SOME PROPERTIES OF LOCALLY COMPACT GROUPS

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In this paper a number of questions about locally compact groups are studied. The structure of finite dimensional connected locally compact groups is investigated, and a fairly simple representation of such groups is obtained. Using this it is proved that finite dimensional arcwise connected locally compact groups are Lie groups, and that in general arcwise connected locally compact groups are locally connected. Semi-simple locally compact groups are then investigated, and it is shown that under suitable restrictions these satisfy many of the properties of semi-simple Lie groups. For example, a factor group of a semi-simple locally compact group is semi-simple. A result of Zassenhaus, Auslander and Wang is reformulated, and in this new formulation it is shown to be true under more general conditions. This fact is used in the study of (C)-groups in the sense of K. Iwasawa.

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NOTATIONAL CONVENTIONS. Topological spaces will always be Hausdorff. If G is a topological group, then G_0 will denote the connected component of the identity. Our groups will be written multiplicatively and we shall use 1 to denote the identity. When we use the word group, we shall always mean a topological group. A homomorphism of a group into another group will always mean a continuous homomorphism. We shall use the term algebraic homomorphism when we do not require continuity. Likewise the word isomorphism will always imply bicontinuity. In using the word subgroup, we do not restrict ourselves to closed subgroups, although we shall never consider the factor space G/H unless H is a closed subgroup of G.

1. Preliminaries

We collect here some of the known results to which we shall make constant reference.

1.1 THEOREM. (*Structure of locally compact groups*). *Let G be a locally compact group such that G/G_0 is compact. Let U be a neighbourhood of the*

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identity in G. Then there is a compact normal subgroup K of G, $K \subset U$, such that G/K is a Lie group with finitely many components.

PROOF. See p. 175 of [10].

1.2 THEOREM. *Let K be a compact group, and denote by $A(K)$ its automorphism group furnished with the compact open topology. Then the identity component of $A(K)$ is the group of those inner automorphism of K which are generated by elements of the identity component of K.*

PROOF. See [8].

1.3 LEMMA. *If G is a connected locally compact group and if H and N are two closed normal subgroups, one of which is compact, and if $G = HN$, then $G = H_0N_0$.*

PROOF. See [4].

1.4 THEOREM. (*Iwasawa* [8]). *Let G be a connected group and K a compact normal subgroup. Let H be the centralizer of K in G. Then $G = HK_0$. If furthermore G is locally compact, $G = H_0K_0$.*

PROOF. This follows easily from 1.2 and 1.3.

1.5 DEFINITION. For a group G define $C_0(G) = G$, define $C_1(G)$ to be the smallest subgroup of G containing all elements of the form $a^{-1}b^{-1}ab$ for a and b in G. Define inductively $C_{n+1}(G) = C_1(C_n(G))$.

1.6 DEFINITION. For a group G define $D_0(G) = G$, define $D_1(G)$ to be the closure of $C_1(G)$ and define inductively $D_{n+1}(G) = D_1(D_n(G))$.

1.7 DEFINITION. A group G is called *solvable* if for some integer n, $C_n(G) = \{1\}$.

1.8 THEOREM. *If H is a dense subgroup of G, then $D_n(G)$ is the closure in G of $C_n(H)$.*

PROOF. This is obvious for $n = 0$ and the general case follows easily by induction.

1.9 THEOREM. *A group G is solvable if and only if $D_n(G) = \{1\}$ for some n.*

PROOF. This follows immediately from 1.8 with $G = H$.

1.10 THEOREM. *Let M be a group, H a solvable subgroup, and G the closure of H in M. Then G is solvable.*

PROOF. This follows easily from 1.8 and 1.9.

1.11 THEOREM. *Let G be a connected locally compact group. Suppose that $\bigcap_{n=1}^{\infty} D_n(G) = \{1\}$. Then G is solvable.*

PROOF. See [8].

2. Finite dimensional groups

2.1 DEFINITION. A locally compact group G with G/G_0 compact will be called *finite dimensional* if there is a compact normal totally disconnected subgroup K of G such that G/K is a Lie group.

This definition agrees with the usual definition of dimension as is shown in chapter 4 of [10].

Our purpose in this section is to examine the structure of finite dimensional locally compact groups. In [10] it is proved (p. 184) that a metric finite dimensional locally compact group is locally the product of a Lie group and a totally disconnected group. By using a slightly different technique of lifting arcs, we are able to determine the global structure of connected finite dimensional locally compact groups. We do not need to assume that the group is metrisable, but we prove that it is as a corollary.

2.2 THEOREM. *Let M be a connected Lie group with a discrete central subgroup Z which is free abelian on generators $z_1, \cdots, z_m$ and let N be a compact totally disconnected group which contains a dense subgroup X which is algebraically a free abelian group on generators $x_1, \cdots, x_m$. Let D be the subgroup of $M \times N$ generated by the elements $(z_1, x_1), \cdots, (z_m, x_m)$. Then $(M \times N)/D$ is a connected finite dimensional locally compact group. Conversely every connected finite dimensional locally compact group is isomorphic to a group of this type.*

PROOF. The proof of the first assertion is straightforward, and will be omitted. For the proof of the converse, we first introduce some notation. J will always denote some suitable indexing set, and i, j, k will always be elements of that set. We shall sometimes use d to denote a distinguished element of J. If we have a group G we will use the corresponding Greek letter γ as a homomorphism. In particular, if G has a family of subgroups $\{H_i\}$, γ_i will denote the natural homomorphism of G onto G/H_i. If furthermore, $H_i \subset H_j$, we shall use γ_{ij} for the natural homomorphism of G/H_i onto G/H_j. Likewise, when we use other Latin letters to denote a group, we shall use the corresponding Greek letters as homomorphisms. We shall say that a family $\{H_i\}$ of subgroups of a given group is directed, if given i, j in J, there is a k in J, such that $H_k \subset H_i \cap H_j$.

LEMMA 1. *Let Z be a discrete group and $\{Z_i\}$ a directed family of normal subgroups of Z such that Z/Z_i is of finite order, and such that $\cap Z_i = \{1\}$.*

Then there is a totally disconnected compact group N, a one-one homomorphism α of Z onto a dense subgroup X of N and a directed family $\{N_i\}$ of open subgroups of N such that $\alpha(Z_i) = \alpha(Z) \cap N_i$. Furthermore $N_i \subset N_j$ if and only if $Z_i \subset Z_j$.

Proof. Define K to be the complete direct product of the Z/Z_i, and define the map α so that the i-th component of $\alpha(z)$ is the coset of z in Z/Z_i. Define K_i to be the open subgroup of K consisting of those elements whose i-th component is the identity of Z/Z_i. Define N to be the closure of $\alpha(Z)$ in K. Define $N_i = N \cap K_i$. It is straightforward to verify that the lemma is satisfied.

