_m f(u^\varepsilon(s-\rho)), \theta_m u^\varepsilon(s))\,\mathrm{d}s \\
& + 2\mathrm{Re}\int_t^r (\theta_m g, \theta_m u^\varepsilon(s))\,\mathrm{d}s + \sum_{k\in\mathbb{N}}\int_t^r \|\theta_m h_k + \theta_m\sigma_k(u^\varepsilon(s-\rho))\|^2\mathrm{d}s \\
& + 2\mathrm{Re}\sum_{k\in\mathbb{N}}\int_t^r (\theta_m h_k + \theta_m\sigma_k(u^\varepsilon(s-\rho)), \theta_m u^\varepsilon(s))\,\mathrm{d}W_k(s).
\end{aligned}
\tag{3.34}
$$

For the second term on the right-hand side of (3.34), similar to (3.24), we have

$$
-2\mathrm{Re}\int_t^r \left(iAu^\varepsilon(s), \theta_m^2 u^\varepsilon(s)\right)\,\mathrm{d}s \leqslant \frac{C}{m}\int_t^r \|u^\varepsilon(s)\|^2\,\mathrm{d}s. \tag{3.35}
$$

Then by (3.34)–(3.35) we get

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\leqslant r\leqslant t+T}\|\theta_m u^\varepsilon(r)\|^2\right] \leqslant& \mathbb{E}[\|\theta_m u^\varepsilon(t)\|^2] + \frac{C}{m}\int_t^{t+T}\mathbb{E}[\|u^\varepsilon(s)\|^2]\,\mathrm{d}s \\
& + 2\int_t^{t+T}\mathbb{E}\left[\|\theta_m f(u^\varepsilon(s-\rho))\|\|\theta_m u^\varepsilon(s)\|\right]\,\mathrm{d}s + 2\int_t^{t+T}\mathbb{E}[\|\theta_m g\|\|\theta_m u^\varepsilon(s)\|]\,\mathrm{d}s \\
& + \sum_{k\in\mathbb{N}}\int_t^{t+T}\mathbb{E}[\|\theta_m h_k + \theta_m\sigma_k(u^\varepsilon(s-\rho))\|^2]\,\mathrm{d}s \\
& + 2\mathbb{E}\left[\sup_{t\leqslant r\leqslant t+T}\left|\sum_{k\in\mathbb{N}}\int_t^r (\theta_m h_k + \theta_m\sigma_k(u^\varepsilon(s-\rho)), \theta_m u^\varepsilon(s))\,\mathrm{d}W_k(s)\right|\right].
\end{aligned}
\tag{3.36}
$$

For the second term on the right-hand side of (3.36), by lemma 3.1 we know that there exists $N_2 = N_2(\varepsilon', E) \geqslant N_1$ such that for $m \geqslant N_2$,

$$
\frac{C}{m}\int_t^{t+T}\mathbb{E}[\|u^\varepsilon(s)\|^2]\,\mathrm{d}s \leqslant \varepsilon' T. \tag{3.37}
$$

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For the third term on the right-hand side of (3.36), by **(A3)** and (3.33) we have for $m \geqslant N_2$,

$$
\begin{aligned}
&2\int_t^{t+T} \mathbb{E}[\|\theta_m f(u^\varepsilon(s-\rho))\| \|\theta_m u^\varepsilon(s)\|] \,\mathrm{d}s \\
&\quad \leqslant 2\beta_0^2 \int_{t-\rho}^{t+T-\rho} \mathbb{E}[\|\theta_m u^\varepsilon(s)\|^2] \,\mathrm{d}s + 2\sum_{|n|\geqslant m} |\alpha_n|^2\, T + \int_t^{t+T} \mathbb{E}[\|\theta_m u^\varepsilon(s)\|^2] \,\mathrm{d}s \\
&\quad \leqslant (2\beta_0^2+1)\varepsilon' T + 2\sum_{|n|\geqslant m} |\alpha_n|^2\, T + 2\beta_0^2 \int_{-\rho}^{0} \mathbb{E}[\|\theta_m \xi(s)\|^2] \,\mathrm{d}s. \qquad (3.38)
\end{aligned}
$$

For the fourth term on the right-hand side of (3.36), by (3.33) and Young's inequality we get for $m \geqslant N_2$,

$$
2\int_t^{t+T} \mathbb{E}[\|\theta_m g\| \|\theta_m u^\varepsilon(s)\|] \,\mathrm{d}s \leqslant \sum_{|n|\geqslant m} |g_n|^2 T + \varepsilon' T. \qquad (3.39)
$$

For the fifth term on the right-hand side of (3.36), by **(A4)** we have for $m \geqslant N_2$,

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}} \int_t^{t+T} \mathbb{E}[\|\theta_m h_k + \theta_m \sigma_k(u^\varepsilon(s-\rho))\|^2] \,\mathrm{d}s \\
&\quad \leqslant 2\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}} |h_{k,n}|^2\, T + 4\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}} |\delta_{k,n}|^2 T \\
&\qquad + 4\|\beta\|^2 \varepsilon' T + 4\|\beta\|^2 \int_{-\rho}^{0} \mathbb{E}[\|\theta_m \xi(s)\|^2] \mathrm{d}s. \qquad (3.40)
\end{aligned}
$$

For the last term on the right-hand side of (3.36), by **(A4)** and the BDG inequality we obtain for $m \geqslant N_2$,

$$
\begin{aligned}
&2\mathbb{E}\left[\sup_{t\leqslant r\leqslant t+T} \left| \sum_{k\in\mathbb{N}} \int_t^r (\theta_m h_k + \theta_m \sigma_k(u^\varepsilon(s-\rho)), \theta_m u^\varepsilon(s))\, \mathrm{d}W_k(s) \right| \right] \\
&\quad \leqslant \frac{1}{2}\mathbb{E}[\sup_{t\leqslant r\leqslant t+T} \|\theta_m u^\varepsilon(s)\|^2] + C\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}} |h_{k,n}|^2\, T \\
&\qquad + 2C\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}} |\delta_{k,n}|^2\, T + 2C\|\beta\|^2 \varepsilon' T + 2C\|\beta\|^2 \int_{-\rho}^{0} \mathbb{E}[\|\theta_m \xi(s)\|^2] \,\mathrm{d}s. \quad (3.41)
\end{aligned}
$$

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It follows from (3.36)-(3.41) that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\leqslant r\leqslant t+T}\|\theta_m u^\varepsilon(r)\|^2\right] &\leqslant 4\varepsilon' T + 2(2\beta_0^2+1)\varepsilon' T \\
&+ 4\sum_{|n|\geqslant m}|\alpha_n|^2 T + 2\sum_{|n|\geqslant m}|g_n|^2 T + 2\varepsilon' \\
&+ 2(C+2)\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}}|h_{k,n}|^2 T + 4(C+2)\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}}|\delta_{k,n}|^2 T \\
&+ 4(C+2)\|\beta\|^2\varepsilon' T + \left[4(C+2)\|\beta\|^2 + 4\beta_0^2\right]\int_{-\rho}^0 \mathbb{E}[\|\theta_m\xi(s)\|^2]\,\mathrm{d}s. \qquad (3.42)
\end{aligned}
$$

Similar to (3.29)–(3.30), we obtain that there exists $N_3 = N_3(\varepsilon', E) \geqslant N_2$ such that for all $m \geqslant N_3$,

$$
\sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}}|h_{n,k}|^2 \vee \sum_{|n|\geqslant m}\sum_{k\in\mathbb{N}}|\delta_{n,k}|^2 \vee \sum_{|n|\geqslant m}|g_n|^2 \vee \sum_{|n|\geqslant m}|\alpha_n|^2 \leqslant \varepsilon',
$$

and

$$
\int_{-\rho}^0 \mathbb{E}[\|\theta_m\xi(s)\|^2]\,\mathrm{d}s \leqslant \varepsilon',
$$

which along with (3.42) implies that for all $t \geqslant 0$ and $m \geqslant N_3$,

$$
\mathbb{E}\left[\sup_{t\leqslant r\leqslant t+T}\|\theta_m u^\varepsilon(r)\|^2\right] \leqslant C_T\varepsilon',
$$

where $C_T > 0$ depends only on T but not on $\varepsilon'$, $m$ or $E$. This completes this proof. □

As an immediate consequence of lemma 3.7, we have the following result.

Corollary 3.8. *Suppose* ***(A1)***–***(A5)*** *and* ***(H)*** *hold. If* $(u^0, \xi) \in L^2(\Omega; l^2) \times L^2(\Omega; L^2((-\rho, 0); l^2))$, *then the solution* $u^\varepsilon$ *of* (2.6) *satisfies that for every* $\delta_2 > 0$ *and* $T > 0$,

$$
\limsup_{m\to\infty}\sup_{t\geqslant 0}\mathbb{P}\left(\left\{\sup_{s\in[t,t+T]}\sum_{|n|\geqslant m}|u_n^\varepsilon(s)|^2 > \delta_2\right\}\right) = 0.
$$

*Proof.* By the Chebyshev inequality, we obtain that

$$
\mathbb{P}\left(\left\{\sup_{s\in[t,t+T]}\sum_{|n|\geqslant m}|u_n^\varepsilon(s)|^2 > \delta_2\right\}\right) \leqslant \frac{1}{\delta_2}\mathbb{E}\left[\sup_{s\in[t,t+T]}\sum_{|n|\geqslant m}|u_n^\varepsilon(s)|^2\right],
$$

which together with lemma 3.7 completes the proof. □

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REMARK 3.9. If $(u^0, \xi) \in L^2(\Omega; l^2) \times L^2(\Omega; C([-\rho, 0]; l^2))$, from the proofs of lemma 3.7 and corollary 3.8, we can further obtain that for every $\delta_2 > 0$ and $T > 0$,

$$\limsup_{m\to\infty} \sup_{t\geqslant -\rho} \mathbb{P}\left(\left\{\sup_{s\in[t,t+T]} \sum_{|n|\geqslant m} |u_n^\varepsilon(s)|^2 > \delta_2\right\}\right) = 0.$$

### 3.2. Existence of invariant measures

*3.2.1. Transition semigroup.* In this subsection, we first introduce the transition semigroup of (2.6), and then show the Feller property and the Markov property of the transition semigroup, which will play a crucial role in proving the existence of invariant measures on $l^2 \times L^2((-\rho, 0); l^2)$.

For any initial time $t_0 \geqslant 0$ and initial data $(u^0, \xi) \in L^2(\Omega; l^2) \times L^2(\Omega; L^2((-\rho, 0); l^2))$, we know that (2.6) has a unique solution on $[t_0, \infty)$, which is denoted by $u^\varepsilon(t; t_0, u^0, \xi)$. The segment of $u^\varepsilon(t; t_0, u^0, \xi)$ on $(t-\rho, t)$ with $t \geqslant t_0$ is written as $u_t^\varepsilon(t_0, u^0, \xi)$; that is,

$$u_t^\varepsilon(t_0, u^0, \xi)(s) = u^\varepsilon(t+s; t_0, u^0, \xi), \quad \forall\ s \in (-\rho, 0).$$

Then we have $u_t^\varepsilon(t_0, u^0, \xi) \in L^2(\Omega; L^2((-\rho, 0); l^2))$ for all $t \geqslant t_0$.

