ANZIAM J. 56(2015), 381–396
doi:10.1017/S1446181115000061

TRANSMISSION OF ELASTIC WAVES IN ANISOTROPIC NEMATIC ELASTOMERS

S. S. SINGH[1]

(Received 8 August, 2014; revised 12 March, 2015)

Abstract

The problem of reflection and refraction of elastic waves due to an incident quasi-primary (qP) wave at a plane interface between two dissimilar nematic elastomer half-spaces has been investigated. The expressions for the phase velocities corresponding to primary and secondary waves are given. It is observed that these phase velocities depend on the angle of propagation of the elastic waves. The reflection and refraction coefficients corresponding to the reflected and refracted waves, respectively, are derived by using appropriate boundary conditions. The energy transmission of the reflected and refracted waves is obtained, and the energy ratios and the reflection and refraction coefficients are computed numerically.

2010 *Mathematics subject classification*: 82B24.

Keywords and phrases: nematic elastomers, qP-wave, qSV-wave, reflection, refraction, phase velocity.

1. Introduction

Nematic elastomers are materials combining the elastic properties of rubbers with the anisotropy of liquid crystals. They consist of networks of elastic solid chains formed by the cross linking of nematic crystalline molecules, called mesogens, as the elements of their main chains and pendant side groups. Due to this structure, any stress on the polymer network influences the nematic order and any change in the orientational order will affect the mechanical shape of the elastomer. The interplay between elastic and orientational changes is responsible for many fascinating properties of such materials that are different from the classical elastic solids and liquid crystals. Liquid crystalline elastomers (LCEs) have a number of applications in the fields of mechanical actuators (artificial muscles), optics and coatings of materials, which can dissipate mechanical energy [4, 9]. Several discussions of different problems in liquid nematic elastomers exist in the literature [2, 19, 22].

[1]Department of Mathematics and Computer Science, Mizoram University, Aizawl 796 004, Mizoram, India; e-mail: saratcha32@yahoo.co.uk.

The electroclinic (EC) effect is an electro-optical coupling observed in liquid crystals, which is the rotation of the optical axis about an electric field, perpendicular to the optical axis itself. The tilt is linear in the electric field, and the proportionality coefficient and the EC coefficient are properties of the material. Greco and Ferrarini [18] derived the molecular expressions for the EC coefficient and a computational methodology on the basis of the molecular structure. Finkelmann et al. [14] synthesized side-chain nematic polymer networks, and performed differential scanning calorimetry (DSC), X-ray, birefringence and thermo-mechanical characterizations. Selinger et al. [23] developed a phenomenological theory for the isotropic–nematic transition in liquid-crystalline elastomers through a variation of Landau theory. DeSimone and Dolzmann [10] analysed the soft deformation paths and domain patterns in nematic elastomers through the minimization of a nonconvex free energy. Anderson et al. [3] developed a continuum theory for the mechanical behaviour of rubber materials. Conti et al. [7] showed that the effective energy results from an instability of fine-scale oscillations for deformation gradients in part of the phase space, leading to two distinct macroscopic modes of response, called soft and hard. Clarke et al. [6] reported the theoretical and experimental study of linear visco-elastic response in oriented monodomain nematic elastomers. Nematic elastomers exhibit the remarkable phenomenon of soft or semisoft elasticity in which the effective shear modulus for shears in planes containing the anisotropic axis, respectively, vanishes or is very small.

Fradkin et al. [16] studied the visco-elastic theory of nematic elastomers in the low-frequency limit, which was used to investigate the spectral and polarization properties of acoustic waves propagating in liquid-crystalline nematic elastomers. Gebretsadkan and Kalra [17] investigated the propagation of linear waves in relativistic anisotropic magneto-hydrodynamics, and plotted a Fresnel ray surface. Singh [25] discussed the problem of elastic wave propagation in a nematic elastomer, and obtained the reflection coefficients using the linear visco-elastic theory of nematic elastomers. Terentjev et al. [27] developed a theory of elastic waves in oriented monodomain nematic elastomers, and discussed the effect of soft elasticity combined with the Leslie–Ericksen version of the dissipation function that results in an unusual dispersion and anomalous anisotropy of shear acoustic waves. Some other researchers also contributed in solving problems in nematic elastomers [8, 11, 13, 15, 21, 24].

The problems of wave propagation are very common in the field of earthquake engineering, geophysics and seismology. They give us information about the medium (or material) through which the waves travel. The seismic signals propagating through the interior of the Earth are very helpful in exploration of the valuable minerals, crystals and metals buried inside the Earth's crust. Examples of such problems are given by Achenbach [1], Carcione [5] and Singh [26]. In this article, we discuss the problem of transmission of elastic waves in anisotropic and nematic elastomers. The reflection and refraction coefficients corresponding to the reflected and refracted elastic waves are obtained analytically and numerically along with the energy distribution at the interface.

2. Basic equation

The elastic potential energy density in a nematic solid takes the form [9, 27]

$$F = C_1(\mathbf{n}\cdot\epsilon\cdot\mathbf{n})^2 + 2C_2\mathrm{tr}[\mathbf{e}](\mathbf{n}\cdot\epsilon\cdot\mathbf{n}) + C_3(\mathrm{tr}[\mathbf{e}])^2 + 2C_4(\mathbf{n}\times\epsilon\times\mathbf{n})^2 \\ + 4C_5(\mathbf{n}\times(\epsilon\cdot\mathbf{n}))^2 + \tfrac{1}{2}D_1(\mathbf{n}\times\Theta)^2 + D_2\mathbf{n}\cdot\epsilon\cdot(\mathbf{n}\times\Theta),$$

where the Frank elastic energy that describes the nonuniform directors is not included due to the assumption of uniform director rotations in nematic elastomers. Here, $\mathbf{\Theta} = \mathbf{\Omega} - (\mathbf{n}\times\delta\mathbf{n})$ is an independent rotational variable, $\delta\mathbf{n}$ is a small variation in the undistorted nematic director, $\mathbf{n}\cdot\mathbf{\Omega} = (1/2)$ curl $\mathbf{u}$ is the local rotation vector, $(\mathbf{n}\times\delta\mathbf{n})$ are director rotations, $\epsilon_{ik} = e_{ik} - (1/3)\ \mathrm{tr}[\mathbf{e}]\delta_{ik}$ $(i,k = 1,2,3)$ is the traceless part of the linear symmetric strain, $e_{ik} = (1/2)(\delta_k u_i + \delta_i u_k)$, C_i are elastic constants and D_1, D_2 are coupling constants.