Lemma 2. *Let M be a connected Lie group which contains in its centre a discrete subgroup Z which is free abelian on generators $z_1, \cdots, z_m$. Suppose that we are given a directed family $\{Z_i\}$ of subgroups of Z such that Z/Z_i is finite and such that $\cap Z_i = \{1\}$. Let N be the group constructed in lemma 1, and α the corresponding homomorphism. Define $x_i = \alpha(z_i)$. Then the x_i are generators of a dense subgroup X of N, which algebraically is free abelian on these generators. The subgroup D of $M \times N$ generated by the elements $(z_1, x_1), \cdots, (z_m, x_m)$ consists exactly of the elements of the form $(z, \alpha(z))$ for z in Z. Define G to be the group $(M \times N)/D$ and let π denote the natural homomorphism of $M \times N$ onto G. Define $H = \pi(N)$ and $H_i = \pi(N_i)$. Then $H_i \subset H_j$ if and only if $Z_i \subset Z_j$. There is a homomorphism φ_i of M/Z_i onto G/H_i such that whenever $Z_i \subset Z_j$ we have commutativity in diagram 1.*

$$\begin{tikzcd}
M/Z_i \arrow[r, "\varphi_i"] \arrow[d, "\mu_{ij}"'] & G/H_i \arrow[d, "\gamma_{ij}"] \\
M/Z_j \arrow[r, "\varphi_j"] & G/H_j
\end{tikzcd}$$

Diagram 1

We omit the proof which is straightforward.

Lemma 3. *Let T be a connected finite dimensional locally compact group. Then T contains a compact totally disconnected subgroup F and a directed family $\{F_i\}$ of open subgroups of F such that*

(i) *There exists a Lie group M and a discrete central subgroup Z of M which is free abelian on finitely many generators, and a directed family $\{Z_i\}$ of subgroups of Z such that Z/Z_i is finite, $\bigcap Z_i = \{1\}$ and $Z_i \subset Z_j$ if and only if $F_i \subset F_j$.*

(ii) *There is an isomorphism ψ_i of M/Z_i onto T/F_i.*

(iii) *Whenever $F_i \subset F_j$ we have commutativity in diagram 2.*

$$\begin{array}{ccc} M/Z_i & \xrightarrow{\psi_i} & T/F_i \\ \mu_{ij}\downarrow & & \downarrow\tau_{ij} \\ M/Z_j & \xrightarrow{\psi_j} & T/F_j \end{array}$$

Diagram 2

PROOF. We first construct a Lie group $\tilde{M}$ which satisfies the properties required of M in (ii) and (iii). Since T is by hypothesis finite dimensional there is a compact totally disconnected normal subgroup F_d of T such that T/F_d is a connected Lie group. Let $\tilde{M}$ be the universal covering group of T/F_d. Let the groups F_i be the open subgroups of F_d. Then T/F_d is isomorphic to $(T/F_i)/(F_d/F_i)$ so it follows that T/F_i is a covering group of T/F_d. The natural map τ_{id} is the covering homomorphism. Denote by $\tilde{\rho}_d$ the covering map of $\tilde{M}$ onto T/F_d. The covering map $\tilde{\rho}_d$ lifts to a covering homomorphism $\tilde{\rho}_i$ of $\tilde{M}$ onto T/F_i. Set $\tilde{Z}_d$ equal to the kernel of $\tilde{\rho}_d$ and $\tilde{Z}_i$ the kernel of $\tilde{\rho}_i$. Then $\tilde{\rho}_i$ induces an isomorphism $\tilde{\psi}_i$ of $\tilde{M}/\tilde{Z}_i$ onto T/F_i. It is easily verified that the assertions of the lemma are verified with $\tilde{M}$, $\tilde{Z}_i, \tilde{\psi}_i$, in place of M, Z_i, ψ_i, except that $\cap \tilde{Z}_i = \{1\}$ may not be satisfied, and Z_d may not be a finitely generated free abelian group. We define $M = \tilde{M}/(\cap \tilde{Z}_i)$ and $Z_j = \tilde{Z}_j/(\cap Z_i)$. Let ψ_i be the isomorphism of M/Z_i onto T/F_i induced by $\tilde{\psi}_i$. Being a discrete central subgroup of the connected Lie group M, Z_d is finitely generated, and thus the torsion subgroup of Z_d is finite. Hence there is a k such that Z_k does not contain any torsion elements other than the identity. Then Z_k is a finitely generated free abelian group. Define F to be F_k, and Z to be Z_k, and restrict i to those indices such that $F_i \subset F$. The assertions of the lemma are now seen to be true.

LEMMA 4. *Let G and T be two topological groups with directed families $\{H_i\}$ and $\{F_i\}$ respectively, of compact normal subgroups, such that $H_i \subset H_j$ if and only if $F_i \subset F_j$, and such that $\cap H_i = \{1\}$, $\cap F_i = \{1\}$. Suppose that there are isomorphisms σ_i of G/H_i onto T/F_i such that whenever $H_i \subset H_j$, we have commutativity in diagram* 3. *Then G and T are isomorphic.*

$$\begin{array}{ccc} G/H_i & \xrightarrow{\sigma_i} & T/F_i \\ \gamma_{ij}\downarrow & & \downarrow\tau_{ij} \\ G/H_j & \xrightarrow{\sigma_j} & T/F_j \end{array}$$

Diagram 3

We omit the proof, since the argument is standard, and the result is well known.

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Now to complete the proof of theorem 2.2 we observe that if T is a finite dimensional locally compact connected group, there is a group G of the form $(M \times N)/D$ required in the theorem (see lemmas 2 and 3 above), and there are subgroups H_i of G and F_i of T such that the hypotheses of lemma 4 are satisfied, where we define $\sigma_i = \psi_i \cdot \varphi_i^{-1}$. From lemma 4 we conclude that G and T are isomorphic, and the proof of theorem 2.2 is complete.

Remark. If the group T is a compact abelian group it is not difficult to construct a much simpler proof which uses the duality theory for locally compact abelian groups. We omit the details.

We now consider the problem of showing that finite dimensional connected locally compact groups are metrizable. This was first proved in a different manner by Newburgh [11]. From 2.2 we easily see that it reduces to

2.3 Lemma. *Let K be a totally disconnected compact group with a dense subgroup X which is free abelian on finitely many generators. Then K is metrizable.*

Proof. It suffices to show that there is a countable base for the neighbourhoods of the identity (p. 34 of [10]). Since the open subgroups form a base for the neighbourhoods of the identity (p. 56 of [10]) it suffices to show that there are countably many open subgroups of K. The map which associates with an open subgroup H of K the subgroup $H \cap X$ of X establishes a one-one correspondence between the open subgroups of K, and certain of the subgroups of finite index of X. However there are at most countably many of the latter. This completes the proof.