If $\varphi : l^2 \times L^2((-\rho, 0); l^2) \to \mathbb{C}$ is a bounded Borel function, then for $0 \leqslant r \leqslant t$ and $(u^0, \xi) \in l^2 \times L^2((-\rho, 0); l^2)$, we set

$$(p_{r,t}^\varepsilon \varphi)(u^0, \xi) = \mathbb{E}\left[\varphi\left(u^\varepsilon(t; r, u^0, \xi), u_t^\varepsilon(r, u^0, \xi)\right)\right].$$

The family $\{p_{r,t}^\varepsilon\}_{0\leqslant r\leqslant t}$ is called the transition semigroup of (2.6), and $p_{0,t}^\varepsilon$ is written as $p_t^\varepsilon$ for simplicity. In particular, for $\Gamma \in \mathcal{B}(l^2 \times L^2((-\rho, 0); l^2))$, $0 \leqslant r \leqslant t$ and $(u^0, \xi) \in l^2 \times L^2((-\rho, 0); l^2)$, we set

$$p^\varepsilon(r, (u^0, \xi); t, \Gamma) = (p_{r,t}^\varepsilon I_\Gamma)(u^0, \xi) = \mathbb{P}\left(\left\{\omega \in \Omega : (u^\varepsilon(t; r, u^0, \xi), u_t^\varepsilon(r, u^0, \xi)) \in \Gamma\right\}\right),$$

where $I_\Gamma$ is the characteristic function of $\Gamma$. Recall that a probability measure $\mu^\varepsilon$ on $l^2 \times L^2((-\rho, 0); l^2)$ is called an invariant measure of (2.6), if

$$\int_{l^2\times L^2((-\rho,0);l^2)} (p_t^\varepsilon \varphi)(u^0, \xi)\,\mathrm{d}\mu^\varepsilon = \int_{l^2\times L^2((-\rho,0);l^2)} \varphi(u^0, \xi)\,\mathrm{d}\mu^\varepsilon, \quad \forall\ t \geqslant 0, \tag{3.43}$$

for every bounded Borel function $\varphi : l^2 \times L^2((-\rho, 0); l^2) \to \mathbb{C}$.

Given $(u^n, \xi^n)$, $(u^0, \xi) \in l^2 \times L^2((-\rho, 0); l^2)$, $R > 0$ and $r_0 \geqslant 0$, define

$$T_R^n = \inf\{t \geqslant r_0 : \|u^\varepsilon(t; r_0, u^0, \xi)\| > R \text{ or } \|u^\varepsilon(t; r_0, u^n, \xi^n)\| > R\}.$$

Next we show the continuity of $(u^\varepsilon(t_0; r_0, u^0, \xi), u_{t_0}^\varepsilon(r_0, u^0, \xi))$ with respect to initial data in $l^2 \times L^2((-\rho, 0); l^2)$, which is useful for proving the Feller property of $\{p_{r,t}^\varepsilon\}_{0\leqslant r\leqslant t}$.

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LEMMA 3.10. *Suppose that* ***(A1)***–***(A5)*** *and* ***(H)*** *hold. If* $(u^n,\, \xi^n) \to (u^0,\, \xi)$ *in* $l^2 \times L^2((-\rho,\, 0); l^2)$*, then for every* $0 \leqslant r_0 \leqslant t_0$*,*

$$\lim_{n\to\infty} \mathbb{E}\Big[ \left\| u^\varepsilon \left(t_0 \wedge T_R^n; r_0, u^n, \xi^n\right) - u^\varepsilon \left(t_0 \wedge T_R^n; r_0, u^0, \xi\right) \right\|^2 + \int_{t_0-\rho}^{t_0} \left\| u^\varepsilon \left(t \wedge T_R^n; r_0, u^n, \xi^n\right) - u^\varepsilon \left(t \wedge T_R^n; r_0, u^0, \xi\right) \right\|^2 \, \mathrm{d}t \Big] = 0. \tag{3.44}$$

*Proof.* For simplicity, we write $u^\varepsilon(t; r_0,\, u^n,\, \xi^n)$ as $u^{n,\varepsilon}(t)$ and $u^\varepsilon(t; r_0,\, u^0,\, \xi)$ as $u^\varepsilon(t)$. By (2.6) and Itô's formula, we get for all $r_0 \leqslant t \leqslant t_0$,

$$\begin{aligned} &\left\| u^{n,\varepsilon} \left(t \wedge T_R^n\right) - u^\varepsilon \left(t \wedge T_R^n\right) \right\|^2 \\ &\leqslant \left\| u^n - u^0 \right\|^2 + 2 \left| \int_{r_0}^{t\wedge T_R^n} i \left( |u^{n,\varepsilon}(s)|^2 u^{n,\varepsilon}(s) - |u^\varepsilon(s)|^2 u^\varepsilon(s), u^{n,\varepsilon}(s) - u^\varepsilon(s) \right) \mathrm{d}s \right| \\ &\quad + 2 \left| \int_{r_0}^{t\wedge T_R^n} \left( f(u^{n,\varepsilon}(s-\rho)) - f(u^\varepsilon(s-\rho)), u^{n,\varepsilon}(s) - u^\varepsilon(s) \right) \mathrm{d}s \right| \\ &\quad + \varepsilon \sum_{k\in\mathbb{N}} \int_{r_0}^{t\wedge T_R^n} \left\| \sigma_k(u^{n,\varepsilon}(s-\rho)) - \sigma_k(u^\varepsilon(s-\rho)) \right\|^2 \mathrm{d}s \\ &\quad + 2\sqrt{\varepsilon} \left| \sum_{k\in\mathbb{N}} \int_{r_0}^{t\wedge T_R^n} \left( \sigma_k \left(u^{n,\varepsilon}(s-\rho)\right) - \sigma_k \left(u^\varepsilon(s-\rho)\right), u^{n,\varepsilon}(s) - u^\varepsilon(s) \right) \mathrm{d}W_k(s) \right|. \end{aligned} \tag{3.45}$$

For the second term on the right-hand side of (3.45), we know that there exists $C_{1,R} > 0$ depending only on $R$ such that

$$\begin{aligned} &2 \left| \int_{r_0}^{t\wedge T_R^n} i \left( |u^{n,\varepsilon}(s)|^2 u^{n,\varepsilon}(s) - |u^\varepsilon(s)|^2 u^\varepsilon(s), u^{n,\varepsilon}(s) - u^\varepsilon(s) \right) \mathrm{d}s \right| \\ &\quad \leqslant C_{1,R} \int_{r_0}^{t\wedge T_R^n} \left\| u^{n,\varepsilon}(s) - u^\varepsilon(s) \right\|^2 \mathrm{d}s. \end{aligned} \tag{3.46}$$

For the third term on the right-hand side of (3.45), by (2.3) we have

$$\begin{aligned} &2 \left| \int_{r_0}^{t\wedge T_R^n} \left( f\left(u^{n,\varepsilon}(s-\rho)\right) - f\left(u^\varepsilon(s-\rho)\right), u^{n,\varepsilon}(s) - u^\varepsilon(s) \right) \mathrm{d}s \right| \\ &\quad \leqslant (1 + C_{2,R}) \int_{r_0}^{t\wedge T_R^n} \left\| u^{n,\varepsilon}(s) - u^\varepsilon(s) \right\|^2 \mathrm{d}s + \int_{-\rho}^{0} \left\| f\left(\xi^n(s)\right) - f\left(\xi(s)\right) \right\|^2 \mathrm{d}s, \end{aligned} \tag{3.47}$$

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where $C_{2,R} > 0$ depends only on $R$. For the fourth term on the right-hand side of (3.45), by (2.5) we get

$$\begin{aligned}
&\varepsilon \sum_{k\in\mathbb{N}} \int_{r_0}^{t\wedge T_R^n} \left\|\sigma_k\left(u^{n,\varepsilon}(s-\rho)\right) - \sigma_k\left(u^{\varepsilon}(s-\rho)\right)\right\|^2 \mathrm{d}s \\
&\quad \leqslant C_{3,R} \int_{r_0}^{t\wedge T_R^n} \|u^{n,\varepsilon}(s) - u^{\varepsilon}(s)\|^2 \mathrm{d}s + \sum_{k\in\mathbb{N}} \int_{-\rho}^{0} \|\sigma_k\left(\xi^n(s)\right) - \sigma_k\left(\xi(s)\right)\|^2 \mathrm{d}s,
\end{aligned} \tag{3.48}$$

where $C_{3,R} > 0$ depends only on $R$. It follows from (3.45)–(3.48) that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{r_0\leqslant r\leqslant t} \|u^{n,\varepsilon}\left(r\wedge T_R^n\right) - u^{\varepsilon}\left(r\wedge T_R^n\right)\|^2\right] \\
&\quad \leqslant \left\|u^n - u^0\right\|^2 + \int_{-\rho}^{0} \|f\left(\xi^n(s)\right) - f\left(\xi(s)\right)\|^2 \mathrm{d}s \\
&\qquad + \sum_{k\in\mathbb{N}} \int_{-\rho}^{0} \|\sigma_k\left(\xi^n(s)\right) - \sigma_k\left(\xi(s)\right)\|^2 \mathrm{d}s \\
&\qquad + \left(C_{1,R} + C_{2,R} + C_{3,R} + 1\right) \int_{r_0}^{t} \mathbb{E}\left[\sup_{r_0\leqslant s\leqslant r} \|u^{n,\varepsilon}\left(s\wedge T_R^n\right) - u^{\varepsilon}\left(s\wedge T_R^n\right)\|^2\right] \mathrm{d}r \\
&\qquad + 2\sqrt{\varepsilon}\,\mathbb{E}\Bigg[\sup_{r_0\leqslant r\leqslant t\wedge T_R^n} \Bigg|\sum_{k\in\mathbb{N}} \int_{r_0}^{r} \Big(\sigma_k\left(u^{n,\varepsilon}(s-\rho)\right) \\
&\qquad\qquad - \sigma_k\left(u^{\varepsilon}(s-\rho)\right), u^{n,\varepsilon}(s) - u^{\varepsilon}(s)\Big)\, \mathrm{d}W_k(s)\Bigg|\Bigg].
\end{aligned} \tag{3.49}$$

By (2.5) and the BDG inequality, we obtain that there exists a constant $C_{4,R} > 0$ depending only on $R$ such that

$$\begin{aligned}
&2\sqrt{\varepsilon}\,\mathbb{E}\left[\sup_{r_0\leqslant r\leqslant t\wedge T_R^n} \left|\sum_{k\in\mathbb{N}} \int_{r_0}^{r} \left(\sigma_k\left(u^{n,\varepsilon}(s-\rho)\right) - \sigma_k\left(u^{\varepsilon}(s-\rho)\right), u^{n,\varepsilon}(s) - u^{\varepsilon}(s)\right) \mathrm{d}W_k(s)\right|\right] \\
&\quad \leqslant \frac{1}{2}\mathbb{E}\left[\sup_{r_0\leqslant r\leqslant t} \|u^{n,\varepsilon}\left(r\wedge T_R^n\right) - u^{\varepsilon}\left(r\wedge T_R^n\right)\|^2\right] \\
&\qquad + C_{4,R} \int_{r_0}^{t} \mathbb{E}\left[\sup_{r_0\leqslant s\leqslant r} \|u^{n,\varepsilon}\left(s\wedge T_R^n\right) - u^{\varepsilon}\left(s\wedge T_R^n\right)\|^2\right] \mathrm{d}r \\
&\qquad + C_{4,R} \sum_{k\in\mathbb{N}} \int_{-\rho}^{0} \|\sigma_k\left(\xi^n(s)\right) - \sigma_k\left(\xi(s)\right)\|^2 \mathrm{d}s.
\end{aligned} \tag{3.50}$$

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By (3.49)–(3.50) we have for all $t \in [r_0, t_0]$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r_0 \leqslant r \leqslant t} \|u^{n,\varepsilon}(r \wedge T_R^n) - u^{\varepsilon}(r \wedge T_R^n)\|^2\right] \\
&\quad \leqslant 2(C_{1,R} + C_{2,R} + C_{3,R} + C_{4,R} + 1) \\
&\qquad \cdot \int_{r_0}^{t} \mathbb{E}\left[\sup_{r_0 \leqslant s \leqslant r} \|u_1^{\varepsilon}(s \wedge T_R^n) - u_2^{\varepsilon}(s \wedge T_R^n)\|^2\right] \mathrm{d}r \\
&\qquad + 2\left\|u^n - u^0\right\|^2 + 2\int_{-\rho}^{0} \|f(\xi^n(s)) - f(\xi(s))\|^2 \,\mathrm{d}s \\
&\qquad + 2(C_{4,R} + 1)\sum_{k \in \mathbb{N}} \int_{-\rho}^{0} \|\sigma_k(\xi^n(s)) - \sigma_k(\xi(s))\|^2 \,\mathrm{d}s.
\end{aligned} \tag{3.51}
$$