Using the Leslie–Ericksen theory [12, 20] of anisotropic viscous dissipation in a nematic liquid, the Rayleigh dissipation function (for entropy production density) can be written in the quadratic form of corresponding velocities [27] as

$$T\dot{s} = A_1(\mathbf{n}\cdot\dot{\epsilon}\cdot\mathbf{n})^2 + 2A_2\mathrm{tr}[\dot{\mathbf{e}}](\mathbf{n}\cdot\dot{\epsilon}\cdot\mathbf{n}) + A_3(\mathrm{tr}[\dot{\mathbf{e}}])^2 + 2A_4(\mathbf{n}\times\dot{\epsilon}\times\mathbf{n})^2 \\ + 4A_5(\mathbf{n}\times(\dot{\epsilon}\cdot\mathbf{n}))^2 + \tfrac{1}{2}\gamma_1(\mathbf{n}\times\dot{\Theta})^2 + \gamma_2\mathbf{n}\cdot\dot{\epsilon}\cdot(\mathbf{n}\times\dot{\Theta}),$$

where A_i are viscous coefficients and the superimposed dots represent the derivative with respect to time. This equation describes two types of dissipation; dissipation by shear flow and dissipation by rotation of the director (which vanishes for rigid rotations).

The equations of motion of a viscous nematic solid after neglecting the effects of Frank elasticity on the director gradient [16] are

$$\nabla\cdot\tau = \rho\ddot{\mathbf{u}}, \\ \mathbf{n}\times[(D_1+\gamma_1\delta_t)\mathbf{n}\times\mathbf{\Theta} + (D_2+\gamma_2\delta_t)\mathbf{n}\cdot\epsilon] = 0, \tag{2.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and equation (2.1) gives the balance of the torques.

The components of the visco-elastic symmetric stress tensors with the choice of the coordinate axis x_3 to lie in the direction of the undistorted director $\mathbf{n}$ are

$$\begin{aligned} \tau_{11} &= (1+\tau_R\partial_t)(c_{11}\epsilon_{11} + c_{12}\epsilon_{22} + c_{13}\epsilon_{33}), \\ \tau_{22} &= (1+\tau_R\partial_t)(c_{12}\epsilon_{11} + c_{11}\epsilon_{22} + c_{13}\epsilon_{33}), \\ \tau_{33} &= (1+\tau_R\partial_t)(c_{13}\epsilon_{11} + c_{13}\epsilon_{22} + c_{33}\epsilon_{33}), \\ \tau_{12} &= \tau_{21} = 2(1+\tau_R\partial_t)c_{66}\epsilon_{12}, \\ \tau_{13} &= 2(1+\tau_R\partial_t)c_{44}\epsilon_{13} - \tfrac{1}{2}D_1(1+\tau_2\partial_t)\Theta_2, \\ \tau_{23} &= 2(1+\tau_R\partial_t)c_{44}\epsilon_{23} + \tfrac{1}{2}D_2(1+\tau_2\partial_t)\Theta_1, \end{aligned} \tag{2.2}$$

where τ_R is the characteristic time of rubber relaxation, and τ_1, τ_2 are director rotation times. We have the following relations [25]:

$$A_i = C_i\tau_R, \quad \gamma_1 = D_1\tau_1, \quad \gamma_2 = D_2\tau_2.$$

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

The Rayleigh dissipation function is positive if

$$\tau_2^2 \le \frac{8C_5 D_1}{D_2^2}\tau_R \tau_1,$$

where C_5 is the shear modulus. Using equations (2.2) and (2.1), the components of the rotational variable Θ are given by the author [25] as

$$\Theta_1 = -\frac{D_2}{D_1}\frac{1+\imath\omega\tau_2}{1+\imath\omega\tau_1}\epsilon_{23}, \quad \Theta_2 = \frac{D_2}{D_1}\frac{1+\imath\omega\tau_2}{1+\imath\omega\tau_1}\epsilon_{13},$$

where ω is the angular velocity.

3. Problem formulation

Let us consider two-dimensional wave propagation in the x_1x_3-plane with the x_1-axis lying horizontally and the x_3-axis vertically downward. The dissimilar anisotropic nematic elastomer half-spaces, $M = \{x_3 \mid x_3 > 0\}$ and $M' = \{x_3 \mid x_3 < 0\}$, are separated by $x_3 = 0$. Note that the corresponding parameters in M' are denoted by inserting a $(')$ to that of M.

The equations of motion in the nematic elastomer M after neglecting the effects of Frank elasticity on the director gradient are written as

$$\rho\ddot{u}_1 = (1+\imath\omega\tau_R)\{c_{11}u_{1,11} + c_{44}^R u_{1,33} + (c_{13}+c_{44}^R)u_{3,13}\},$$

$$\rho\ddot{u}_3 = (1+\imath\omega\tau_R)\{c_{33}u_{3,33} + c_{44}^R u_{3,11} + (c_{13}+c_{44}^R)u_{1,13}\},$$

where

$$c_{44}^R(\omega) = 2C_5 - \frac{1}{4}\frac{D_2^2}{D_1}\frac{(1+\imath\omega\tau_2)^2}{(1+\imath\omega\tau_1)(1+\imath\omega\tau_R)}.$$

Similarly, the equations of motion in the nematic elastomer M' can be written as

$$\rho\ddot{u}_1' = (1+\imath\omega\tau_R')\{c_{11}'u_{1,11}' + c_{44}'^R u_{1,33}' + (c_{13}'+c_{44}'^R)u_{3,13}'\},$$
$$\rho\ddot{u}_3' = (1+\imath\omega\tau_R')\{c_{33}'u_{3,33}' + c_{44}'^R u_{3,11}' + (c_{13}'+c_{44}'^R)u_{1,13}'\},$$

where

$$c_{44}'^R(\omega) = 2C_5' - \frac{1}{4}\frac{D_2'^2}{D_1'}\frac{(1+\imath\omega\tau_2')^2}{(1+\imath\omega\tau_1')(1+\imath\omega\tau_R')}.$$

Suppose that a plane wave propagating in the half-space M is incident at the plane interface $x_0 = 0$, in which a part of the incident energy is reflected to the half-space M and another part is refracted to the half-space M'. The displacement of the elastic waves may be represented as

$$u_1^{(\beta)}(x_1, x_3, t) = A^{(\beta)}d_1^{(\beta)}\exp\{\imath(\omega t - k_1^{(\beta)}x_1 - k_3^{(\beta)}x_3)\}, \tag{3.1}$$
$$u_3^{(\beta)}(x_1, x_3, t) = A^{(\beta)}d_3^{(\beta)}\exp\{\imath(\omega t - k_1^{(\beta)}x_1 - k_3^{(\beta)}x_3)\}, \tag{3.2}$$

where $A^{(\beta)}$ is the amplitude constant, $d_1^{(\beta)}$ is the component of the unit displacement vector, ω is the angular frequency and $k_1^{(\beta)}$ and $k_3^{(\beta)}$ are corresponding wavenumbers