Remark. By way of giving another proof of 2.3, notice that the character group of K is a torsion subgroup of the n-torus and hence countable, so K is metrizable.

We now make use of 2.2 to examine arcwise connected locally compact groups.

2.4 Theorem. *Let G be a connected finite dimensional locally compact group. Then there is a Lie group M and a continuous one-one homomorphism π of M onto the arc component of the identity of G which is dense in G. G is arcwise connected if and only if it is a Lie group.*

Proof. According to theorem 2.2, G can be written as $(M \times N)/D$ where M is a Lie group, N is a totally disconnected compact abelian group and D is discrete. If we denote by π the natural homomorphism of $M \times N$ onto G, then we see from the construction in 2.2 that π is one-one both on M and on N. Since M is arcwise connected, $\pi(M)$ is contained in the arc component of the identity of G. Since D is discrete, π is a local homeomorphism, so the arc component of the identity in G is in the image of the arc component of the identity of $M \times N$. Thus $\pi(M)$ is the arc component of the identity in G. Since if G is not a Lie group, N is uncountable, while D is countable, so $\pi(M)$ is not the whole of G, and thus G is not arc connected. The fact that $\pi(M)$ is dense in G is obvious from the construction in theorem 2.2.

We shall use 2.4 to show that an arcwise connected locally compact group is locally connected. We first need the following.

2.5 THEOREM. *Let G be a connected locally compact group. Then G is locally connected if and only if every finite dimensional factor group is locally connected.*

PROOF. In one direction the argument is trivial. To prove the other direction, suppose that every finite dimensional factor group is locally connected. Let U be a neighbourhood of the identity in G. We must show that U contains a connected neighbourhood of the identity. Choose a compact symmetric neighbourhood V of the identity such that $V^2 \subset U$. Let K be a compact normal subgroup of G contained in V such that G/K is a Lie group. Then G/K_0 is finite dimensional, and by hypothesis locally connected. Denote by π the natural map of G onto G/K_0. Then $\pi(V)$ contains a connected neighbourhood of the identity. Denote by W the inverse image under π of this neighbourhood. Thus $WK_0 = W$, K_0 is connected, and $\pi(W)$ is connected. From this it is easily seen that W is a connected neighbourhood of the identity in G. Furthermore $W \subset U$, since

$$W \subset \pi^{-1}(\pi(V)) = VK_0 \subset V^2 \subset U.$$

Thus G is locally connected.

2.6 COROLLARY. *An arcwise connected locally compact group is locally connected.*

PROOF. If G is arcwise connected, every finite dimensional factor group of G is arcwise connected, and (by 2.4) a Lie group, so locally connected. Thus we conclude from 2.5 that G itself is locally connected.

3. Semi-simple groups

3.1 DEFINITION. A group G will be called *semi-simple* if G has no connected normal solvable subgroup. On account of 1.10 this is equivalent to saying that G has no connected normal solvable closed subgroup.

3.2 DEFINITION. If G is a group, a subgroup R of G is called the *radical* of G if R is a connected normal solvable subgroup, and R is not properly contained in any connected normal solvable subgroup.

Clearly the radical of G, if it exists, is unique, since the product of two connected normal solvable subgroups is again a connected normal solvable subgroup. It follows from 1.10 that the radical, if it exists, is closed. Clearly a connected closed normal solvable subgroup R of G is the radical if and only if G/R is semi-simple.

For locally compact groups Iwasawa [8] has shown that the radical always exists.

If G is a semi-simple Lie group, it is well known that every factor group of G is also semi-simple (see for example chapter 2 of [6]). We shall prove that this is true for all locally compact groups.

3.3 LEMMA. *Let G be a locally compact group. Then G is semi-simple if and only if G_0 is.*

PROOF. If G is not semi-simple, obviously G_0 is not semi-simple. If G_0 is not semi-simple, let R denote the radical of G_0. R is a characteristic subgroup of G_0, and hence a normal subgroup of G. Hence G is not semi-simple.

We remark that the radicals of G and G_0 coincide.

3.4 LEMMA. *Let G be a connected semi-simple locally compact group, and K a compact normal subgroup. Then G/K is semi-simple.*

PROOF. Denote by N the identity component of the centralizer of K in G. From 1.4 we know that $G = KN$. Let R be the radical of $N/(N \cap K)$. Let R_1 be that subgroup of N such that $R_1/(N \cap K) = R$. Since $N \cap K$ is in the centre of G, R_1 is solvable. Since R_1 is a normal subgroup of the semi-simple group G it follows that R_1 is totally disconnected. So therefore is R. Hence R is the trivial group and $N/(N \cap K)$ is semi-simple. Thus $G/K = KN/K$ is semi-simple.

3.5 LEMMA. *Let G be a semi-simple locally compact group and K a compact normal subgroup. Then G/K is semi-simple.*

PROOF. By 3.3 G_0 is semi-simple, so by 3.4 $G_0/(G_0 \cap K)$ is semi-simple. Hence G_0K/K is semi-simple. But (p. 63 of [7]) this is the identity component of G/K, so it follows from 3.3 that that G/K is semi-simple.

The next lemma is essentially a converse to 3.5.

3.6 LEMMA. *Let G be a topological group which is not semi-simple. Then there is a neighbourhood U of the identity of G such that for any closed normal subgroup N of G contained in U, G/N is not semi-simple.*

PROOF. Since G is not semi-simple, there is a non-trivial solvable normal connected subgroup S of G. Let x be a point of S different from the identity. Let U be the complement of the set whose only point is x. Then if $N \subset U$ is a closed normal subgroup of G, S maps onto a non-trivial normal solvable connected subgroup of G/N, so G/N is not semi-simple.

We are now ready to proceed with the proof of the result mentioned above.

3.7 THEOREM. *Let G be a semi-simple locally compact group and H a closed normal subgroup. Then G/H is semi-simple.*

PROOF. Denote by π the natural map of G onto G/H. Assume firstly that G/G_0 is compact. Suppose that G/H is not semi-simple. Let U be a neighbourhood of the identity in G/H as in **3.6**. There is a compact normal subgroup K of G in $\pi^{-1}(U)$ such that G/K is a Lie group. Then KH/H is contained in U, so $(G/H)/(KH/H)$ is not semi-simple. Hence G/KH is not semi-simple. On the other hand, **3.5** implies that G/K is a semi-simple Lie group, so from the known result for Lie groups, $(G/K)/(KH/K)$ is semi-simple, so G/KH is semi-simple. This establishes the result in the special case G/G_0 compact. We consider now the general case. Let F be an open subgroup of G such that $F/F_0(= F/G_0)$ is compact. Since G is semi-simple, so is $G_0 = F_0$ and so therefore is F. From the special case we have proved, $F/(F \cap H)$ is semi-simple. Thus FH/H is semi-simple. That is $\pi(F)$ is semi-simple. Since F is open $\pi(F)$ is open in G/H, so the identity component of G/H is identical with the identity component of $\pi(F)$. Hence the identity component of G/H is semi-simple, so G/H is semi-simple.