By Gronwall's inequality and (3.51) we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r_0 \leqslant t \leqslant t_0} \|u^{n,\varepsilon}(t \wedge T_R^n) - u^{\varepsilon}(t \wedge T_R^n)\|^2\right] \\
&\quad \leqslant e^{2(C_{1,R} + C_{2,R} + C_{3,R} + C_{4,R} + 1)(t_0 - r_0)} \\
&\qquad \cdot \Bigg(2\left\|u^n - u^0\right\|^2 + 2\int_{-\rho}^{0} \|f(\xi^n(s)) - f(\xi(s))\|^2 \,\mathrm{d}s \\
&\qquad + 2(C_{4,R} + 1)\sum_{k \in \mathbb{N}} \int_{-\rho}^{0} \|\sigma_k(\xi^n(s)) - \sigma_k(\xi(s))\|^2 \,\mathrm{d}s\Bigg).
\end{aligned} \tag{3.52}
$$

Since $(u^n, \xi^n) \to (u^0, \xi)$ in $l^2 \times L^2((-\rho, 0); l^2)$, by (2.2), (2.4) and the Vitali convergence theorem, we infer that $\int_{-\rho}^{0} \|f(\xi^n(s)) - f(\xi(s))\|^2 \,\mathrm{d}s \to 0$ and $\sum_{k \in \mathbb{N}} \int_{-\rho}^{0} \|\sigma_k(\xi^n(s)) - \sigma_k(\xi(s))\|^2 \,\mathrm{d}s \to 0$. Thus by (3.52) we know that

$$
\mathbb{E}\left[\sup_{r_0 \leqslant t \leqslant t_0} \|u^{n,\varepsilon}(t \wedge T_R^n) - u^{\varepsilon}(t \wedge T_R^n)\|^2\right] \to 0. \tag{3.53}
$$

Since $\|\xi^n - \xi\|_{L^2((-\rho,0);l^2)} \to 0$, it follows from (3.53) that

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_0 - \rho}^{t_0} \|u^{n,\varepsilon}(t \wedge T_R^n) - u^{\varepsilon}(t \wedge T_R^n)\|^2 \,\mathrm{d}t + \|u^{n,\varepsilon}(t_0 \wedge T_R^n) - u^{\varepsilon}(t_0 \wedge T_R^n)\|^2\right] \\
&\quad \leqslant \mathbb{E}\Bigg[\int_{-\rho}^{0} \|\xi^n(t) - \xi(t)\|^2 \,\mathrm{d}t + \int_{r_0}^{t_0} \|u^{n,\varepsilon}(t \wedge T_R^n) - u^{\varepsilon}(t \wedge T_R^n)\|^2 \,\mathrm{d}t \\
&\qquad + \|u^{n,\varepsilon}(t_0 \wedge T_R^n) - u^{\varepsilon}(t_0 \wedge T_R^n)\|^2\Bigg] \\
&\quad \leqslant \int_{-\rho}^{0} \|\xi^n(t) - \xi(t)\|^2 \mathrm{d}t + \mathbb{E}\left[\sup_{r_0 \leqslant t \leqslant t_0} \|u^{n,\varepsilon}(t \wedge T_R^n) - u^{\varepsilon}(t \wedge T_R^n)\|^2\right](t_0 - r_0 + 1) \\
&\quad \longrightarrow 0 \quad \text{as} \quad n \to 0,
\end{aligned}
$$

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as desired. □

By lemmas 3.2 and 3.10 and the arguments of [**17**, pp. 250-252], we can obtain the following properties of $\{p^{\varepsilon}_{r,t}\}_{0\leqslant r\leqslant t}$.

Lemma 3.11. *Suppose that **(A1)**–**(A5)** and **(H)** hold. Then we have:*

(1) *$\{p^{\varepsilon}_{r,t}\}_{0\leqslant r\leqslant t}$ is Feller; that is, if $\varphi : l^2 \times L^2((-\rho, 0); l^2) \to \mathbb{C}$ is bounded and continuous, then for any $0 \leqslant r \leqslant t$, the function $p^{\varepsilon}_{r,t}\varphi : l^2 \times L^2((-\rho, 0); l^2) \to \mathbb{C}$ is also bounded and continuous.*

(2) *$\{p^{\varepsilon}_{r,t}\}_{0\leqslant r\leqslant t}$ is homogeneous; that is, for all $0 \leqslant r \leqslant t$,*

$$p^{\varepsilon}(r, (u^0, \xi); t, \cdot) = p^{\varepsilon}(0, (u^0, \xi); t - r, \cdot), \quad \forall\ (u^0, \xi) \in l^2 \times L^2((-\rho, 0); l^2).$$

(3) *For any $0 \leqslant s \leqslant r \leqslant t$, the Chapman–Kolmogorov equation holds true:*

$$p^{\varepsilon}(s, (u^0, \xi); t, \Gamma) = \int_{l^2\times L^2((-\rho,0);l^2)} p^{\varepsilon}(s, (u^0, \xi); r, dx) p^{\varepsilon}(r, x; t, \Gamma),$$

*where $(u^0, \xi) \in l^2 \times L^2((-\rho, 0); l^2)$ and $\Gamma \in \mathcal{B}(l^2 \times L^2((-\rho, 0); l^2))$.*

*3.2.2. Proof of theorem 2.1.* Now we are in a position to present the proof of theorem 2.1.

*Proof.* For simplicity, we now write $u^{\varepsilon}(t; 0, \mathbf{0}, \mathbf{0})$ as $u^{\varepsilon}(t)$ and $u^{\varepsilon}_t(0, \mathbf{0}, \mathbf{0})$ as $u^{\varepsilon}_t$. By remark 3.3 we see that for given $\varepsilon' > 0$, there exists $R_1 = R_1(\varepsilon') > 0$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left(\left\{\|u^{\varepsilon}_t\|_{C([-\rho,0];l^2)} > R_1\right\}\right) < \frac{\varepsilon'}{3}. \tag{3.54}$$

By remark 3.5, we know that for given $\varepsilon' > 0$ and $m \in \mathbb{N}$, there exists $\eta_{m,\varepsilon'} > 0$ depending only on $m$ and $\varepsilon'$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left(\left\{\sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|<\eta_{m,\varepsilon'}} \|u^{\varepsilon}_t(t_1) - u^{\varepsilon}_t(t_2)\| > \frac{1}{2^m}\right\}\right) < \frac{\varepsilon'}{4^m},$$

and thus

$$\mathbb{P}\left(\bigcup_{m=1}^{\infty}\left\{\sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|<\eta_{m,\varepsilon'}} \|u^{\varepsilon}_t(t_1) - u^{\varepsilon}_t(t_2)\| > \frac{1}{2^m}\right\}\right) < \sum_{m=1}^{\infty} \frac{\varepsilon'}{4^m} \leqslant \frac{\varepsilon'}{3}. \tag{3.55}$$

It follows from remark 3.9 that for given $\varepsilon' > 0$ and $m \in \mathbb{N}$, there exists an integer $n_{m,\varepsilon'} > 0$ depending only on $m$ and $\varepsilon'$ such that for all $t \geqslant 0$,

$$\mathbb{P}\left(\left\{\sup_{t-\rho\leqslant r\leqslant t} \sum_{|n|\geqslant n_{m,\varepsilon'}} |u^{\varepsilon}_n(r)|^2 > \frac{1}{2^m}\right\}\right) < \frac{\varepsilon'}{4^m},$$

and hence we obtain for all $t \geqslant 0$,

$$\mathbb{P}\left(\bigcup_{m=1}^{\infty}\left\{\sup_{t-\rho\leqslant r\leqslant t}\sum_{|n|\geqslant n_{m,\varepsilon'}}|u_n^\varepsilon(r)|^2>\frac{1}{2^m}\right\}\right)<\sum_{m=1}^{\infty}\frac{\varepsilon'}{4^m}\leqslant\frac{\varepsilon'}{3}. \tag{3.56}$$

Given $\varepsilon' > 0$, denote by

$$Z_{1,\varepsilon'} = \{\xi \in C([-\rho,0];l^2) : \|\xi\|_{C([-\rho,0];l^2)} \leqslant R_1\}, \tag{3.57}$$

$$Z_{2,\varepsilon'} = \left\{\xi \in C([-\rho,0];l^2) : \sup_{t_1,t_2\in[-\rho,0],|t_1-t_2|<\eta_{m,\varepsilon'}}\|\xi(t_1)-\xi(t_2)\| \leqslant \frac{1}{2^m}\right.$$
$$\left.\text{for all } m \in \mathbb{N}\right\}, \tag{3.58}$$

$$Z_{3,\varepsilon'} = \left\{\xi \in C([-\rho,0];l^2) : \sup_{-\rho\leqslant s\leqslant 0}\sum_{|n|\geqslant n_{m,\varepsilon'}}|\xi_n(s)|^2 \leqslant \frac{1}{2^m} \text{ for all } m \in \mathbb{N}\right\}, \tag{3.59}$$

and

$$Z_{\varepsilon'} = Z_{1,\varepsilon'}\bigcap Z_{2,\varepsilon'}\bigcap Z_{3,\varepsilon'}. \tag{3.60}$$

It follows from (3.54)–(3.56) that for all $t \geqslant 0$,

$$\mathbb{P}(\{u_t^\varepsilon \in Z_{\varepsilon'}\}) > 1-\varepsilon'. \tag{3.61}$$

By (3.57), (3.59)–(3.60), we know that the set $\{z(0) : z \in Z_{\varepsilon'}\}$ is precompact in $l^2$. Moreover, according to the Ascoli–Arzalà theorem and (3.57)-(3.60), one can show that $Z_{\varepsilon'}$ is a precompact subset of $C([-\rho,\, 0]; l^2)$. Since the embedding $C([-\rho,\, 0]; l^2) \hookrightarrow L^2((-\rho,\, 0); l^2)$ is continuous, $Z_{\varepsilon'}$ is precompact in $L^2((-\rho,\, 0); l^2)$. Thus we conclude that $\tilde{Z}_{\varepsilon'} = \{(z(0),\, z) : z \in Z_{\varepsilon'}\}$ is precompact in $l^2 \times L^2((-\rho,\, 0); l^2)$.

On the other hand, by (3.61) we obtain that for all $t \geqslant 0$,

$$\mathbb{P}\left(\left\{(u^\varepsilon(t), u_t^\varepsilon) \in \tilde{Z}_{\varepsilon'}\right\}\right) = \mathbb{P}(\{u_t^\varepsilon \in Z_{\varepsilon'}\}) > 1-\varepsilon',$$

which along with the precompactness of $\tilde{Z}_{\varepsilon'}$ implies that the distributions of the family $\{(u^\varepsilon(t),\, u_t^\varepsilon) : t \geqslant 0\}$ are tight on $l^2 \times L^2((-\rho,\, 0); l^2)$.

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We denote the distributions of the family $\{(u^\varepsilon(t), u^\varepsilon_t) : t \geqslant 0\}$ by $\{L^\varepsilon_t\}_{t\geqslant 0}$ for simplicity. For given $k \in \mathbb{N}$, we set

$$\mu^\varepsilon_k = \frac{1}{k}\int_0^k L^\varepsilon_t(\cdot)\,\mathrm{d}t. \tag{3.62}$$

By (3.61), we know that for all $k \in \mathbb{N}$,

$$\mu^\varepsilon_k(\tilde{Z}_{\varepsilon'}) > 1 - \varepsilon'. \tag{3.63}$$

Consequently, it follows from (3.62)–(3.63) that $\{\mu^\varepsilon_k\}_{k=1}^\infty$ is tight, and hence there exists a probability measure $\mu^\varepsilon$ on $l^2 \times L^2((-\rho, 0); l^2)$ such that, up to a subsequence, $\mu^\varepsilon_k$ weakly converges to $\mu^\varepsilon$ as $k \to \infty$. Then by lemma 3.11, one can verify that $\mu^\varepsilon$ is an invariant measure of (2.6) by the argument of [**12**, Theorem 4.3]. □

## 4. The large deviation principle

In this section, we will investigate the LDP of the family $\{u^\varepsilon\}_{\varepsilon>0}$ by the weak convergence method. We first review the basic concepts of weak convergence theory in the next subsection.