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

with $\beta = 0$ for the incident wave, $\beta = 1$ for the reflected qP-wave, $\beta = 2$ for the reflected quasi-shear vertical qSV-wave, $\beta = 3$ for the refracted qP-wave and $\beta = 4$ for the refracted qSV-wave. The relation of the angles of incident and reflected as well as refracted waves is given by Snell's law [25]

$$\frac{\sin\alpha}{c_0(\alpha)} = \frac{\sin\alpha_1}{c_1(\alpha_1)} = \frac{\sin\alpha_2}{c_2(\alpha_2)} = \frac{\sin\alpha_3}{c_1'(\alpha_3)} = \frac{\sin\alpha_4}{c_2'(\alpha_4)} = \frac{1}{c_a}, \tag{3.3}$$

where c_a is the apparent velocity, $c_0(\alpha)$ is the phase velocity of the incident wave, $c_1(\alpha_1)$ is the phase velocity of the reflected qP-wave, $c_2(\alpha_2)$ is the phase velocity of the reflected qSV-wave, $c_1'(\alpha_3)$ is the phase velocity of the refracted qP-wave and $c_2'(\alpha_4)$ is the phase velocity of the refracted qSV-wave.

The phase velocity of the incident qP-wave is given by

$$c_0^2(\alpha) = \frac{B + E + \sqrt{(B-E)^2 + 4D}}{2\rho}, \tag{3.4}$$

where $\mathbf{p} = (p_1, p_2, p_3)$, $B = (1 + \imath\omega\tau_R)(c_{11}p_1^2 + c_{44}^R p_3^2)$, $E = c_{44}^R p_1^2 + c_{33}p_3^2$ and $D = (1 + \imath\omega\tau_R)(c_{13} + c_{44}^R)p_1 p_3$. The expressions for the phase velocity corresponding to the reflected and refracted waves are given by

$$\begin{aligned}
c_1^2(\alpha_1) &= \frac{B^{(\alpha_1)} + E^{(\alpha_1)} + \sqrt{(B^{(\alpha_1)} - E^{(\alpha_1)})^2 + 4D^{(\alpha_1)}}}{2\rho},\\
c_2^2(\alpha_2) &= \frac{B^{(\alpha_2)} + E^{(\alpha_2)} - \sqrt{(B^{(\alpha_2)} - E^{(\alpha_2)})^2 + 4D^{(\alpha_2)}}}{2\rho},\\
c_1'^2(\alpha_3) &= \frac{B^{(\alpha_3)} + E^{(\alpha_3)} + \sqrt{(B^{(\alpha_3)} - E^{(\alpha_3)})^2 + 4D^{(\alpha_3}}}{2\rho'},\\
c_2'^2(\alpha_4) &= \frac{B^{(\alpha_4)} + E^{(\alpha_4)} - \sqrt{(B^{(\alpha_4)} - E^{(\alpha_4)})^2 + 4D^{(\alpha_4)}}}{2\rho'},
\end{aligned}$$

where $\mathbf{p}^{(\beta)} = (p_1^{(\beta)}, p_2^{(\beta)}, p_3^{(\beta)})$. For $\beta = 1$ and 2,

$$B^{(\alpha_\beta)} = (1 + \imath\omega\tau_R)(c_{11}p_1^{(\beta)^2} + c_{44}^R p_3^{(\beta)^2}), \quad E^{(\alpha_\beta)} = c_{44}^R p_1^{(\beta)^2} + c_{33}p_3^{(\beta)^2},$$
$$D^{(\alpha_\beta)} = (1 + \imath\omega\tau_R)(c_{13} + c_{44}^R)p_1^{(\beta)}p_3^{(\beta)}$$

and, for $\beta = 3$ and 4,

$$B^{(\alpha_\beta)} = (1 + \imath\omega\tau_R')(c_{11}'p_1^{(\beta)^2} + c_{44}'^R p_3^{(\beta)^2}), \quad E^{(\alpha_\beta)} = c_{44}'^R p_1^{(\beta)^2} + c_{33}'p_3^{(\beta)^2},$$
$$D^{(\alpha_\beta)} = (1 + \imath\omega\tau_R')(c_{13}' + c_{44}'^R)p_1^{(\beta)}p_3^{(\beta)}.$$

Thus, we have seen that the phase velocity of the elastic waves in the nematic elastomers depends on the angle of propagation. Consequently, the elastic waves in anisotropic nematic elastomers are quasi in nature.

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

4. Boundary conditions

The boundary conditions of the problem are the continuity of displacement and stress tractions at $x_3 = 0$, given by:

(a) continuity of displacements at $x_3 = 0$

$$u_1^{(\beta)}(M) = u_1^{(\beta)}(M') \quad \text{and} \quad u_3^{(\beta)}(M) = u_3^{(\beta)}(M'); \tag{4.1}$$

(b) continuity of stress tractions at $x_3 = 0$

$$\tau_{33}(M) = \tau_{33}(M') \quad \text{and} \quad \tau_{13}(M) = \tau_{13}(M'). \tag{4.2}$$

Using equations (3.1)–(3.3) and the stress tractions in (4.1) and (4.2), we get

$$AX = G, \tag{4.3}$$

where A is a matrix of order 4×4 with elements

$$\begin{gathered}
a_{11} = d_1^{(1)}, \quad a_{12} = d_1^{(2)}, \quad a_{13} = -d_1^{(3)}, \quad a_{14} = -d_1^{(4)}, \\
a_{21} = d_3^{(1)}, \quad a_{22} = d_3^{(2)}, \quad a_{23} = -d_3^{(3)}, \quad a_{24} = -d_3^{(4)}, \\
a_{31} = k_1^{(1)} d_1^{(1)} c_{13} + k_3^{(1)} d_3^{(1)} c_{33}, \quad a_{32} = k_1^{(2)} d_1^{(2)} c_{13} + k_3^{(2)} d_3^{(2)} c_{33}, \\
a_{33} = -\tau_0 (k_1^{(3)} d_1^{(3)} c'_{13} + k_3^{(3)} d_3^{(3)} c'_{33}), \quad a_{34} = -\tau_0 (k_1^{(4)} d_1^{(4)} c'_{13} + k_3^{(4)} d_3^{(4)} c'_{33}), \\
a_{41} = k_3^{(1)} d_1^{(1)} + k_1^{(1)} d_3^{(1)}, \quad a_{42} = k_3^{(2)} d_1^{(2)} + k_1^{(2)} d_3^{(2)}, \\
a_{43} = -\tau'_0 (k_3^{(1)} d_1^{(3)} + k_1^{(3)} d_3^{(1)}), \quad a_{44} = -\tau'_0 (k_3^{(4)} d_1^{(4)} + k_1^{(4)} d_3^{(4)}), \\
\tau_0 = (1 + \imath\omega\tau'_R)/(1 + \imath\omega\tau_R), \quad \tau'_0 = (1 + \imath\omega\tau'_R) c'_{44}(\omega)/\{(1 + \imath\omega\tau_R) c_{44}(\omega)\};
\end{gathered}$$