3.8 COROLLARY. *Let G be a locally compact group with radical R. Let H be a closed normal subgroup of G. Denote by π the natural homomorphism of G onto G/H. Then the closure of $\pi(R)$ is the radical of G/H.*

PROOF. Denote by N the closure of $\pi(R)$. Since N is a solvable normal connected subgroup of G/H, it will suffice to show that $(G/H)/N$ is semi-simple. But $(G/H)/N$ is isomorphic to $G/\pi^{-1}(N)$ which in turn is isomorphic to $(G/R)/(\pi^{-1}(N)/R)$. The latter group is semi-simple, being a factor group of the semi-simple group G/R. This completes the proof.

We do not know of any correct proofs of **3.7** in the literature. The proof in [4] is erroneous.

Another consequence of **3.7** is the following theorem.

3.9 THEOREM. *Let G be a semi-simple locally compact group which has a compact normal subgroup K such that G/K is connected. Let A denote the identity component of the automorphism group of G (with the compact-open topology). Then A consists of those automorphisms which are inner automorphisms by an element of the identity component of G.*

Proof. By enlarging K if necessary, we may assume that G/K has no compact normal subgroups (see [8]). Then K is sent into itself by all continuous automorphisms of G. Furthermore G/K is a semi-simple Lie group. Let σ be an element of A. Then σ induces an automorphism of G/K, which is in the identity component of the automorphism group of G/K. Thus σ induces an inner automorphism of G/K. Hence we may find an x in G such that the image of x in G/K induces an inner automorphism of G/K equal to the automorphism of G/K induced by σ. We may choose x to be in the identity component of G, since G/K is connected. Then by applying the inner automorphism of G generated by x^{-1} we see that we may assume without loss of generality that σ induces the identity automorphism on G/K. The restriction of automorphisms of G to automorphisms of K maps A continuously into the identity component of the automorphism group of K. Thus from 1.2 we conclude that there is a k in the identity component of K which induces an inner automorphism on K which is equal to the restriction of σ. Thus, by applying the inner automorphism of G by k^{-1} we see that we may assume that σ gives rise to the identity automorphisms of K and of G/K. We shall prove that in this case σ is the identity automorphism of G. Define $f(x) = x^{-1}\sigma(x)$. Clearly f is continuous. But (p. 108 of [15]) f maps G into the centre of K, and f is constant on cosets of K. Since G is semi-simple the centre of K is totally disconnected. Thus f induces a continuous map of the connected group G/K into the totally disconnected centre of K. Thus f is constant, and since $f(1) = 1$, it follows that $f(x) = 1$ for all x in G. That is $x = \sigma(x)$. This completes the proof.

3.10 Corollary. *Let T be a group, and G a subgroup. Suppose that T is connected, G is semi-simple and locally compact, and G possesses a compact normal subgroup K such that G/K is connected. Let H be the centraliser of G in T. Then $T = HG_0$.*

We remark that 3.10 is stated, but without correct proof, in [4].

We now wish to give some conditions that guarantee that a subgroup of a locally compact group is closed. The most basic condition will be that of the next lemma. The next few pages are devoted to obtaining a generalization.

3.11 Lemma. *Let G be a semi-simple Lie group and H a connected semi-simple Lie subgroup of G. Then H is closed in G.*

Proof. Denote by Ad_G the adjoint representation of G. Since G is semi-simple, the kernel of Ad_G is discrete, so Ad_G is a local isomorphism. In [2] Goto has shown that a connected semi-simple matrix group is closed. Thus $Ad_G H$ is closed in $Ad_G G$. Hence $Ad^{-1}(Ad_G H)$ is closed in G. So therefore is its identity component. That is H is closed in G.

3.12 LEMMA. *Let G and H be connected semi-simple locally compact groups. Let π be a continuous homomorphism of H into G. Then the closure of $\pi(H)$ is semi-simple. If H is finite dimensional, so is the closure of $\pi(H)$.*

PROOF. Denote by L the closure of $\pi(H)$. Suppose L is not semi-simple. According to 3.6 there is a neighbourhood V of the identity in L such that for any closed normal subgroup T of L with $T \subset V$, L/T is not semi-simple. Let U be a neighbourhood of the identity of G such that $L \cap U = V$. Let K be a compact normal subgroup of G contained in U such that G/K is a semi-simple Lie group. The group $H/\pi^{-1}(K)$ is semi-simple, and π induces a continuous one-one homomorphism of it into the Lie group G/K. Thus (p. 130 of [1]) $H/\pi^{-1}(K)$ is a semi-simple Lie group. Now 3.11 implies that its image in G/K is closed. That is $\pi(H)K/K = LK/K$. Since the former is semi-simple, so is the latter. Thus $L/(L \cap K)$ is semi-simple. But $L \cap K \subset V$, contradicting our choice of V. Thus L is semi-simple. Next assume that H is finite dimensional. We must show that L is finite dimensional. Let K be a compact totally disconnected normal subgroup of H such that H/K is a Lie group. Then $\pi(K)$ is a compact subgroup of L which is normalised by the dense subgroup $\pi(H)$ of L. Hence it is normalised by L. By replacing H by H/K and L by $L/\pi(K)$ we may, and shall, assume that H is a Lie group. Suppose that the dimension of H is n. Let N be a closed normal subgroup of L such that L/N is a Lie group. Then π induces a one-one homomorphism of $H/\pi^{-1}(N)$ onto a dense subgroup of L/N. From 3.11 we conclude that the image of $H/\pi^{-1}(N)$ is closed in L/N, and therefore is the whole of L/N. Thus L/N is isomorphic to a factor group of an n-dimensional Lie group. Thus the dimension of every Lie factor group of L is at most n. Hence L is finite dimensional (see the discussion of dimension in chapter 4 of [10]).