### 4.1. Preliminaries

In this subsection, we recall some definitions and results from the theory of large deviations. Let $\mathcal{E}$ be a polish space, and $\{X^\varepsilon\}$ be a family of random variables defined on the space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ and taking values in $\mathcal{E}$.

Definition 4.1. *A function $I : \mathcal{E} \to [0, \infty]$ is called to be a rate function, if it is lower semicontinuous on $\mathcal{E}$. A rate function $I$ is called a good rate function, if for each $a \in [0, \infty)$, the level set $\{x \in \mathcal{E} : I(x) \leqslant a\}$ is a compact subset of $\mathcal{E}$.*

Definition 4.2. *Let $I$ be a rate function on $\mathcal{E}$. The family $\{X^\varepsilon\}$ is said to satisfy the LDP on $\mathcal{E}$ with rate function $I$ if the following two conditions hold:*

(1) *Large deviation upper bound. For each closed subset $F$ of $\mathcal{E}$,*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leqslant -\inf_{x\in F} I(x),$$

(2) *Large deviation lower bound. For each open subset $G$ of $\mathcal{E}$,*

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geqslant -\inf_{x\in G} I(x).$$

Definition 4.3. *Let $I$ be a rate function on $\mathcal{E}$. The family $\{X^\varepsilon\}$ is said to satisfy the Laplace principle on $\mathcal{E}$ with rate function $I$ if for all bounded continuous functions $h : \mathcal{E} \to \mathbb{R}$,*

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\left[\exp\left(-\frac{h(X^\varepsilon)}{\varepsilon}\right)\right] = -\inf_{x\in\mathcal{E}}\{h(x) + I(x)\}.$$

Since $\mathcal{E}$ is a polish space, the family $\{X^\varepsilon\}$ satisfies the large deviation principle on $\mathcal{E}$ with a rate function $I$ if and only if the family $\{X^\varepsilon\}$ satisfies the Laplace

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principle on $\mathcal{E}$ with the same rate function. In view of this equivalent result, we will focus on the Laplace principle hereafter. In what follows, we introduce some notations and a criteria for the Laplace principle, which is useful for proving the Laplace principle for the family of solutions $\{u^\varepsilon\}$ of (2.6) on $t \in [0, T]$. Let

$$H = \left\{ u = (u_j)_{j=1}^{\infty} : \sum\nolimits_{j=1}^{\infty} |u_j|^2 < \infty \right\}.$$

For every $k \in \mathbb{N}$, let $e_k = (\delta_{k,j})_{j=1}^{\infty}$ with $\delta_{k,j} = 1$ for $j = k$ and $\delta_{k,j} = 0$ otherwise. Then $\{e_k, k \in \mathbb{N}\}$ is an orthonormal basis of $H$. Let $W$ be the cylindrical Wiener process on $H$ (which does not take values in $H$), given by

$$W(t) = \sum_{k \in \mathbb{N}} W_k(t) e_k, \quad t \in \mathbb{R}^+,$$

where the series converges in $L^2(\Omega; C([0, T]; U))$ with $U$ being a larger separable Hilbert space such that the embedding $H \hookrightarrow U$ is Hilbert–Schmidt.

For each $a > 0$, define

$$S_a = \left\{ v \in L^2([0, T]; H) : \int_0^T \|v(s)\|_H^2 \, \mathrm{d}s \leqslant a \right\}.$$

Then $S_a$ is a Polish space under the weak topology of $L^2([0, T]; H)$. Henceforth, wherever we refer to $S_a$, we will consider it endowed with this topology. Let $\mathcal{A}$ denote the class of $H$-valued $\mathcal{F}_t$-predictable processes $v$ which satisfy $\int_0^T \|v\|_H^2 \, \mathrm{d}s < \infty$, $\mathbb{P}$-almost surely, and for each $a \in (0, \infty)$, we define

$$\mathcal{A}_a = \{v \in \mathcal{A} : v(\omega) \in S_a, \mathbb{P}\text{-almost surely}\}.$$

For each $\varepsilon \in (0, 1)$, let $G^\varepsilon : C([0, T]; U) \to C([0, T]; l^2)$ be a measurable map. The following lemma gives sufficient conditions for the Laplace principle to hold for the family $\{G^\varepsilon(W)\}$ as $\varepsilon \to 0$.

Lemma 4.4 [7], theorem 4.4. *Suppose that there exists a measurable map $G : C([0, T]; U) \to C([0, T]; l^2)$ such that the following two conditions hold:*

(H1) *for each $a \in (0, \infty)$, the set $\left\{ G(\int_0^\cdot v(s) \, \mathrm{d}s) : v \in S_a \right\}$ is a compact subset of $C([0, T]; l^2)$,*

(H2) *if $\{v^\varepsilon\} \subset \mathcal{A}_a$ for some $a > 0$, and $v^\varepsilon$ converges in distribution to $v$ as $S_a$-valued random variables, then $G^\varepsilon(W + \varepsilon^{-1/2} \int_0^\cdot v^\varepsilon(t) \, \mathrm{d}t)$ converges in distribution to $G(\int_0^\cdot v(t) \, \mathrm{d}t)$.*

*Then $\{G^\varepsilon(W)\}$ satisfies the Laplace principle on $C([0, T]; l^2)$ with rate function $I : C([0, T]; l^2) \to [0, \infty]$ defined by*

$$I(x) = \inf \left\{ \frac{1}{2} \int_0^T \|v(t)\|_H^2 \, \mathrm{d}t : x = G\left( \int_0^\cdot v(t) \, \mathrm{d}t \right), v \in L^2([0, T]; H) \right\}, \tag{4.1}$$

*where we use the usual convention $\inf(\emptyset) = \infty$.*

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### 4.2. The LDP for solution processes

This subsection is devoted to formulating the Laplace principle for the family of solutions $\{u^\varepsilon\}$ of system (2.6) on the finite time interval $[0, T]$ as $\varepsilon \to 0$, from which we can establish the LDP for the family $\{u^\varepsilon\}$. We first specify the maps $G^\varepsilon$ and $G$ in the context of system (2.6), and we then use lemma 4.4 to deduce an analogous criterion of the Laplace principle for the family $\{u^\varepsilon\}$.

Given $u \in l^2$, define $\sigma(u) : H \to l^2$ by

$$\sigma(u)(v) = \sum_{k\in\mathbb{N}} (h_k + \sigma_k(u))v_k, \quad \forall\ v = (v_k)_{k=1}^\infty \in H. \tag{4.2}$$

We find that $\sigma(u)$ is well-defined by (2.1) and (2.4). Moreover, the operator is Hilbert–Schmidt and

$$\|\sigma(u)\|_{L(H;l^2)} \leqslant \|\sigma(u)\|_{L_2(H;l^2)} = \left(\sum_{k\in\mathbb{N}} \|h_k + \sigma_k(u)\|^2\right)^{1/2} < \infty,$$

where $L(H; l^2)$ denotes the space of bounded linear operators from $H$ to $l^2$ with norm $\|\cdot\|_{L(H;l^2)}$, and $L_2(H; l^2)$ denotes the space of Hilbert–Schmidt operators from $H$ to $l^2$ with norm $\|\cdot\|_{L_2(H;l^2)}$. In terms of (4.2), system (2.6) on the finite time interval $[0, T]$ can be reformulated as

$$\begin{cases} \mathrm{d}u^\varepsilon(t) = -iAu^\varepsilon(t)\,\mathrm{d}t - i|u^\varepsilon(t)|^2u^\varepsilon(t)\,\mathrm{d}t - \lambda u^\varepsilon(t)\,\mathrm{d}t + f(u^\varepsilon(t-\rho))\,\mathrm{d}t \\ \qquad\qquad + g\,\mathrm{d}t + \sqrt{\varepsilon}\sigma(u^\varepsilon(t-\rho))\,\mathrm{d}W(t),\ t \in [0, T], \\ u^\varepsilon(0) = u^0,\ u^\varepsilon(s) = \xi(s),\ s \in (-\rho, 0), \end{cases} \tag{4.3}$$

Given $(u^0, \xi) \in l^2 \times L^2((-\rho, 0), l^2)$, $\varepsilon \in (0, 1)$ and $T > 0$, by the existence and uniqueness of solutions of system (2.6), we infer that there exists a Borel measurable map $G^\varepsilon : C([0, T]; U) \to C([0, T]; l^2)$ such that $u^\varepsilon = G^\varepsilon(W)$, $\mathbb{P}$-almost surely.

Moreover, for any $v \in \mathcal{A}_a$ with $a \in (0, \infty)$, the Girsanov theorem shows that the stochastic process

$$\widetilde{W}(t) := W(t) + \varepsilon^{-1/2}\int_0^t v(s)\,\mathrm{d}s$$

is a cylindrical Wiener process with identity covariance operator under the probability $\mathbb{P}_v^\varepsilon$ as given by

$$\frac{\mathrm{d}\mathbb{P}_v^\varepsilon}{\mathrm{d}\mathbb{P}} = \exp\left\{-\varepsilon^{-1/2}\int_0^T v(t)\,\mathrm{d}W(t) - \frac{1}{2}\varepsilon^{-1}\int_0^T \|v(t)\|_H^2\,\mathrm{d}t\right\}.$$

Let $u_v^\varepsilon = G^\varepsilon(\widetilde{W})$. Then $u_v^\varepsilon$ is the unique solution of (4.3) with $W$ replaced by $\widetilde{W}$, which implies that $u_v^\varepsilon$ is the unique solution of the following controlled stochastic

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delay system:

$$\begin{cases} \mathrm{d}u_v^\varepsilon(t) = -iAu_v^\varepsilon(t)\,\mathrm{d}t - i|u_v^\varepsilon(t)|^2u_v^\varepsilon(t)\,\mathrm{d}t - \lambda u_v^\varepsilon(t)\,\mathrm{d}t + f(u_v^\varepsilon(t-\rho))\mathrm{d}t \\ \qquad\qquad +g\mathrm{d}t + \sigma(u_v^\varepsilon(t-\rho))v^\varepsilon(t)\mathrm{d}t + \sqrt{\varepsilon}\sigma(u_v^\varepsilon(t-\rho))\mathrm{d}W(t),\ t\in[0,T], \\ u_v^\varepsilon(0) = u^0,\ u_v^\varepsilon(s) = \xi(s),\ s\in(-\rho,0). \end{cases} \tag{4.4}$$

To define the map $G$, we introduce a controlled deterministic delay system associated with (4.3) as follows:

$$\begin{cases} \mathrm{d}u_v(t) = -iAu_v(t)\,\mathrm{d}t - i|u_v(t)|^2u_v(t)\,\mathrm{d}t - \lambda u_v(t)\,\mathrm{d}t + f(u_v(t-\rho))\,\mathrm{d}t \\ \qquad\qquad +g(t)\,\mathrm{d}t + \sigma(u_v(t-\rho))v(t)\,\mathrm{d}t,\ t\in[0,T], \\ u_v(0) = u^0,\ u_v(s) = \xi(s),\ s\in(-\rho,0). \end{cases} \tag{4.5}$$

By a solution $u_v$ of (4.5), we mean $u_v$ is a map from $[-\rho,\, T]$ to $l^2$ such that $u_v(t)$ is continuous for $t\in[0,\, T]$, $u_v(0) = u^0$ and $u_v = \xi$ on $(-\rho,\, 0)$.