X and G are column matrices given by

$$X = \frac{1}{A^{(0)}} [A^{(1)} \quad A^{(2)} \quad A^{(3)} \quad A^{(4)}]^t, \quad G = -[b_1 \quad b_2 \quad b_3 \quad b_4]^t,$$

with $b_1 = d_1^{(0)},\ b_2 = d_3^{(0)},\ b_3 = k_1^{(0)} d_1^{(0)} c_{13} + k_3^{(0)} d_3^{(0)} c_{33},\ b_4 = k_3^{(0)} d_1^{(0)} + k_1^{(0)} d_3^{(0)}$.

In the next section, equation (4.3) will be used in finding the reflection and refraction coefficients of the reflected and refracted qP- and qSV-waves.

5. Reflection and refraction coefficients

Solving the equations in the matrix form (4.3), we get the reflection and refraction coefficients of the reflected and refracted waves as

$$\begin{gathered}
r^{(1)} = \frac{A^{(1)}}{A_0} = \frac{\Delta_1}{\Delta}, \quad r^{(2)} = \frac{A^{(2)}}{A_0} = \frac{\Delta_2}{\Delta}, \\
r^{(3)} = \frac{A^{(3)}}{A_0} = \frac{\Delta_3}{\Delta}, \quad r^{(4)} = \frac{A^{(4)}}{A_0} = \frac{\Delta_4}{\Delta},
\end{gathered}$$

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

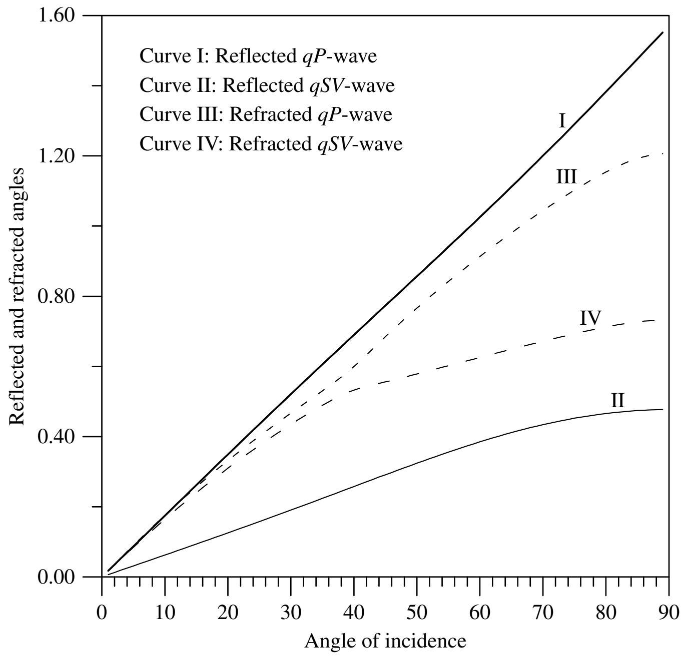

FIGURE 1. Variation of angles of the reflected and refracted waves with angle of incidence.

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

The expressions for Δ_1, Δ_2, Δ_3 and Δ_4 are obtained by replacing the first, second, third and fourth columns, respectively, of the determinant Δ with the column matrix G. The coefficients $r^{(1)}$ and $r^{(2)}$ correspond to the reflection coefficients of the reflected qP-wave and the reflected qSV-wave, respectively, while $r^{(3)}$ and $r^{(4)}$ correspond to the refraction coefficients of the refracted qP-wave and the refracted qSV-wave, respectively.

5.1. Special case In the absence of the upper half-space M', the problem reduces to the reflection of elastic waves due to a free surface. Considering the boundary condition (4.2), the reflection coefficients are given by

$$r^{(1)} = \frac{b_4 a_{32} - b_3 a_{42}}{a_{41} a_{32} - a_{31} a_{42}} \quad \text{and} \quad r^{(2)} = \frac{b_3 a_{41} - b_4 a_{31}}{a_{41} a_{32} - a_{42} a_{31}}.$$

These results are similar to the results obtained by Singh [25] for the relevant problem.

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

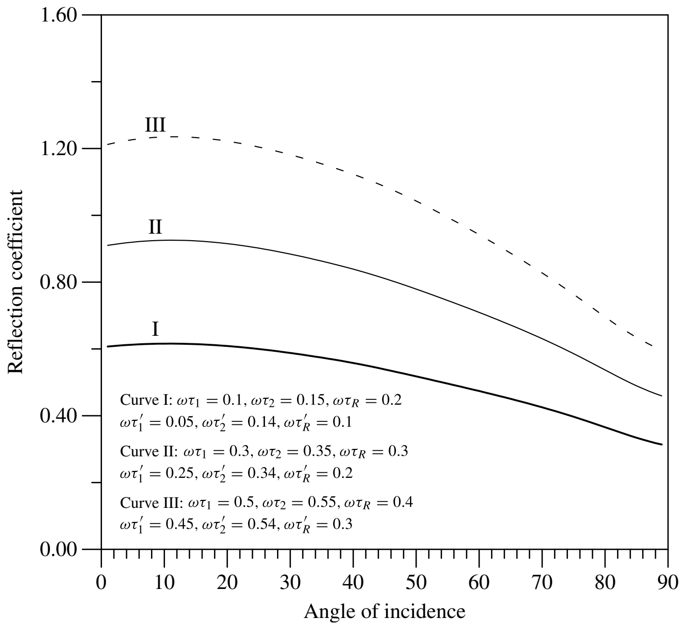

FIGURE 2. Variation of $r^{(1)}$ with angle of incidence α.