3.13 THEOREM. *Let G and H be semi-simple connected locally compact groups, and suppose that there is a homomorphism π of H onto a dense subgroup of G. Assume G is locally connected. Then $\pi(H)$ is closed in G and hence equal to G. Conversely if G is not locally connected, there is a semi-simple connected locally compact H, and a homomorphism π of H onto a proper dense subgroup of G.*

PROOF. Assume G is locally connected, and that we are given H and π. Let K be a compact normal subgroup of H such that H/K is a Lie group. Then $\pi(K)$ is normalised by the dense subgroup $\pi(H)$ of G, and hence by all of G. Thus π induces a homomorphism of the semi-simple Lie group H/K (and hence of $H/\pi^{-1}(\pi(K))$) onto a dense subgroup of $G/\pi(K)$. It will suffice to show that the image of $H/\pi^{-1}(\pi(K))$ is all of $G/\pi(K)$. But $G/\pi(K)$ is semi-simple, so **3.12** implies it is finite dimensional. Since G is locally connected, $G/\pi(K)$ is a Lie group. But 3.11 now implies that the image of $H/\pi^{-1}(\pi(K))$ is closed in $G/\pi(K)$, and therefore is $G/\pi(K)$. For the converse, assume that G is not locally connected. Let E be the largest compact normal subgroup of G, and let L be the identity component of the centralizer of E in G. Then L is finite dimensional (it has no connected compact normal subgroup, since such a group would have to be in E and thus in the centre of G, contradicting semi-simplicity). Write $L = (M \times N)/D$ as in 2.2, where M is a semi-simple Lie group. Since $G = EL$, it follows that there is a homomorphism π of $E \times M \times N$ onto G, and the image of $E \times M$ is dense. Set $H = E \times M$. Then the proof will be complete provided we can show that $\pi(H)$ is a proper subgroup of G. But if $\pi(H)$ were the whole of G, G would be isomorphic to a factor group of H, and hence locally connected, since H is locally connected (E is locally connected, being a factor group of a product of simple Lie groups. See pp. **88**—**93** of [12]).

3.14 Corollary. *Let G be a connected semi-simple compact group and let H be a connected semi-simple locally compact group. Suppose that there is a one-one homomorphism π of H into G. Then H is compact.*

Proof. From **3.12** we conclude that the closure of $\pi(H)$ is a semi-simple connected compact group. Hence we may assume without loss of generality that $\pi(H)$ is dense in G. But since G is compact it is locally connected (and in fact a factor group of a product of Lie groups, see pp. 88—**93** of [12]). Thus from **3.13** we see that $\pi(H) = G$. We may now appeal to (5.29) of [7] to conclude that π is an isomorphism, so H is compact.

3.15 Definition. A semi-simple locally compact group G will be said to *have no compact factor* if every compact normal subgroup of G is totally disconnected.

If G is a Lie group, G is locally isomorphic to a product of simple Lie groups (see chapter 2 of [6]). The condition of the above definition is then that none of these simple factors is compact. This is also equivalent to G having no compact factor group other than the trivial G/G. This last equivalence is true without G being a Lie group, as the next lemma shows.

3.16 Lemma. *Let G be a connected semi-simple locally compact group. Then G has no compact factor if and only if, for any closed normal subgroup H of G, G/H is not compact unless $G = H$.*

Proof. First assume that for a closed normal H, G/H compact implies $G = H$. Let K be the largest compact connected normal subgroup of G. We must show that K is trivial. Denote by H the identity component of the centraliser of K in G. Then $G = KH$, by **1.4**. Thus G/H is isomorphic to

$K/(K \cap H)$ which is compact. Thus $G = H$. Hence G centralizes K, so since G is semi-simple, K is the trivial subgroup. Now conversely, suppose that G has no compact factor. Choose a compact normal subgroup K of G such that G/K is a Lie group. Our assumptions on G imply that K is totally disconnected. Furthermore G/K has no compact factor (obvious). Suppose that H is a closed normal subgroup of G such that G/H is compact. We must show that $G = H$. First observe that G/HK is compact. Thus $(G/K)/(HK/K)$ is compact. Since G/K is a Lie group with no compact factor, it follows that $HK/K = G/K$, and thus $G = HK$. Thus $G/H = HK/H$. But G/H is connected, while HK/H is totally disconnected (being isomorphic to $K/(K \cap H)$). This is only possible if $G = H$, as required.

3.17 LEMMA. *Let G be a connected semi-simple locally compact group, K the largest compact connected normal subgroup of G, and N the identity component of the centralizer of K in G. Let H be a connected semi-simple locally compact group with no compact factor, and let π be a homomorphism of H into G. Then $\pi(H) \subset N$.*

PROOF. G/N is compact, being isomorphic to $K/(K \cap N)$. But π induces a one-one homomorphism of $H/\pi^{-1}(N)$ into G/N. It follows from 3.14 that $H/\pi^{-1}(N)$ is compact, and so 3.16 implies that $\pi^{-1}(N) = H$. That is $\pi(H) \subset N$.

We are now ready to give a best possible generalisation of 3.11 to connected locally compact semi-simple groups.

3.18 THEOREM. *Let G be a connected semi-simple locally compact group. Let K be the largest compact connected normal subgroup, and let N be the identity component of the centraliser of K in G. Then the following conditions on G are equivalent.*

(*i*) *N is a Lie group.*

(*ii*) *G can be written L/T, where L is a product of semi-simple connected Lie groups, and T is a compact totally disconnected central subgroup.*

(*iii*) *Every closed connected semi-simple subgroup of G is locally connected.*

(*iv*) *Whenever H is a connected semi-simple locally compact group and π is a homomorphism of H into G, $\pi(H)$ is closed.*

(*v*) *Whenever H is a connected semi-simple Lie group and π is a homomorphism of H into G, $\pi(H)$ is closed.*

PROOF.

(*iii*) $\Rightarrow$ (*iv*). Obvious from 3.12 and 3.13.

(*iv*) $\Rightarrow$ (*v*). Obvious.

(*v*) $\Rightarrow$ (*i*). Suppose that (*i*) is false. According to 2.4 there is a connected Lie group H and a one-one homomorphism of H onto a proper dense subgroup of N (namely the arc component of the identity of N). Hence (*v*) is not satisfied.

(i) $\Rightarrow$ *(ii)*. Since N is a semi-simple Lie group, $N \cap K$ is finite. Since $G = KN$ it follows that G is isomorphic to a factor group of $K \times N$ by a finite group. Being a connected compact semi-simple group, K is isomorphic to a factor group of a product of semi-simple connected Lie groups by a compact totally disconnected central subgroup. It follows that the same is true of G.