For any $v\in L^2([0,\, T]; H)$ and $(u^0,\, \xi)\in l^2\times L^2((-\rho,\, 0); l^2)$, we will prove the existence and uniqueness of solutions of (4.5) in lemma 4.7 in Subsection 4.3. As a consequence of lemma 4.7, we will see that the solution of (4.5) is continuous in $C([0,\, T]; \ell^2)$ with respect to the control term $v$ in $L^2([0,\, T]; H)$. Hence we can define $G : C([0,\, T]; U)\to C([0,\, T]; l^2)$ by

$$G(\varphi) = \begin{cases} u_v, \text{ if } \ \varphi = \int_0^\cdot v(t)\,\mathrm{d}t \ \text{ for } \ \text{some } v\in L^2([0,T]; H); \\ 0, \ \ \text{otherwise}, \end{cases} \tag{4.6}$$

where $u_v$ is the unique solution of (4.5) corresponding to the control term $v$.

By lemma 4.4, we deduce the following result.

Corollary 4.5. *If $G^\varepsilon$ and $G$ defined in this subsection satisfy conditions* ***(H1)*** *and* ***(H2)*** *presented in lemma 4.4, then the family $\{u^\varepsilon\}$ satisfies the Laplace principle on $C([0,\, T]; l^2)$ with the rate function $I$ given by (4.1).*

In the following, we will prove theorem 2.3 by verifying that $G^\varepsilon$ and $G$ defined in this subsection satisfy the conditions **(H1)** and **(H2)** in lemma 4.4.

## 4.3. Proof of theorem 2.3

To prove theorem 2.3, we need the following priori estimates for the solutions of (4.5).

Lemma 4.6. *Suppose that* ***(A1)–(A5)*** *hold and $T > 0$. If $(u^0,\, \xi)\in l^2\times L^2((-\rho,\, 0); l^2)$, $v\in L^2([0,\, T]; H)$ and $u_v$ is a solution of system (4.5), then*

$$\|u_v\|^2_{C([0,T];l^2)} \leqslant C_T\left(\|u^0\|^2 + \int_{-\rho}^0 \|\xi(t)\|^2\,\mathrm{d}t + 1\right) e^{C_T(1+\|v\|^2_{L^2([0,T];H)})},$$

*where $C_T > 0$ is a constant depending only on $T$.*

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*Proof.* By (4.5) we obtain that for all $t \in [0, T]$,

$$\frac{d}{\mathrm{d}t}\|u_v(t)\|^2 \leqslant -2\lambda\|u_v(t)\|^2 + 2\mathrm{Re}\ (f(u_v(t-\rho)), u_v(t)) + 2\mathrm{Re}\ (g, u_v(t)) + 2\mathrm{Re}\ (\sigma(u_v(t-\rho))v(t), u_v(t)). \tag{4.7}$$

For the second term on the right-hand side of (4.7), by (2.2) we get

$$\begin{aligned} 2\mathrm{Re}\ (f(u_v(t-\rho)), u_v(t)) &\leqslant \|u_v(t)\|^2 + \|f(u_v(t-\rho))\|^2 \\ &\leqslant 2\beta_0^2\|u_v(t-\rho)\|^2 + 2\|\alpha\|^2 + \|u_v(t)\|^2. \end{aligned} \tag{4.8}$$

For the last term on the right-hand side of (4.7), by (2.4) we have

$$\begin{aligned} &2\mathrm{Re}\ (\sigma(u_v(t-\rho))v(t), u_v(t)) \\ &\quad\leqslant \|\sigma(u_v(t-\rho))\|^2_{L(H;l^2)} + \|v(t)\|_H^2\|u_v(t)\|^2 \\ &\quad\leqslant \sum_{k\in\mathbb{N}}\|h_k + \sigma_k(u_v(t-\rho))\|^2 + \|v(t)\|_H^2\|u_v(t)\|^2 \\ &\quad\leqslant 2\|h\|^2 + 4\|\beta\|^2\|u_v(t-\rho)\|^2 + 4\|\delta\|^2 + \|v(t)\|_H^2\|u_v(t)\|^2. \end{aligned} \tag{4.9}$$

By (4.7)–(4.9) and Young's inequality we obtain that for all $t \in [0, T]$,

$$\frac{d}{\mathrm{d}t}\|u_v(t)\|^2 \leqslant (2 + \|v(t)\|_H^2)\|u_v(t)\|^2 + (2\beta_0^2 + 4\|\beta\|^2)\|u_v(t-\rho)\|^2 + \|g\|^2 + 2\|\alpha\|^2 + 4\|\delta\|^2 + 2\|h\|^2,$$

which implies

$$\begin{aligned} \|u_v(t)\|^2 \leqslant \|u^0\|^2 &+ \int_0^t (2 + \|v(s)\|_H^2 + 2\beta_0^2 + 4\|\beta\|^2)\|u_v(s)\|^2\,\mathrm{d}s \\ &+ (2\beta_0^2 + 4\|\beta\|^2)\int_{-\rho}^0 \|\xi(s)\|^2\,\mathrm{d}s + (\|g\|^2 + 2\|\alpha\|^2 + 4\|\delta\|^2 + 2\|h\|^2)T. \end{aligned} \tag{4.10}$$

By (4.10) we have for all $t \in [0, T]$,

$$\begin{aligned} \sup_{r\in[0,t]}\|u_v(r)\|^2 \leqslant \|u^0\|^2 &+ \int_0^t (2 + \|v(s)\|_H^2 + 2\beta_0^2 + 4\|\beta\|^2)\sup_{r\in[0,s]}\|u_v(r)\|^2\,\mathrm{d}s \\ &+ (2\beta_0^2 + 4\|\beta\|^2)\int_{-\rho}^0 \|\xi(s)\|^2\,\mathrm{d}s \\ &+ (\|g\|^2 + 2\|\alpha\|^2 + 4\|\delta\|^2 + 2\|h\|^2)T. \end{aligned} \tag{4.11}$$

By Gronwall's inequality, it follows from (4.11) that for all $t \in [0, T]$,

$$\begin{aligned}
&\sup_{r\in[0,t]} \|u_v(r)\|^2 \\
&\quad\leqslant \left( \|u^0\|^2 + (\|g\|^2 + 2\|\alpha\|^2 + 4\|\delta\|^2 + 2\|h\|^2)T + (2\beta_0^2 + 4\|\beta\|^2) \int_{-\rho}^{0} \|\xi(s)\|^2 \, \mathrm{d}s \right) \\
&\qquad \cdot e^{\int_0^T (2+\|v(s)\|_H^2 + 2\beta_0 + 4\|\beta\|^2)\, \mathrm{d}s} \\
&\quad\leqslant \left( \|u^0\|^2 + (\|g\|^2 + 2\|\alpha\|^2 + 4\|\delta\|^2 + 2\|h\|^2)T + (2\beta_0^2 + 4\|\beta\|^2) \int_{-\rho}^{0} \|\xi(s)\|^2 \, \mathrm{d}s \right) \\
&\qquad \cdot e^{(2+2\beta_0+4\|\beta\|^2)T + \|v\|^2_{L^2([0,T];H)}},
\end{aligned}$$

which completes the proof. □

Based on above priori estimates of solutions, we next prove the well-posedness of system (4.5).

LEMMA 4.7. *Suppose that* ***(A1)–(A5)*** *hold. Then for every* $(u^0, \xi) \in l^2 \times L^2((-\rho, 0); l^2)$ *and* $v \in L^2([0, T]; H)$*, system* (4.5) *has a unique solution* $u_v$ *in* $C([0, T]; l^2)$*. Moreover, if* $v_1, v_2 \in L^2([0, T]; H)$ *with* $\|v_1\|_{L^2([0,T];H)} \vee \|v_2\|_{L^2([0,T];H)} \leqslant R_1$ *for some* $R_1 > 0$ *and* $\|u^0\|^2 \vee \int_{-\rho}^0 \|\xi(s)\|^2 \, \mathrm{d}s \leqslant R_2$ *for some* $R_2 > 0$*, then the solutions* $u_{v_1}$ *and* $u_{v_2}$ *of* (4.5) *with initial data* $(u^0, \xi)$ *satisfy*

$$\|u_{v_1} - u_{v_2}\|^2_{C([0,T];l^2)} \leqslant C_1 \|v_1 - v_2\|^2_{L^2([0,T];H)}, \tag{4.12}$$

*where* $C_1 > 0$ *is a constant depending on* $R_1$,$R_2$ *and* $T$*.*

*Proof.* Note that system (4.5) on $[0, \rho]$ is equivalent to the following system without delay:

$$\begin{cases} \mathrm{d}u_v(t) = -iAu_v(t)\, \mathrm{d}t - i|u_v(t)|^2 u_v(t)\, \mathrm{d}t - \lambda u_v(t)\, \mathrm{d}t + f(\xi(t-\rho))\, \mathrm{d}t \\ \qquad\qquad +g\, \mathrm{d}t + \sigma(\xi(t-\rho))v(t)\, \mathrm{d}t, \ \ t \in [0, \rho], \\ u_v(0) = u^0. \end{cases} \tag{4.13}$$

Let $F(t, u) = -iAu - i|u|^2 u - \lambda u + f(\xi(t-\rho)) + g + \sigma(\xi(t-\rho))v(t)$. By (2.2) and (2.4) we find that for every $R > 0$, there exists $C_R > 0$ depending only on $R$ such that for all $t \in [0, T]$ and $u \in l^2$ with $\|u\| \leqslant R$,

$$\|F(t,u)\| \leqslant C_R \left[ (1 + \|v(t)\|_H)\|\xi(t-\rho)\| + \|u\| + \|v(t)\|_H + 1 \right], \tag{4.14}$$

and for all $t \in [0, T]$ and $u_1, u_2 \in l^2$ with $\|u_1\| \vee \|u_2\| \leqslant R$,

$$\|F(t, u_1) - F(t.u_2)\| \leqslant C_R \|u_1 - u_2\|. \tag{4.15}$$

Hence, by (4.14)–(4.15) and lemma 4.6, system (4.13) has a unique solution $u_v$ defined on $[0, \rho]$. Repeating this argument, one can extend the solution $u_v$ to the whole interval $[0, T]$.

Next, we are going to prove (4.12). By lemma 4.6, for $v_1, v_2 \in L^2([0, T]; H)$ with $\|v_1\|_{L^2([0,T];H)} \vee \|v_2\|_{L^2([0,T];H)} \leqslant R_1$ for some $R_1 > 0$ and

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$\|u^0\|^2 \vee \int_{-\rho}^{0} \|\xi(s)\|^2 \, \mathrm{d}s \leqslant R_2$ for some $R_2 > 0$, there exists $K = K(R_1, R_2, T) > 0$ such that $\sup_{t\in[0,T]}(\|u_{v_1}(t)\| + \|u_{v_2}(t)\|) \leqslant K$. By (4.5) we get for all $t \in [0, T]$,

$$
\begin{aligned}
\|u_{v_1}(t) - u_{v_2}(t)\|^2 &\leqslant 2\int_0^t \|u_{v_1}(s) - u_{v_2}(s)\| \|i|u_{v_1}(s)|u_{v_1}(s) - i|u_{v_2}(s)|u_{v_2}(s)\| \, \mathrm{d}s \\
&+ 2\int_0^t \|u_{v_1}(s) - u_{v_2}(s)\| \|f(u_{v_1}(s-\rho)) - f(u_{v_2}(s-\rho))\| \, \mathrm{d}s \\
&+ 2\int_0^t \|\sigma(u_{v_1}(s-\rho))v_1(s) - \sigma(u_{v_2}(s-\rho))v_2(s)\| \|u_{v_1}(s) - u_{v_2}(s)\| \, \mathrm{d}s. \qquad (4.16)
\end{aligned}
$$