6. Energy partition

Let us consider the energy distribution of the incident longitudinal wave due to the reflection and refraction of elastic waves at the plane interface between two dissimilar nematic elastomers. The rate of energy transmission is given by Achenbach [1] as

$$\wp^* = \tau_{31} \cdot \dot{u}_1 + \tau_{33} \cdot \dot{u}_3.$$

The energy due to an incident longitudinal wave may be represented as

$$\begin{aligned} E_{\text{inc}} &= f_0 \omega A_0^2 \exp[2\imath\{\omega t - k_1^{(0)} x_1 - k_3^{(0)} x_3\}], \quad \text{where} \\ f_0 &= (1 + \imath\omega\tau_R)[c_{13} d_1^{(0)} k_1^{(0)} + c_{33} d_3^{(0)} k_3^{(0)} + c_{44}^R (d_1^{(0)} k_3^{(0)} + d_3^{(0)} k_1^{(0)})]. \end{aligned}$$

The incident energy is distributed to various reflected and refracted waves. The energy ratios of the various waves are defined as the ratios of the energies of the corresponding waves to the energy of the incident wave. These ratios of the reflected and refracted waves are given as

$$E_\beta = \left|\frac{f_\beta}{f_0}\right| \left|\frac{A_1^\beta}{A_0}\right|^2 \quad (\beta = 1, 2 \text{ for reflection and } \beta = 3, 4 \text{ for refraction}),$$

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

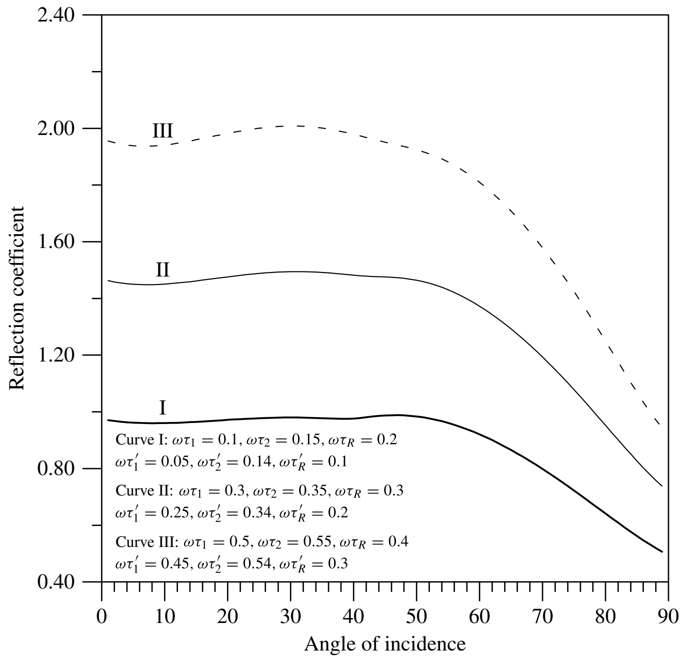

FIGURE 3. Variation of $r^{(2)}$ with angle of incidence α.

where

$$\begin{aligned}
f_1 &= (1 + \iota\omega\tau_R)[c_{13}d_1^{(1)}k_1^{(1)} + c_{33}d_3^{(1)}k_3^{(1)} + c_{44}^R(d_1^{(1)}k_3^{(1)} + d_3^{(1)}k_1^{(1)})],\\
f_2 &= (1 + \iota\omega\tau_R)[c_{13}d_1^{(2)}k_1^{(2)} + c_{33}d_3^{(2)}k_3^{(2)} + c_{44}^R(d_1^{(2)}k_3^{(2)} + d_3^{(2)}k_1^{(2)})],\\
f_3 &= (1 + \iota\omega\tau_R')[c_{13}'d_1^{(3)}k_1^{(3)} + c_{33}'d_3^{(3)}k_3^{(3)} + c_{44}'^R(d_1^{(3)}k_3^{(3)} + d_3^{(3)}k_1^{(3)})],\\
f_4 &= (1 + \iota\omega\tau_R')[c_{13}'d_1^{(4)}k_1^{(4)} + c_{33}d_3^{(4)}k_3^{(4)} + c_{44}'^R(d_1^{(4)}k_3^{(4)} + d_3^{(4)}k_1^{(4)})].
\end{aligned}$$

The energy ratio E_1 corresponds to the reflected qP-wave and E_2 corresponds to the reflected qSV-wave, while the energy ratio E_3 corresponds to the refracted qP-wave and E_4 corresponds to the refracted qSV-wave. Thus, we have seen that energy ratios corresponding to the reflected and refracted waves are functions of elastic constants, the coupling constants, the characteristic time of rubber relaxation and the director rotation-time elastic parameter. These energy ratios satisfy

$$E_1 + E_2 + E_3 + E_4 = 1.$$

The sum of the energy ratios is equal to the energy of the incident wave, which proves the conservation of energy.

7. Numerical results and discussion

In order to compute the reflection and refraction coefficients and energy ratios of the reflected and refracted waves, the following parameter values are used.

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

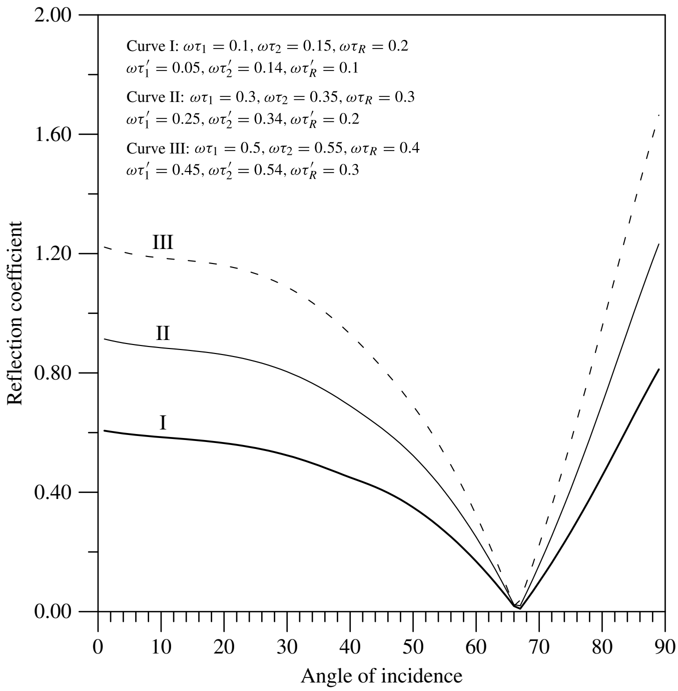

FIGURE 4. Variation of $r^{(3)}$ with angle of incidence α.