(ii) $\Rightarrow$ *(iii)*. Since *(ii)* is true we may write $G = L/T$ where T is compact, and L is a product of semi-simple Lie groups. Denote by λ the natural map of L onto L/T. Suppose that H is a closed connected semi-simple subgroup of G. Set $H_1 = \lambda^{-1}(H)$, and set H' the identity component of H_1. Since $\lambda(H') = H$, H is isomorphic to a factor group of H'. Thus it will suffice to show that H' is locally connected. Let K' be the largest compact connected normal subgroup of H', and let N' be the identity component of the centralizer of K' in H'. From the implication *(i)* $\Rightarrow$ *(ii)* which we have already proved, it will suffice to show that N' is a Lie group. But N' is a connected semi-simple locally compact group with no compact factor. Thus it follows from 3.17 that N' lies in the identity component of the centralizer of the largest compact connected normal subgroup of L. This subgroup of L is a Lie group, and in fact is the product of the non-compact simple factors of L. Thus N' is a closed subgroup of a Lie group, so is itself a Lie group. This completes the proof of the theorem.

4. Some generalizations of a theorem of Zassenhaus, Auslander and Wang

Suppose that G is a Lie group with radical R, such that G/R is compact. Let H be a closed subgroup of G. We would like to be able to conclude that H has a normal solvable subgroup S such that H/S is compact. This will be proved (theorem 5.4). If H were discrete we could conclude the result we desire from a theorem of Zassenhaus and Auslander. If at least the identity component of H is solvable, we could conclude the result from Wang's generalization of the Zassenhaus, Auslander result. We wish to give a further generalization which will allow us to prove 5.4. We first state the Wang result.

4.1 THEOREM (*Zassenhaus, Auslander, Wang*). *Let G be a connected Lie group with radical R, and let H be a closed subgroup of G. Then if H_0 (the identity component of H) is solvable, so is the identity component of the closure of HR.*

PROOF. This is theorem A of the appendix to [13] in case R is simply connected. The general case follows easily by lifting to the universal covering group G.

This theorem may be formulated in the following equivalent form.

4.2 THEOREM. *Let G be a connected Lie group with radical R. Let H be a closed subgroup of G, and suppose that H_0 is solvable. Then if HR is dense in G, $H_0R = G$.*

PROOF OF EQUIVALENCE OF 4.1 AND 4.2. Obviously 4.2 follows immediately from 4.1. On the other hand if we assume 4.2, and if G and H satisfy the hypotheses of 4.1, set L the closure of RH, and R' the radical of L_0. Set $H' = H \cap L_0$. Then the identity component of H' is solvable, and $R'H'$ is dense in L_0. Then 4.2 implies that $L_0 = (H')_0R'$, which is clearly solvable. Thus 4.1 is true.

We now state our generalisation of 4.2.

4.3 THEOREM. *Let G be a Lie group with radical R. Suppose that H is a closed subgroup of G such that RH is dense in G. Then $RH = G$, and $RH_0 = G_0$.*

PROOF. We assume G connected. It is easily seen that the general case will then also be true. In this case we have only to show that $RH_0 = G$. Denote by π the natural map of G onto G/R. Then $\pi(H_0)$ is a normal subgroup of $\pi(H)$. However since $\pi(H_0)$ is a connected Lie subgroup of G/R, its normaliser is closed (it consists of those x in G/R such that $Ad_{G/R}x$ leaves the Lie subalgebra corresponding to $\pi(H_0)$ invariant). Thus $\pi(H_0)$ is a normal subgroup of G/R. Since G/R is semi-simple there is a connected normal (hence semi-simple) subgroup N of G/R such that $G/R = \pi(H_0)N$ and $\pi(H_0) \cap N$ is discrete. Being semi-simple, N is closed (see 3.11), so if we define $L = H \cap \pi^{-1}(N)$ it will follow that L is a closed subgroup of G. Since $\pi(H_0) \subset \pi(H)$, since $\pi(H)$ is dense in G/R, and since G/R is the product of $\pi(H_0)$ and N, it is easily seen that $\pi(H) \cap N$ is dense in N. That is $\pi(L)$ is dense in N. Also L_0 is solvable. For since $\pi(L_0)$ is in both $\pi(H_0)$ and N and L_0 is connected, $\pi(L_0)$ is the trivial group, and $L_0 \subset R$, so L_0 is solvable. Now obviously the radical of $\pi^{-1}(N)$ is R, and since $\pi(L)$ is dense in N, LR is dense in $\pi^{-1}(N)$. We therefore conclude that $L_0R = \pi^{-1}(N)$ (apply 4.2). But since $L_0 \subset R$, it follows that $R = \pi^{-1}(N)$, so N is trivial, and $\pi(H_0) = G/R$. That is $H_0R = G$. This completes the proof.

In the next few pages we now try to consider to what extent 4.1, 4.2, and 4.3 are true if we allow any locally compact groups, and not just Lie groups. As above, if we can prove 4.1 holds for arbitrary locally compact connected G, so will 4.2. Our next result will show that indeed 4.1 is true in this extra generality, although as we shall later see, the same is not true of 4.3.

4.4 THEOREM. *Let G be a connected locally compact group, and H a closed subgroup. Let R be the radical of G. Assume that the identity component of H is solvable. Then the identity component of the closure of RH is solvable.*

PROOF. Denote by M the closure of RH, and by M_0 its identity component. Let U be a neighbourhood of the identity in G. Choose a compact normal subgroup K contained in U, such that G/K is a Lie group. Applying 4.1 to G/K, we easily conclude that M_0 is solvable modulo K. Thus $\bigcap_{n=1}^{\infty} D_n(M_0) \subset K \subset U$. Since U is arbitrary, it follows that $\bigcap_{n=1}^{\infty} D_n(M_0) = \{1\}$. Thus 1.11 implies that M_0 is solvable.

Before we consider possible generalisations of 4.3 to locally compact groups, we prove a lemma which will allow us to restrict our attention to connected groups for much of what follows.

4.5 LEMMA. *Let G be a locally compact group, with radical R. Let H be a closed subgroup of G such that RH is dense in G. Then RH_0 is dense in G_0.*

PROOF. It is easily seen that we may pass to an open subgroup of G. Thus we may assume that G/G_0 is compact. Let K be a compact normal subgroup of G such that G/K is a Lie group. Applying 4.3 to G/K, we see that $RH_0 = G_0$ modulo K. That is every point in G_0 is the product of an element in RH_0 and an element in K. Since K may be chosen in an arbitrarily small neighbourhood of the identity, it follows that elements of G_0 have elements of RH_0 arbitrarily close. That is RH_0 is dense in G_0.

We now treat the case of connected groups.

4.6 LEMMA. *Let G be a locally compact connected group such that G/R is locally connected, R being the radical of G. Let H be a connected locally compact group, and π a homomorphism of H into G such that $R\pi(H)$ is dense in G. Then $R\pi(H) = G$.*

PROOF. Set $S = \pi^{-1}(R)$. Then π induces a homomorphism ρ of H/S onto a dense subgroup of G/R. But H/S is semi-simple, for otherwise the closure of the image under ρ of the radical of H/S would be a non-trivial connected solvable normal subgroup of G/R, contradicting the semi-simplicity of G/R. But we may now use 3.13 to conclude that $\rho(H/S) = G/R$. That is $R\pi(H) = G$.