By (2.3) and Young's inequality we have

$$
\begin{aligned}
&2\int_0^t \|u_{v_1}(s) - u_{v_2}(s)\| \|f(u_{v_1}(s-\rho)) - f(u_{v_2}(s-\rho))\| \, \mathrm{d}s \\
&\quad + 2\int_0^t \|u_{v_1}(s) - u_{v_2}(s)\| \|i|u_{v_1}(s)|u_{v_1}(s) - i|u_{v_2}(s)|u_{v_2}(s)\| \, \mathrm{d}s \\
&\quad \leqslant (K_1 + 1)\int_0^t \|u_{v_1}(s) - u_{v_2}(s)\|^2 \, \mathrm{d}s, \qquad (4.17)
\end{aligned}
$$

where $K_1 > 0$ depends on $R_1$, $R_2$ and $T$. For the last term on the right-hand side of (4.16), by (2.5) we get that there exists $K_2 = K_2(R_1, R_2, T) > 0$ such that

$$
\begin{aligned}
&2\int_0^t \|\sigma(u_{v_1}(s-\rho))v_1(s) - \sigma(u_{v_2}(s-\rho))v_2(s)\| \|u_{v_1}(s) - u_{v_2}(s)\| \, \mathrm{d}s \\
&\leqslant 2\int_0^t \|\sigma\left(u_{v_1}(s-\rho)\right)v_1(s) - \sigma(u_{v_2}(s-\rho))v_1(s)\| \, \|u_{v_1}(s) - u_{v_2}(s)\| \, \mathrm{d}s \\
&\quad + 2\int_0^t \|\sigma(u_{v_2}(s-\rho))(v_1(s) - v_2(s))\| \|u_{v_1}(s) - u_{v_2}(s)\| \, \mathrm{d}s \\
&\leqslant \int_0^t \sum_{k\in\mathbb{N}} \|\sigma_k(u_{v_1}(s-\rho)) - \sigma_k(u_{v_2}(s-\rho))\|^2 \, \mathrm{d}s \\
&\quad + \int_0^t \|v_1(s)\|_H^2 \|u_{v_1}(s) - u_{v_2}(s)\|^2 \mathrm{d}s \\
&\quad + 2K\int_0^t \|\sigma(u_{v_2}(s-\rho))(v_1(s) - v_2(s))\| \mathrm{d}s \\
&\leqslant K_2\int_0^t \|u_{v_1}(s) - u_{v_2}(s)\|^2 \, \mathrm{d}s + \int_0^t \|v_1(s)\|_H^2 \|u_{v_1}(s) - u_{v_2}(s)\|^2 \mathrm{d}s \\
&\quad + 2K\int_0^t \|\sigma(u_{v_2}(s-\rho))(v_1(s) - v_2(s))\| \, \mathrm{d}s. \qquad (4.18)
\end{aligned}
$$

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Since $\int_{-\rho}^0 \|\xi(s)\|^2 \,\mathrm{d}s \leqslant R_2$ and $\sup_{t\in[0,T]} \|u_{v_2}(t)\| \leqslant K$, we obtain that there exists $K_3 = K_3(R_1, R_2, T) > 0$ such that for all $t \in [0, T]$,

$$\begin{aligned}
&\int_0^t \|\sigma(u_{v_2}(s-\rho))(v_1(s) - v_2(s))\| \,\mathrm{d}s \\
&\quad \leqslant \left(\int_0^t \|\sigma(u_{v_2}(s-\rho))\|^2_{L_2(H;l^2)} \,\mathrm{d}s\right)^{1/2} \left(\int_0^t \|v_1(s) - v_2(s)\|^2_H \,\mathrm{d}s\right)^{1/2} \\
&\quad \leqslant K_3 \|v_1 - v_2\|_{L^2([0,T];H)}.
\end{aligned} \tag{4.19}$$

It follows from (4.16)–(4.19) that

$$\begin{aligned}
\|u_{v_1}(t) - u_{v_2}(t)\|^2 \leqslant &\int_0^t (K_1 + K_2 + 1 + \|v_1(s)\|^2_H) \|u_{v_1}(s) - u_{v_2}(s)\|^2 \,\mathrm{d}s \\
&+ 2KK_3 \|v_1 - v_2\|_{L^2([0,T];H)}.
\end{aligned}$$

Then by Gronwall's inequality we know that for all $t \in [0, T]$,

$$\|u_{v_1}(t) - u_{v_2}(t)\|^2 \leqslant 2KK_3 \|v_1 - v_2\|_{L^2([0,T];H)} e^{(K_1+K_2+1)T + \|v_1\|^2_{L^2([0,T];H)}},$$

which completes the proof. □

The following lemma shows the continuity of an integral operator.

Lemma 4.8. *Suppose that* ***(A1)**–**(A5)*** *hold. For a fixed* $\varphi \in L^\infty([0, T]; l^2) \cap L^2((-\rho, T); l^2)$, *define the operator* $\Gamma : L^2([0, T]; H) \to C([0, T]; l^2)$ *by*

$$\Gamma(v)(t) = \int_0^t \sigma(\varphi(s-\rho)) v(s) \,\mathrm{d}s, \quad \forall\ v \in L^2([0, T]; H). \tag{4.20}$$

*Then* $\Gamma$ *is continuous from the weak topology of* $L^2([0, T]; H)$ *to the strong topology of* $C([0, T]; l^2)$.

*Proof.* Note that the operator $\Gamma : L^2([0, T]; H) \to C([0, T]; l^2)$ is well-defined. In fact, by (2.1) and (2.4) we get for every $v \in L^2([0, T]; H)$,

$$\begin{aligned}
&\int_0^T \|\sigma(\varphi(s-\rho)) v(s)\| \,\mathrm{d}s \\
&\quad \leqslant \int_0^T \|\sigma(\varphi(s-\rho))\|_{L(H;l^2)} \|v(s)\|_H \,\mathrm{d}s \\
&\quad \leqslant \left(\int_0^T \|\sigma(\varphi(s-\rho))\|^2_{L_2(H;l^2)} \,\mathrm{d}s\right)^{1/2} \|v\|_{L^2([0,T];H)}
\end{aligned}$$

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$$\leqslant \left(2\|h\|^2 T + 4\|\beta\|^2 \|\varphi\|^2_{L^\infty([0,T];l^2)} T + 4\|\delta\|^2 T + 4\|\beta\|^2 \int_{-\rho}^{0} \|\varphi(s)\|^2 \mathrm{d}s\right)^{1/2}$$
$$\cdot \|v\|_{L^2([0,T];H)} < \infty, \tag{4.21}$$

which implies that $\Gamma(v) \in C([0, T]; l^2)$ for all $v \in L^2([0, T]; H)$. Moreover, by (4.21) we get that $\Gamma : L^2([0, T]; H) \to C([0, T]; l^2)$ is bounded. On the other hand, from (4.2) and (4.20) it is easy to see that $\Gamma : L^2([0, T]; H) \to C([0, T]; l^2)$ is linear. Since the operator $\Gamma : L^2([0, T]; H) \to C([0, T]; l^2)$ is strongly continuous and linear, one can deduce that $\Gamma$ is weakly continuous. Following the argument of [**33**, Lemma 4.3] with small modification and the Ascoli–Arzelà theorem, one can further show that $\Gamma$ is continuous from the weak topology of $L^2([0, T]; H)$ to the strong topology of $C([0, T]; l^2)$. □

Thanks to this lemma, we can proceed to prove the continuity of $u_v$ in $C([0, T]; l^2)$ with respect to $v \in L^2([0, T]; H)$ in the weak topology of $L^2([0, T]; H)$, which is crucial to verify condition **(H1)** for the Laplace principle of $\{u^\varepsilon\}$.

LEMMA 4.9. *Suppose that **(A1)**–**(A5)** hold. If $v_n \to v$ weakly in $L^2([0, T]; H)$, then $u_{v_n} \to u_v$ strongly in $C([0, T]; l^2)$, where $u_{v_n}$ and $u_v$ are solutions of (4.5) corresponding to $v_n$ and $v$, respectively.*

*Proof.* Since $v_n \to v$ weakly in $L^2([0, T]; H)$, there exists a constant $N_1 > 0$ such that $\|v\|_{L^2([0,T];H)} \leqslant N_1$ and $\|v_n\|_{L^2([0,T];H)} \leqslant N_1$ for all $n \in \mathbb{N}$. Then by lemma 4.6 there exists a constant $N_2 = N_2(N_1, T, u^0, \xi) > 0$ such that

$$\sup_{t\in[0,T]} (\|u_{v_n}(t)\| \vee \|u_v(t)\|) \leqslant N_2, \quad \forall\, n \in \mathbb{N}.$$

By (4.5) we have

$$\begin{aligned}\frac{d}{\mathrm{d}t}(u_{v_n}(t) - u_v(t)) = &-iA(u_{v_n}(t) - u_v(t)) - i(|u_{v_n}(t)|^2 u_{v_n}(t) - |u_v(t)|^2 u_v(t)) \\ &- \lambda(u_{v_n}(t) - u_v(t)) + f(u_{v_n}(t-\rho)) - f(u_v(t-\rho)) \\ &+ \sigma(u_{v_n}(t-\rho))v_n(t) - \sigma(u_v(t-\rho))v(t).\end{aligned} \tag{4.22}$$

We set

$$\Theta_n(t) = \int_0^t \sigma(u_v(s-\rho))(v_n(s) - v(s))\,\mathrm{d}s.$$

Since $v_n \to v$ weakly in $L^2([0, T]; H)$, by lemma 4.8 we obtain

$$\Theta_n(t) \to 0 \text{ in } C([0,T]; l^2), \text{ as } n \to \infty. \tag{4.23}$$

Then by (4.22) and (4.23) one can show $u_{v_n} \to u_v$ strongly in $C([0, T]; l^2)$. The details are similar to [**33**] and hence omitted here. □

We now prove the map $G$ given by (4.6) fulfills condition **(H1)** in lemma 4.4.

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LEMMA 4.10. *Suppose that **(A1)**–**(A5)** hold. Then for every $a \in (0, \infty)$, the set*

$$\Xi_a = \left\{ G\left( \int_0^{\cdot} v(t)\,\mathrm{d}t \right) : v \in S_a \right\} \tag{4.24}$$

*is a compact subset in $C([0, T]; l^2)$.*

*Proof.* Let $\{u_{v_n}\}$ be any sequence in $\Xi_a$ and $\{v_n\} \subset S_a \subset L^2([0, T]; H)$. Since $S_a$ is a polish space under the weak topology of $L^2([0, T]; H)$, there exist $v \in S_a$ and a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to v$ weakly. Then, by lemma 4.9 we know that $u_{v_{n_k}} \to u_v$ in $C([0, T]; l^2)$, which implies that $\Xi_a$ is compact in $C([0, T]; l^2)$. □

Next, we derive the uniform estimates for the solutions of (4.4).