(1) For the half-space M:
$C_1 = 1.42 \times 10^5$ N m^{-2}, $C_2 = 2.25 \times 10^5$ N m^{-2}, $C_3 = 4.88 \times 10^5$ N m^{-2}, $C_4 = 2.15 \times 10^5$ N m^{-2}, $C_5 = 1.06 \times 10^5$ N m^{-2}, $D_1 = 0.12$, $D_2 = 0.05$, $\rho = 1.66 \times 10^3$ kg m^{-3}.

(2) For the half-space M':
$C_1' = 3.52 \times 10^5$ N m^{-2}, $C_2' = 2.28 \times 10^5$ N m^{-2}, $C_3 = 1.65 \times 10^5$ N m^{-2}, $C_4 = 1.60 \times 10^5$ N m^{-2}, $C_5 = 4.34 \times 10^5$ N m^{-2}, $D_1 = 0.15$, $D_2 = 0.17$, $\rho = 1.26 \times 10^3$ kg m^{-3}.

Using equations (3.3) and (3.4), we obtained the angles corresponding to the reflected and refracted waves. The variation of these angles with the angle of incidence is depicted in Figure 1. Curves I–IV correspond to the angle of the reflected qP-wave, reflected qSV-wave, refracted qP-wave and refracted qSV-wave, respectively. All these angles increase with the increase of the angle of incidence α.

The variations of reflection and refraction coefficients with angle of incidence for different values of $\omega\tau_1$, $\omega\tau_2$, $\omega\tau_r$ and $\omega\tau_1'$, $\omega\tau_2'$, $\omega\tau_r'$ are shown in Figures 2–5, while the variations of energy ratios with α are depicted in Figures 6–9.

In all these figures, we magnify the coefficients or energy ratios in Curves II and III by multiplying with 1.5 and 2, respectively, in order to see their variation clearly and assign the following parameter values for the three curves:

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

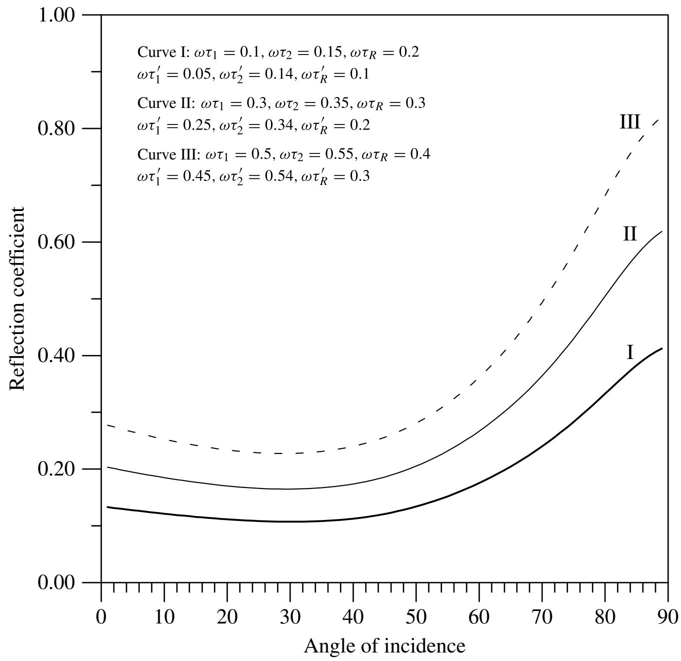

FIGURE 5. Variation of $r^{(4)}$ with angle of incidence α.

Curve I:
$(\omega\tau_1 = 0.1,\ \omega\tau_2 = 0.15,\ \omega\tau_r = 0.2)$ and $(\omega\tau_1' = 0.05,\ \omega\tau_2' = 0.14,\ \omega\tau_r' = 0.1)$;
Curve II:
$(\omega\tau_1 = 0.3,\ \omega\tau_2 = 0.35,\ \omega\tau_r = 0.3)$ and $(\omega\tau_1' = 0.25,\ \omega\tau_2' = 0.34,\ \omega\tau_r' = 0.2)$;
Curve III:
$(\omega\tau_1 = 0.5,\ \omega\tau_2 = 0.55,\ \omega\tau_r = 0.4)$ and $(\omega\tau_1' = 0.55,\ \omega\tau_2' = 0.54,\ \omega\tau_r' = 0.3)$.

In Figure 2, the reflection coefficient, $r^{(1)}$, corresponding to the reflected qP-wave starts from a certain value at normal incidence, increases up to $\alpha = 14°$ and decreases thereafter with the increase of the angle of incidence. We have observed that the director rotation-time parameters $\omega\tau_1$, $\omega\tau_2$, $\omega\tau_r$ and $\omega\tau_1'$, $\omega\tau_2'$, $\omega\tau_r'$ are affected much near the grazing angle of incidence. Figure 3 shows that $r^{(2)}$ of the reflected qSV-wave starts from a certain value and decreases up to $\alpha = 9°$; thereafter it increases up to $\alpha = 30°$ and then decreases with the increase of α. In Figure 4, the refraction coefficient, $r^{(3)}$, of the refracted qP-wave decreases with the increase of the angle of incidence up to $\alpha = 67°$ and increases thereafter. Figure 5 shows that the values of $r^{(4)}$ decreases with the increase of angle of incidence up to $\alpha = 30°$ and then it increases with α. We have observed that the values of $r^{(4)}$ increase with the increase of director rotation-time parameters.

In Figure 6, the energy ratio E_1 corresponding to the reflected qP-wave decreases with the increase of α. Figure 7 shows that the energy ratio, E_2, corresponding to

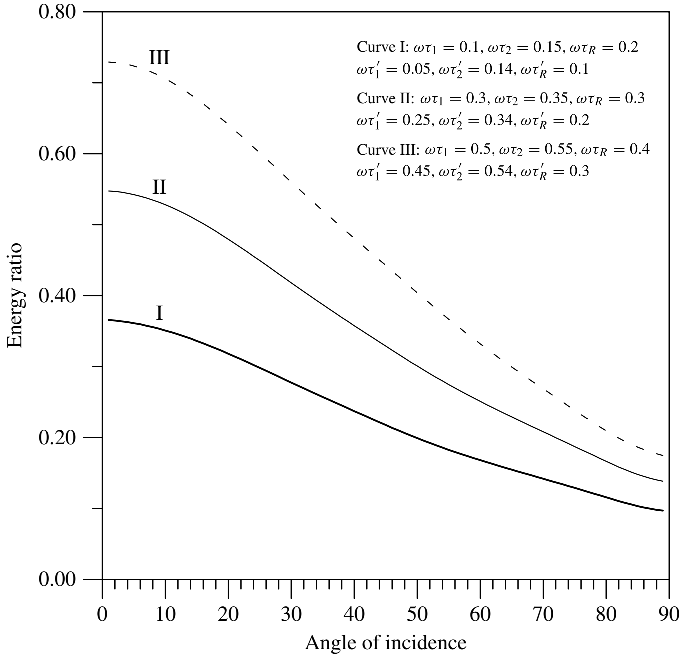

FIGURE 6. Variation of E_1 with angle of incidence α.