4.7 THEOREM. *Let G be a locally compact group with radical R, such that G/R is locally connected. Let H be a closed subgroup of G such that RH is dense in G. Then $RH_0 = G_0$ and $RH = G$.*

PROOF. From 4.5 and 4.6 we may easily conclude that $RH_0 = G_0$. But since G/R is locally connected, G_0 is open. Thus RH is an open, therefore closed subgroup of G. Since RH is dense in G, it follows that $RH = G$.

We now give two counter examples to the possibility of weakening the hypotheses of 4.7. The first example shows that we cannot weaken the assumption that G/R be locally connected to the assumption that G_0/R is locally connected, even if we assume that G/R is compact. The second example shows that the assumption G/R locally connected cannot be weakened to the assumption that G_0 is open, or even that $G = G_0$.

4.8 Example. Let R be the group of positive real numbers under multiplication, and let t be an element different from the identity. Let K be an infinite totally disconnected compact monothetic group, and let x be a generator of a dense infinite cyclic subgroup. Let G be the group $R \times K$, and H the subgroup consisting of elements of the form (t^n, x^n). It is easily seen that R is the radical of G, and that RH is a dense proper subgroup of G.

4.9 Example. Let L be the universal covering group of $SL(2, R)$ and let z be a generator of its centre, which is well known to be infinite cyclic. Let M be the group of positive real numbers under multiplication, and let y be an element of M different from the identity. Let N be a totally disconnected infinite compact monothetic group, and let x be a generator of the dense infinite cyclic subgroup. Let D be the subgroup of $L \times M \times N$ generated by the element (z, y, x). Set $G = (L \times M \times N)/D$, and denote by π the natural map of $L \times M \times N$ onto G. It is easily seen that $\pi(L \times M)$ is dense in G, so G is connected. Set $R = \pi(M)$ and $H = \pi(L)$. It is straightforward to verify that R is the radical of G, that H is closed in G, and that RH is a proper dense subgroup of G.

5. On (C)-groups

5.1 Definition. We call a connected Lie group G with radical R a *(C)-group* if G/R is compact. We call a Lie group a *(C)-group* if its identity component is a (C)-group. We call a locally compact group G a *(C)-group* if every neighbourhood of the identity contains a compact normal subgroup K of G such that G/K is a Lie group which is a (C)-group.

(C)-groups were defined by Iwasawa in [8]. We shall examine certain of their properties here. We shall be concerned mostly with locally compact groups G for which G/G_0 is compact.

5.2 Lemma. *Let G be a locally compact group with G/G_0 compact, and let R be the radical of G. Assume that G is a (C)-group, or more generally, assume that G has a compact normal subgroup K such that G/K is a (C)-group which is a Lie group. Then G/R is compact.*

Proof. Denote by π the natural map of G onto G/K. From 3.8 we know that $\pi(R)$ is the radical of G/K. Thus the identity component of $(G/K)/\pi(R)$ is compact, so since G/G_0 is compact, $(G/K)/\pi(R)$ is compact. Thus $(G/K)/(RK/K)$ is compact, so G/RK is compact, so $(G/R)/(RK/R)$ is compact, so since RK/R is compact, G/R is compact.

5.3 Lemma. *Let G be a locally compact group with radical R, and assume that G/R is compact. Then G/G_0 is compact, and G is a (C)-group. In fact for each closed normal subgroup H of G such that G/H is a Lie group, G/H is a (C)-group.*

Proof. Since G/G_0 is a homomorphic image of G/R it is compact. Suppose now that we are given a closed normal subgroup H such that G/H is a Lie group. Let S be the closure of RH. Denote by π the natural map of G onto G/H. From **3.8**, $\pi(S)$ is the radical of G/H. We shall show that $(G/H)/\pi(S)$ is compact. The desired conclusions will follow immediately. But $(G/H)/\pi(S)$ is isomorphic to G/S, and therefore to $(G/R)/(S/R)$, which is compact, being a homomorphic image of the compact group G/R. This completes the proof.

We now use **4.3** to investigate subgroups of (C)-groups.

5.4 Theorem. *Let G be a Lie group such that G/R is compact. Let H be a closed subgroup of G. Then H has a closed normal subgroup S which is solvable and such that H/S is compact.*

Proof. Let G' be the closure of RH and let R' be the radical of G'. Clearly $R' \supset R$, and G'/R' is compact. Thus HR' is dense in G' so **4.3** implies that $HR' = G'$. Then since HR'/R' is compact, so is $H/(H \cap R')$. Then $S = H \cap R'$ is as desired.

We remark that we could choose the group S to be a characteristic subgroup of H, for we could choose S to be the largest solvable normal subgroup of H. Zassenhaus has shown in [**14**] that this exists if G is a matrix group, and the general result follows by taking the adjoint representation.

If we only require that G be locally compact, but not necessarily a Lie group, **5.4** fails. We shall give an example at the end of this paper. If we allow G to be connected and finite dimensional, the conclusion of **5.4** still holds, as may be easily deduced from **5.4**.

The following slight extension of **5.4** is possible.

5.5 Theorem. *Let G be a locally compact group with radical R, such that G/R is compact. Let H be a discrete subgroup of G. Then H has a solvable normal subgroup S such that H/S is finite.*

Proof. We can find a compact normal subgroup K of G, such that H is mapped isomorphically into G/K, and such that G/K is a Lie group. Then apply **5.4** and the result follows.

We give another direction now in which **5.4** may be extended.

5.6 Theorem. *Let G be a locally compact group with radical R, such that G/R is compact. Let H be a connected locally compact group, and π a one-one homomorphism of H into G. Let S be the radical of H. Then H/S is compact.*

PROOF. By replacing G with the closure of $R\pi(H)$ we may assume without loss of generality that $R\pi(H)$ is dense in G, and thus G is connected. The image in G/R of $\pi(H)$ is thus dense, so the image in G/R of $\pi(S)$ is a solvable connected subgroup which is normalised by a dense subgroup of G/R. Thus the closure of the image in G/R of $\pi(S)$ is connected solvable normal, hence trivial, so $S \subset \pi^{-1}(R)$. Thus $H/\pi^{-1}(R)$ is semi-simple, and π induces a one-one homomorphism of it into G/R. Thus we may conclude from 3.14 that $H/\pi^{-1}(R)$ is compact. Note that since π is one-one, $\pi^{-1}(R)$ is solvable, and since S is the radical of H, $\pi^{-1}(R)/S$ is totally disconnected. Set $L = H/S$, and $T = \pi^{-1}(R)/S$. Then from L/T compact we must show that L is compact. This is shown in the next lemma.