LEMMA 4.11. *Suppose that **(A1)**–**(A5)** hold, $v \in \mathcal{A}_a$ for some $a \in (0, \infty)$ and $(u^0, \xi) \in l^2 \times L^2((-\rho, 0); l^2)$ with $\|u^0\|^2 \vee \int_{-\rho}^0 \|\xi(s)\|^2\,\mathrm{d}s \leqslant R$ for some $R > 0$. Let $u_v^\varepsilon$ be the unique solution of system (4.4) with $v$. Then there exists a constant $C_2 > 0$ depending only on $a$, $R$ and $T$ such that*

$$\sup_{\varepsilon \in (0,1)} \mathbb{E}\left[ \sup_{s \in [0,T]} \|u_v^\varepsilon(s)\|^2 \right] \leqslant C_2. \tag{4.25}$$

*Proof.* Applying Itô's formula to (4.4), we obtain that for all $t \in [0, T]$,

$$\begin{aligned}
\|u_v^\varepsilon(t)\|^2 &\leqslant \|u^0\|^2 + 2\mathrm{Re} \int_0^t (u_v^\varepsilon(s), f(u_v^\varepsilon(s-\rho)))\,\mathrm{d}s + 2\mathrm{Re} \int_0^t (u_v^\varepsilon(s), g)\,\mathrm{d}s \\
&\quad + 2\mathrm{Re} \int_0^t (u_v^\varepsilon(s), \sigma(u_v^\varepsilon(s-\rho)) v^\varepsilon(s))\,\mathrm{d}s + \varepsilon \int_0^t \|\sigma(u_v^\varepsilon(s-\rho))\|_{L_2(H;l^2)}^2\,\mathrm{d}s \\
&\quad + 2\sqrt{\varepsilon}\mathrm{Re} \int_0^t (u_v^\varepsilon(s), \sigma(u_v^\varepsilon(s-\rho))\,\mathrm{d}W(s)).
\end{aligned} \tag{4.26}$$

By (2.4) and the Hölder inequality we obtain

$$\begin{aligned}
&2\mathrm{Re} \int_0^t (u_v^\varepsilon(s), \sigma(u_v^\varepsilon(s-\rho)) v(s))\,\mathrm{d}s \\
&\quad \leqslant 2 \sup_{0 \leqslant s \leqslant t} \|u_v^\varepsilon(s)\| \int_0^t \|\sigma(u_v^\varepsilon(s-\rho))\|_{L_2(H;l^2)} \|v(s)\|_H\,\mathrm{d}s \\
&\quad \leqslant 2 \sup_{0 \leqslant s \leqslant t} \|u_v^\varepsilon(s)\| \left( \int_0^t \|\sigma(u_v^\varepsilon(s-\rho))\|_{L_2(H;l^2)}^2\,\mathrm{d}s \right)^{1/2} \left( \int_0^t \|v(s)\|_H^2\,\mathrm{d}s \right)^{1/2}
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \frac{1}{4} \sup_{0\leqslant s\leqslant t} \|u_v^\varepsilon(s)\|^2 + 4a \int_0^t \|\sigma(u_v^\varepsilon(s-\rho))\|^2_{L_2(H;l^2)}\,\mathrm{d}s \\
&\leqslant \frac{1}{4} \sup_{0\leqslant s\leqslant t} \|u_v^\varepsilon(s)\|^2 + 8aT\|h\|^2 + 16a\|\delta\|^2 T + 16a\|\beta\|^2 \int_0^t \|u_v^\varepsilon(s)\|^2\,\mathrm{d}s \\
&\quad + 16a\|\beta\|^2 \int_{-\rho}^0 \|\xi(s)\|^2\,\mathrm{d}s. \qquad (4.27)
\end{aligned}
$$

It follows from (2.4), (4.8), (4.26)–(4.27) and Young's inequality that for all $t \in [0, T]$ and $\varepsilon \in (0, 1)$,

$$
\begin{aligned}
&\frac{3}{4}\mathbb{E}\left[\sup_{0\leqslant s\leqslant t} \|u_v^\varepsilon(s)\|^2\right] \\
&\quad\leqslant \|u^0\|^2 + (2\beta_0^2 + 4\|\beta\|^2 + 2 + 16a\|\beta\|^2) \int_0^t \mathbb{E}\left[\sup_{0\leqslant r\leqslant s} \|u_v^\varepsilon(r)\|^2\right]\mathrm{d}s \\
&\qquad + (2\beta_0^2 + 4\|\beta\|^2 + 16a\|\beta\|^2) \int_{-\rho}^0 \|\xi(s)\|^2\,\mathrm{d}s + 2\|\alpha\|^2 T + 4\|\delta\|^2 T \\
&\qquad + \|g\|^2 T + 2\|h\|^2 T + 8aT\|h\|^2 + 16a\|\delta\|^2 T \\
&\qquad + 2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{0\leqslant r\leqslant t} \left|\mathrm{Re}\int_0^r (u_v^\varepsilon(s), \sigma(u_v^\varepsilon(s-\rho))\,\mathrm{d}W(s))\right|\right]. \qquad (4.28)
\end{aligned}
$$

For the last term on the right-hand side of (4.28), by the Burkholder inequality we have for all $t \in [0, T]$ and $\varepsilon \in (0, 1)$,

$$
\begin{aligned}
&2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{0\leqslant r\leqslant t} \left|\mathrm{Re}\int_0^r (u_v^\varepsilon(s), \sigma(u_v^\varepsilon(s-\rho))\,\mathrm{d}W(s))\right|\right] \\
&\quad\leqslant 6\mathbb{E}\left[\left(\int_0^t \|u_v^\varepsilon(s)\|^2 \|\|\sigma(u_v^\varepsilon(s-\rho)\|^2_{L_2(H;l^2)}\,\mathrm{d}s\right)^{1/2}\right] \\
&\quad\leqslant \frac{1}{4}\mathbb{E}\left[\sup_{0\leqslant s\leqslant t} \|u_v^\varepsilon(s)\|^2\right] + 36\int_0^t \|\sigma(u_v^\varepsilon(s-\rho))\|^2_{L_2(H;l^2)}\,\mathrm{d}s \\
&\quad\leqslant \frac{1}{4}\mathbb{E}\left[\sup_{0\leqslant s\leqslant t} \|u_v^\varepsilon(s)\|^2\right] + 36\Bigg(2\|h\|^2 T + 4\|\delta\|^2 T \\
&\qquad + 4\|\beta\|^2 \int_{-\rho}^0 \|\xi(s)\|^2\,\mathrm{d}s + 4\|\beta\|^2 \int_0^t \mathbb{E}\left[\sup_{0\leqslant r\leqslant s} \|u_v^\varepsilon(r)\|^2\right]\mathrm{d}s\Bigg). \qquad (4.29)
\end{aligned}
$$

Then (4.25) follows from (4.28)–(4.29) and Gronwall's inequality. □

We now prove $G$ and $G^\varepsilon$ satisfy condition **(H2)** in lemma 4.4.

LEMMA 4.12. *Suppose that* ***(A1)***–***(A5)*** *hold and* $\{v^\varepsilon\} \subseteq \mathcal{A}_a$ *for some* $a \in (0, \infty)$. *If* $\{v^\varepsilon\}$ *converges in distribution to* $v$ *as* $S_a$*-valued random variables, then* $G^\varepsilon(W + 1/\sqrt{\varepsilon}\int_0^\cdot v^\varepsilon(t)\,\mathrm{d}t)$ *converges to* $G(\int_0^\cdot v(t)\,\mathrm{d}t)$ *in distribution.*

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*Proof.* Notice that $u_v = G(\int_0^\cdot v(t)\,\mathrm{d}t)$ is the solution of (4.5) with the control $v$. Let $u^\varepsilon_{v^\varepsilon} = G^\varepsilon(W + 1/\sqrt{\varepsilon}\int_0^\cdot v^\varepsilon(t)\,\mathrm{d}t)$. Then $u^\varepsilon_{v^\varepsilon}$ is the solution of the following system:

$$\begin{cases} \mathrm{d}u^\varepsilon_{v^\varepsilon}(t) = -iAu^\varepsilon_{v^\varepsilon}(t)\,\mathrm{d}t - i|u^\varepsilon_{v^\varepsilon}(t)|^2 u^\varepsilon_{v^\varepsilon}(t)\,\mathrm{d}t - \lambda u^\varepsilon_{v^\varepsilon}(t)\,\mathrm{d}t + f(u^\varepsilon_{v^\varepsilon}(t-\rho))\,\mathrm{d}t \\ \qquad\qquad +g\mathrm{d}t + \sigma(u^\varepsilon_{v^\varepsilon}(t-\rho))v^\varepsilon(t)\mathrm{d}t + \sqrt{\varepsilon}\sigma(u^\varepsilon_{v^\varepsilon}(t-\rho))\mathrm{d}W(t),\ t\in[0,T], \\ u^\varepsilon_{v^\varepsilon}(0) = u^0,\ u^\varepsilon_{v^\varepsilon}(s) = \xi(s),\ s\in(-\rho,0). \end{cases} \tag{4.30}$$

In order to show that $u^\varepsilon_{v^\varepsilon}$ converges to $u_v$ in $C([0,\,T];l^2)$ in distribution, we first establish the convergence of $u^\varepsilon_{v^\varepsilon} - u_{v^\varepsilon}$, where $u_{v^\varepsilon} = G(\int_0^\cdot v^\varepsilon(t)\,\mathrm{d}t)$ is the solution of the following system:

$$\begin{cases} \mathrm{d}u_{v^\varepsilon}(t) = -iAu_{v^\varepsilon}(t)\,\mathrm{d}t - i|u_{v^\varepsilon}(t)|^2 u_{v^\varepsilon}(t)\,\mathrm{d}t - \lambda u_{v^\varepsilon}(t)\,\mathrm{d}t + f(u_{v^\varepsilon}(t-\rho))\,\mathrm{d}t \\ \qquad\qquad +g\,\mathrm{d}t + \sigma(u_{v^\varepsilon}(t-\rho))v^\varepsilon(t)\,\mathrm{d}t,\ t\in[0,T], \\ u_{v^\varepsilon}(0) = u^0,\ u_{v^\varepsilon}(s) = \xi(s),\ s\in(-\rho,0). \end{cases} \tag{4.31}$$

Thus by (4.30)–(4.31) we have

$$\begin{aligned} d(u^\varepsilon_{v^\varepsilon}(t) - u_{v^\varepsilon}(t)) = & -iA\left(u^\varepsilon_{v^\varepsilon}(t) - u_{v^\varepsilon}(t)\right)\,\mathrm{d}t - \lambda\left(u^\varepsilon_{v^\varepsilon}(t) - u_{v^\varepsilon}(t)\right)\mathrm{d}t \\ & - i\left(|u^\varepsilon_{v^\varepsilon}(t)|^2 u^\varepsilon_{v^\varepsilon}(t) - |u_{v^\varepsilon}(t)|^2 u_{v^\varepsilon}(t)\right)\mathrm{d}t \\ & + \left(f(u^\varepsilon_{v^\varepsilon}(t-\rho)) - f(u_{v^\varepsilon}(t-\rho))\right)\mathrm{d}t \\ & + \left(\sigma(u^\varepsilon_{v^\varepsilon}(t-\rho))v^\varepsilon(t) - \sigma(u_{v^\varepsilon}(t-\rho))v^\varepsilon(t)\right)\mathrm{d}t \\ & + \sqrt{\varepsilon}\sigma(u^\varepsilon_{v^\varepsilon}(t-\rho))\,\mathrm{d}W(t). \end{aligned} \tag{4.32}$$

For a given constant $M > 0$, we define a stopping time $\tau^\varepsilon$ by

$$\tau^\varepsilon = \inf\{t \geqslant 0 : \|u^\varepsilon_{v^\varepsilon}(t)\| \geqslant M\} \wedge T,$$

and the infimum of the empty set is taken to be $\infty$. Applying Itô's formula to (4.32) yields that for all $t \in [0,\,T]$,

$$\begin{aligned} & \sup_{0\leqslant r\leqslant t} \|u^\varepsilon_{v^\varepsilon}(r\wedge\tau^\varepsilon) - u_{v^\varepsilon}(r\wedge\tau^\varepsilon)\|^2 \\ & \quad \leqslant 2\int_0^{t\wedge\tau^\varepsilon} \left\||u^\varepsilon_{v^\varepsilon}(s)|^2 u^\varepsilon_{v^\varepsilon}(s) - |u_{v^\varepsilon}(s)|^2 u_{v^\varepsilon}(s)\right\| \|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|\ \mathrm{d}s \end{aligned}$$

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$$
\begin{aligned}
&+ 2\int_0^{t\wedge\tau^\varepsilon} \|f(u^\varepsilon_{v^\varepsilon}(s-\rho)) - f(u_{v^\varepsilon}(s-\rho))\|\|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|\,\mathrm{d}s \\
&+ 2\int_0^{t\wedge\tau^\varepsilon} \|\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))v^\varepsilon(s) - \sigma(u_{v^\varepsilon}(s-\rho))v^\varepsilon(s)\|\|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|\,\mathrm{d}s \\
&+ 2\sqrt{\varepsilon}\sup_{0\leqslant r\leqslant t}\left|\int_0^{r\wedge\tau^\varepsilon} (u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s), \sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))\,\mathrm{d}W(s))\right| \\
&+ \varepsilon\int_0^{t\wedge\tau^\varepsilon} \|\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))\|^2_{L_2(H;l^2)}\,\mathrm{d}s. \qquad (4.33)
\end{aligned}
$$