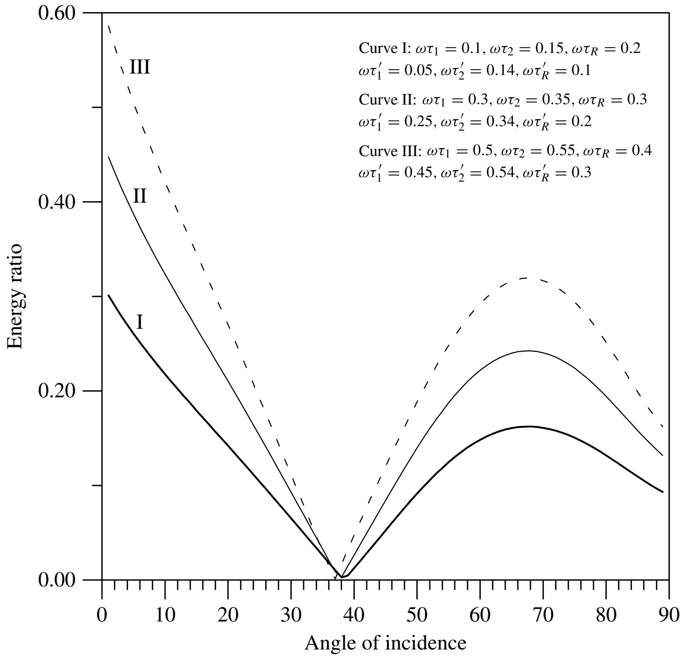

FIGURE 7. Variation of E_2 with angle of incidence α.

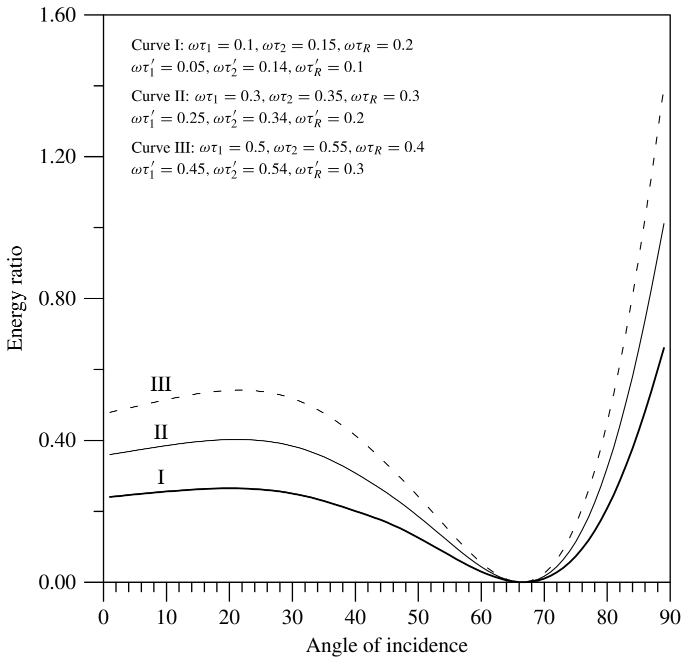

FIGURE 8. Variation of E_3 with angle of incidence α.

the reflected qSV-wave starts at a certain value at the normal angle of incidence, decreases up to $\alpha = 38°$ and forms a parabolic region at $38° \le \alpha \le 90°$. In Figure 8, the energy ratio corresponding to the refracted qP-wave forms two parabolic regions at $0 \le \alpha \le 45°$ and $45° \le \alpha \le 90°$. Figure 9 shows that E_4 decreases up to $\alpha = 33°$ and, thereafter, it increases with the increase of α. We have observed that the sum of the energy ratios is close to one.

8. Conclusion

Using appropriate boundary conditions, the reflection and refraction of elastic waves due to an incident qP-wave at a plane interface between two dissimilar nematic elastomer half-spaces has been investigated. The reflection and refraction coefficients and energy ratios corresponding to the reflected and refracted waves were obtained analytically and numerically for a particular model. The analysis concludes with the following points.

(i) The phase velocities of the elastic waves depend on the angle of incidence.
(ii) The reflection and refraction coefficients and energy ratios are functions of elastic constants, coupling constants, the characteristic time of rubber relaxation, the director rotation-time parameter and the angle of incidence.

https://doi.org/10.1017/S1446181115000061 Published online by Cambridge University Press

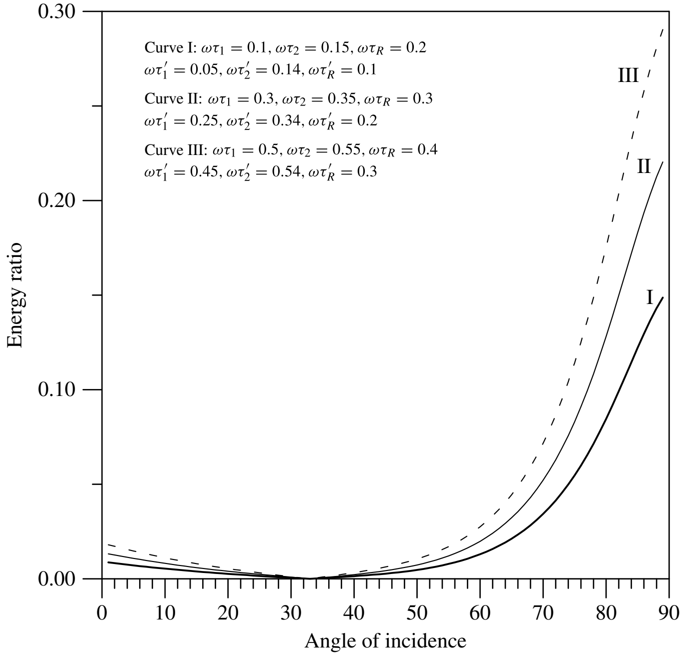

FIGURE 9. Variation of E_4 with angle of incidence α.

(iii) The angles corresponding to the reflected and refracted waves increase with the increase in the angle of incidence.
(iv) The effect of director rotation-time parameters on the reflection coefficient, $r^{(1)}$, is prominent near the glazing angle of incidence.
(v) The values of the refraction coefficient, $r^{(4)}$, increase with an increase in the director rotation-time parameters.
(vi) The energy ratio, E_1, corresponding to the reflected qP-wave decreases with the increase in the angle of incidence.
(vii) The sum of the energy ratios is close to one.