For fixed $(u^0, \xi) \in l^2 \times L^2((-\rho, 0); l^2)$ and $\{v^\varepsilon\} \subset \mathcal{A}_a$, by lemma 4.6 there exists a positive constant $C_3 = C_3(a, u^0, \xi, T)$ such that for all $\varepsilon \in (0, 1)$, $\mathbb{P}$-almost surely,

$$
\sup_{t\in[0,T]} \|u_{v^\varepsilon}(t)\| \leqslant C_3. \qquad (4.34)
$$

For the first term on the right-hand side of (4.33), by (4.34) we get

$$
\begin{aligned}
&2\int_0^{t\wedge\tau^\varepsilon} \left\||u^\varepsilon_{v^\varepsilon}(s)|^2 u^\varepsilon_{v^\varepsilon}(s) - |u_{v^\varepsilon}(s)|^2 u_{v^\varepsilon}(s)\right\| \|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|\,\mathrm{d}s \\
&\quad\leqslant C_4\int_0^{t\wedge\tau^\varepsilon} \|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|^2\,\mathrm{d}s \\
&\quad\leqslant C_4\int_0^{t} \sup_{0\leqslant r\leqslant s}\|u^\varepsilon_{v^\varepsilon}(r\wedge\tau^\varepsilon) - u_{v^\varepsilon}(r\wedge\tau^\varepsilon)\|^2\,\mathrm{d}s, \qquad (4.35)
\end{aligned}
$$

where $C_4 > 0$ depends on $a$, $u^0$, $\xi$, $T$ and $M$. For the second term on the right-hand side of (4.33), by (2.3) we get

$$
\begin{aligned}
&2\int_0^{t\wedge\tau^\varepsilon} \|f(u^\varepsilon_{v^\varepsilon}(s-\rho)) - f(u_{v^\varepsilon}(s-\rho))\|\|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|\,\mathrm{d}t \\
&\quad\leqslant \int_0^{t\wedge\tau^\varepsilon} \|f(u^\varepsilon_{v^\varepsilon}(s-\rho)) - f(u_{v^\varepsilon}(s-\rho))\|^2\,\mathrm{d}s + \int_0^{t\wedge\tau^\varepsilon} \|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|^2\,\mathrm{d}s \\
&\quad\leqslant (C_5+1)\int_0^{t} \sup_{0\leqslant r\leqslant s}\|u^\varepsilon_{v^\varepsilon}(r\wedge\tau^\varepsilon) - u_{v^\varepsilon}(r\wedge\tau^\varepsilon)\|^2\,\mathrm{d}s, \qquad (4.36)
\end{aligned}
$$

where $C_5 > 0$ depends on $a$, $u^0$, $\xi$, $T$ and $M$. For the third term on the right-hand side of (4.33), by (2.5) we obtain

$$
\begin{aligned}
&2\int_0^{t\wedge\tau^\varepsilon} \|\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))v^\varepsilon(s) - \sigma(u_{v^\varepsilon}(s-\rho))v^\varepsilon(s)\|\|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|\,\mathrm{d}s \\
&\quad\leqslant \int_0^{t\wedge\tau^\varepsilon} \|v^\varepsilon(s)\|^2_H\|u^\varepsilon_{v^\varepsilon}(s) - u_{v^\varepsilon}(s)\|^2\,\mathrm{d}s
\end{aligned}
$$

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$$
\begin{aligned}
&+\int_0^{t\wedge\tau^\varepsilon}\sum_{k\in\mathbb{N}}\|\sigma_k(u^\varepsilon_{v^\varepsilon}(s-\rho))-\sigma_k(u_{v^\varepsilon}(s-\rho))\|^2\,\mathrm{d}s\\
&\leqslant\int_0^t(\|v^\varepsilon(s)\|_H^2+C_6)\sup_{0\leqslant r\leqslant s}\|u^\varepsilon_{v^\varepsilon}(r\wedge\tau^\varepsilon)-u_{v^\varepsilon}(r\wedge\tau^\varepsilon)\|^2\,\mathrm{d}s,
\end{aligned}\tag{4.37}
$$

where $C_6$ depends on $a$, $u^0$, $\xi$, $T$ and $M$. For the last term on the right-hand side of (4.33), by (2.4) we get

$$
\begin{aligned}
&\varepsilon\int_0^{t\wedge\tau^\varepsilon}\|\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))\|^2_{L_2(H;l^2)}\,\mathrm{d}s\\
&\quad\leqslant 2\varepsilon\int_0^{t\wedge\tau^\varepsilon}(\|h\|^2+2\|\beta\|^2\|u^\varepsilon_{v^\varepsilon}(s-\rho)\|^2+2\|\delta\|^2)\,\mathrm{d}s\\
&\quad\leqslant 2\varepsilon\|h\|^2T+4\varepsilon\|\delta\|^2T+4\varepsilon\|\beta\|^2M^2T+4\varepsilon\|\beta\|^2\int_{-\rho}^0\|\xi(s)\|^2\,\mathrm{d}s.
\end{aligned}\tag{4.38}
$$

It follows from (4.33)–(4.38) that for all $t\in[0,T]$,

$$
\begin{aligned}
&\sup_{0\leqslant r\leqslant t}\|u^\varepsilon_{v^\varepsilon}(r\wedge\tau^\varepsilon)-u_{v^\varepsilon}(r\wedge\tau^\varepsilon)\|^2\\
&\quad\leqslant\int_0^t(C_4+C_5+C_6+1+\|v^\varepsilon(s)\|_H^2)\sup_{0\leqslant r\leqslant s}\|u^\varepsilon_{v^\varepsilon}(r\wedge\tau^\varepsilon)-u_{v^\varepsilon}(r\wedge\tau^\varepsilon)\|^2\,\mathrm{d}s\\
&\qquad+2\sqrt{\varepsilon}\sup_{0\leqslant r\leqslant T}\left|\int_0^{r\wedge\tau^\varepsilon}(u^\varepsilon_{v^\varepsilon}(s)-u_{v^\varepsilon}(s),\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))\,\mathrm{d}W(s))\right|\\
&\qquad+2\varepsilon\|h\|^2T+4\varepsilon\|\delta\|^2T+4\varepsilon\|\beta\|^2M^2T+4\varepsilon\|\beta\|^2\int_{-\rho}^0\|\xi(s)\|^2\,\mathrm{d}s.
\end{aligned}\tag{4.39}
$$

By (4.39) and Gronwall's inequality, we obtain that for all $t\in[0,T]$,

$$
\begin{aligned}
&\sup_{0\leqslant r\leqslant t}\|u^\varepsilon_{v^\varepsilon}(r\wedge\tau^\varepsilon)-u_{v^\varepsilon}(r\wedge\tau^\varepsilon)\|^2\\
&\quad\leqslant 2C_7\sqrt{\varepsilon}\sup_{0\leqslant r\leqslant T}\left|\int_0^{r\wedge\tau^\varepsilon}(u^\varepsilon_{v^\varepsilon}(s)-u_{v^\varepsilon}(s),\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))\,\mathrm{d}W(s))\right|\\
&\qquad+2C_7\varepsilon\|h\|^2T+4C_7\varepsilon\|\delta\|^2T+4C_7\varepsilon\|\beta\|^2M^2T+4C_7\varepsilon\|\beta\|^2\int_{-\rho}^0\|\xi(s)\|^2\mathrm{d}s,
\end{aligned}\tag{4.40}
$$

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where $C_7 = e^{(C_4+C_5+C_6+1)T+a}$. By Doob's maximal inequality and (4.34), we now estimate the first term on the right-hand side of (4.40),

$$\begin{aligned} &4\varepsilon\mathbb{E}\left[\sup_{0\leqslant r\leqslant T}\left|\int_0^{r\wedge\tau^\varepsilon}\left(u^\varepsilon_{v^\varepsilon}(s)-u_{v^\varepsilon}(s),\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))\,\mathrm{d}W(s)\right)\right|^2\right] \\ &\leqslant 16\varepsilon\mathbb{E}\left[\int_0^{T\wedge\tau^\varepsilon}\|u^\varepsilon_{v^\varepsilon}(s)-u_{v^\varepsilon}(s)\|^2\|\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))\|^2_{L_2(H;l^2)}\,\mathrm{d}s\right] \\ &\leqslant 16\varepsilon(M+C_3)^2\mathbb{E}\left[\int_0^{T\wedge\tau^\varepsilon}\|\sigma(u^\varepsilon_{v^\varepsilon}(s-\rho))\|^2_{L_2(H;l^2)}\,\mathrm{d}s\right] \\ &\leqslant 32\varepsilon(M+C_3)^2\left(\|h\|^2T+2\|\delta\|^2T+2\|\beta\|^2M^2T+2\|\beta\|^2\int_{-\rho}^0\|\xi(s)\|^2\,\mathrm{d}s\right). \end{aligned} \tag{4.41}$$

From (4.40)–(4.41), it follows that

$$\lim_{\varepsilon\to 0}\sup_{0\leqslant r\leqslant T}\|u^\varepsilon_{v^\varepsilon}(r\wedge\tau^\varepsilon)-u_{v^\varepsilon}(r\wedge\tau^\varepsilon)\|^2=0 \quad \text{in probability.} \tag{4.42}$$

Recalling the definition of $\tau^\varepsilon$, by the Chebyshev inequality and lemma 4.11 we obtain

$$\mathbb{P}(\tau^\varepsilon<T)=\mathbb{P}\left(\sup_{t\in[0,T]}\|u^\varepsilon_{v^\varepsilon}(t)\|\geqslant M\right)\leqslant\frac{1}{M^2}\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\varepsilon_{v^\varepsilon}(t)\|^2\right]\leqslant\frac{C_2}{M^2}.$$

Hence it follows that

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|u^\varepsilon_{v^\varepsilon}(t)-u_{v^\varepsilon}(t)\|>\eta\right) \\ &\quad\leqslant\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|u^\varepsilon_{v^\varepsilon}(t)-u_{v^\varepsilon}(t)\|>\eta,\tau^\varepsilon=T\right) \\ &\qquad+\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|u^\varepsilon_{v^\varepsilon}(t)-u_{v^\varepsilon}(t)\|>\eta,\tau^\varepsilon<T\right) \\ &\quad\leqslant\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|u^\varepsilon_{v^\varepsilon}(t\wedge\tau^\varepsilon)-u_{v^\varepsilon}(t\wedge\tau^\varepsilon)\|>\eta\right)+\frac{C_2}{M^2}, \end{aligned} \tag{4.43}$$

which implies that

$$\lim_{\varepsilon\to 0}\sup_{0\leqslant t\leqslant T}\|u^\varepsilon_{v^\varepsilon}(t)-u_{v^\varepsilon}(t)\|^2=0 \quad \text{in probability.} \tag{4.44}$$

Since $\{v^\varepsilon\}$ converges in distribution to $v$ as $S_a$-valued random variables, according to Skorokhod's representation theorem, there exist a probability space $(\widetilde{\Omega}, \widetilde{F}, \widetilde{P})$, and $S_a$-valued random variables $\{\widetilde{v}^\varepsilon\}$ and $\widetilde{v}$ with the same distribution as $\{v^\varepsilon\}$ and $v$, respectively, such that $\{\widetilde{v}^\varepsilon\}\to\widetilde{v}$ $\widetilde{P}$-almost surely in $S_a$. By lemma 4.9 we infer

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that $u_{\widetilde{v}^\varepsilon} \to u_{\widetilde{v}}$ $\widetilde{P}$-almost surely in $C([0,\,T];l^2)$. Then $u_{\widetilde{v}^\varepsilon} \to u_{\widetilde{v}}$ in $C([0,\,T];l^2)$ in distribution, and hence

$$u_{v^\varepsilon} \to u_v \text{ in } C([0,T];l^2) \quad \text{in distribution,} \tag{4.45}$$

which together with (4.44) implies the desired result. □

By lemma 4.10, lemma 4.12 and corollary 4.5, we see that the family $\{u^\varepsilon\}$ satisfies the LDP provided **(A1)**–**(A5)** hold. This completes the proof of theorem 2.3.

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