Acknowledgement

The author acknowledges the Department of Science and Technology (DST), New Delhi, India for financial support through grant no. SR/FTP/MS-017/2010.

References

[1] J. D. Achenbach, *Wave propagation in elastic solids* (North-Holland, New York, 1978).
[2] A. L. Alexe-Ionescu, R. Barberi, G. Barbero and M. Giocondo, "Anchoring energy for nematic liquid crystals: contribution from the spatial variation of the elastic constants", *Phys. Rev. E* **49** (1994) 5378–5388; doi:10.1103/PhysRevE.49.5378.

[3] D. R. Anderson, D. E. Carlson and E. Fried, “A continuum-mechanical theory of nematic elastomers”, *J. Elasticity* **56** (1999) 33–58; doi:10.1023/A:1007647913363.
[4] H. R. Brand and H. Pleiner, “Electrohydrodynamics of nematic liquid crystalline elastomers”, *Physica A* **208** (1994) 359–372; doi:10.1016/0378-4371(94)00060-3.
[5] J. M. Carcione, “Constitutive model and wave-equations for linear, viscoelastic, anisotropic media”, *Geophysics* **60** (1994) 537–548; doi:10.1190/1.1443791.
[6] S. M. Clarke, A. R. Tajbakhsh, E. M. Terentjev and M. Warner, “Anomalous viscoelastic response of nematic elastomers”, *Phys. Rev. Lett.* **86** (2001) 4044–4047; doi:10.1103/PhysRevLett.86.4044.
[7] S. Conti, A. DeSimone and G. Dolzmann, “Soft elastic response of stretched sheets of nematic elastomers: a numerical study”, *J. Mech. Phys. Solids* **50** (2002) 1431–1451; doi:10.1016/S0022-5096(01)00120-X.
[8] F. W. Deeg, K. Diercksen, G. Schwalb, C. Brauchle and H. Reinecke, “Ultrasonic measurements of the anisotropic viscoelastic properties of nematic elastomers”, *Phys. Rev. B* **44** (1991) 2830–2833; doi:10.1103/PhysRevB.44.2830.
[9] P. G. de Gennes, *Liquid crystals of one- and two-dimensional order* (eds W. Helfrich and G. Heppke), (Springer, Berlin, 1980).
[10] A. DeSimone and G. Dolzmann, “Material instabilities in nematic elastomers”, *Physica D* **136** (2000) 175–191; doi:10.1016/S0167-2789(99)00153-0.
[11] A. DeSimone and T. Teresi, “Elastic energies for nematic elastomers”, *Eur. Phys. J. E* **29** (2009) 191–204; doi:10.1140/epje/i2009-10467-9.
[12] J. L. Ericksen, “Anisotropic fluids”, *Arch. Ration. Mech. Anal.* **4** (1959) 231–237; doi:10.1007/BF00281389.
[13] J. L. Ericksen, “Conservation laws for liquid crystals”, *Trans. Soc. Rheol.* **5** (1961) 23–34; doi:10.1122/1.548883.
[14] H. Finkelmann, A. Greve and M. Warner, “The elastic anisotropy of nematic elastomers”, *Eur. Phys. J. E* **5** (2001) 281–293; doi:10.1007/s101890170060.
[15] H. Finkelmann, I. Kundler, E. Terentjev and M. Warner, “Critical stripe domain instability of nematic elastomers”, *J. Phys. II* **7** (1997) 1059–1069; doi:10.1051/jp2:1997171.
[16] L. J. Fradkin, I. V. Kamotski, E. M. Terentjev and D. D. Zakharov, “Low-frequency acoustic waves in nematic elastomers”, *Proc. R. Soc. Lond. A* **459** (2003) 2627–2642; doi:10.1098/rspa.2003.1153.
[17] W. B. Gebretsadkan and G. L. Kalra, “Propagation of linear waves in relativistic anisotropic magnetohydrodynamics”, *Phys. Rev. E* **66** (2002) 057401–4; doi:10.1103/PhysRevE.66.057401.
[18] C. Greco and A. Ferrarini, “Electroclinic effect in nematic liquid crystals: the role of molecular and environmental chirality”, *Phys. Rev. E* **87** (2013) 060501–4; doi:10.1103/PhysRevE.87.060501.
[19] J. Kupfer and H. Finkelmann, “Nematic liquid single-crystal elastomers”, *Makromol. Chem. Rapid Commun.* **12** (1991) 717–726; doi:10.1002/marc.1991.030121211.
[20] F. M. Leslie, “Some constitutive equations for anisotropic fluids”, *Quart. J. Mech. Appl. Math.* **19**(3) (1966) 357–370; doi:10.1093/qjmam/19.3.357.
[21] F. M. Leslie, “Some constitutive equations for liquid crystals”, *Arch. Ration. Mech. Anal.* **28** (1968) 265–283; doi:10.1007/BF00251810.
[22] D. Long and D. C. Morse, “Linear viscoelasticity and director dynamics of nematic liquid crystalline polymer melts”, *Europhys. Lett.* **49** (2000) 255–261; doi:10.1209/epl/i2000-00142-5.
[23] J. V. Selinger, H. G. Jeon and B. R. Ratna, “Isotropic–nematic transition in liquid-crystalline elastomers”, *Phys. Rev. Lett.* **89** (2002) 225701–4; doi:10.1103/PhysRevLett.89.225701.
[24] J. V. Selinger, M. S. Spector, V. A. Greanya, B. T. Weslowski, D. K. Shenoy and R. Shashidhar, “Acoustic realignment of nematic liquid crystal”, *Phys. Rev. E* **66** (2002) 051708–7; doi10.1103/PhysRevE.66.051708.

[25] B. Singh, “Reflection of homogeneous elastic waves from free surface of nematic elastomer half-space”, *J. Phys. D: Appl. Phys.* **40** (2007) 584–592; doi:10.1088/0022-3727/40/2/038.
[26] S. S. Singh, “Effect of initial stresses on incident qSV-waves in pre-stressed elastic half-spaces”, *ANZIAM J.* **52** (2011) 359–371; doi:10.1017/S1446181111000757.
[27] E. M. Terentjev, I. V. Kamotski, D. D. Zakharov and L. J. Fradkin, “Propagation of acoustic waves in nematic elastomers”, *Phys. Rev. E* **66** (2002) 052701; doi:10.1103/PhysRevE.66.052701.