5.7 LEMMA. *Let L be a connected semi-simple locally compact group, and T a totally disconnected closed normal subgroup. Suppose that L/T is compact. Then L is compact.*

PROOF. Let K be a compact normal subgroup of L such that L/K is a Lie group. It will suffice to show that L/K is compact. From the fact that L/T is compact we easily conclude that $(L/K)/(TK/K)$ is compact. But TK/K is isomorphic to $T/(T \cap K)$, so totally disconnected, and being a Lie group, is thus discrete. Hence L/K is a covering group of the compact semi-simple Lie group $(L/K)/(TK/K)$. The compactness of L/K now follows (see chapter 2 of [6]).

We shall now consider groups which contain the free group on two generators. We shall always mean that this group has the discrete topology, and is thus a closed subgroup of any group in which it is embedded.

5.8 LEMMA. *Let G be a locally compact group with radical R such that G/R is compact. Then G does not contain the free group on two generators.*

PROOF. If H is any discrete subgroup of G, then from 5.5 we know that H has a solvable normal subgroup S such that H/S is finite. The free group on two generators does not have this property (see p. 104 of [5]). Thus G does not contain the free group on two generators.

The next few lemmas will lead us to a proof of the converse of this. Namely if G/G_0 is compact, G is locally compact, and G doesn't contain the free group on two generators, G/R is compact.

5.9 LEMMA. *Let G be a group, and N a closed normal subgroup. Then if G/N contains the free group on two generators, so does G.*

PROOF. Let x_1 and x_2 be generators of the free subgroup of G/N. Choose y_1 and y_2 in G which project onto x_1 and x_2 respectively. Clearly a non-trivial relation between y_1 and y_2 would imply a non-trivial relation between x_1 and x_2. Thus y_1 and y_2 generate a free group on two generators

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(its topology is discrete, since the projection onto G/N maps it one-one and continuously to a discrete group in G/N). This completes the proof.

5.10 Lemma. *If G is locally isomorphic to $SL(2, R)$, G contains the free group on two generators.*

Proof. By 5.9 it suffices to show that G/Z contains the free group on two generators, where Z is the centre of G. But this group is isomorphic to the group of fractional linear transformations of the upper half plane. The existence of a subgroup isomorphic to the free group on two generators is now well known from the theory of automorphic functions (it follows from the properties of the elliptic modular function).

5.11 Lemma. *A non-compact semi-simple Lie group contains a closed subgroup locally isomorphic to $SL(2, R)$.*

Proof. If we can find a Lie subgroup locally isomorphic to $SL(2, R)$, 3.11 will imply that it is closed. Thus the problem reduces to showing that a non-compact semi-simple Lie algebra contains a subalgebra isomorphic to the split three dimensional algebra (see [9] for terminology of Lie algebras). But if we can find an element $n \neq 0$ in the Lie algebra such that $ad\ n$ is nilpotent, theorem 17, chapter 3 of [9] can be applied. Thus it suffices to show the existence of such an n. But n may be taken to be any element of the nilpotent factor of the Iwasawa decomposition (see chapter 6 of [6]). This completes the proof.

5.12 Lemma. *Let G be a non-compact semi-simple connected Lie group. Then G contains the free group on two generators.*

Proof. This follows immediately from 5.10 and 5.11.

5.13 Lemma. *Let G be a semi-simple locally compact group. Suppose that G/G_0 is compact but G is not compact. Then G contains the free group on two generators.*

Proof. G_0 is semi-simple, and non-compact. It suffices to show that G_0 contains the free group on two generators. Let K be a compact normal subgroup of G_0 such that G_0/K is a Lie group. Then G_0/K is a non-compact connected semi-simple Lie group. Applying 5.12 and 5.9 now completes the proof.

5.14 Lemma. *Let G be a locally compact group such that G/G_0 is compact, but G/R is not compact, where R is the radical of G. Then G contains the free group on two generators.*

Proof. By 5.13, G/R contains the free group on two generators, so applying 5.9 completes the proof.

We summarize the results of the preceding lemmas in the following theorem.

5.15 THEOREM. *Let G be a locally compact group such that G/G_0 is compact. Let R be the radical of G. Then the following conditions on G are equivalent.*

(*i*) *G is a (C)-group.*

(*ii*) *G has a compact normal subgroup K such that G/K is a Lie group which is a (C)-group.*

(*iii*) *G/R is compact.*

(*iv*) *For every closed normal subgroup H of G such that G/H is a Lie group, G/H is a (C)-group.*

(*v*) *G does not contain the free group on two generators.*

PROOF. By definition, (*i*) $\Rightarrow$ (*ii*). By 5.2, (*ii*) $\Rightarrow$ (*iii*). By 5.3 (*iii*) $\Rightarrow$ (*iv*). Obviously (*iv*) $\Rightarrow$ (*i*). Also 5.8 and 5.13 show that (*iii*) and (*v*) are equivalent.

We finish now by giving the promised example which shows that the hypotheses of 5.4 cannot be weakened to the extent of allowing G to be a locally compact group instead of a Lie group.

5.16 EXAMPLE. Let L be a simple compact group (simple in the algebraic sense of no non-trivial normal subgroup). For example we may take L to be either a finite simple group, of a simple compact connected Lie group with trivial centre. Let L_n be the product of n copies of L. Let σ_n be the automorphism of L_n which sends $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ to $(\lambda_n, \lambda_1, \lambda_2, \cdots, \lambda_{n-1})$. Let M be the complete direct product (with the product topology) of the L_n, $n \geqq 2$, and let N be the semi-direct product of M with a discrete infinite cyclic group whose generator is t, where we require

$$t^n(x_2, x_3, \cdots)t^{-n} = \left(\sigma_2^n(x_2), \sigma_3^n(x_3), \cdots\right).$$

Clearly N is locally compact, and non-compact. It is easily verified that N has no solvable normal subgroup other than the trivial subgroup, so there is no solvable normal S such that N/S is compact. We shall complete the counter example by showing that N may be embedded in a locally compact group G with radical R such that G/R is compact. We first show that there is a compact group K, and a one-one homomorphism of N into K. Let L_n^* be the semi-direct product of L_n with the cyclic group of order n, where the generator of the cyclic group acts on L_n as the automorphism σ_n. Let K be the complete direct product of the L_n^*, $n \geqq 2$. Then it is easy to construct a one-one homomorphism π of N into K. (We remark that we could even choose K to be a product of simple compact Lie groups if we desire, for once we have constructed some suitable compact group, we may use the Peter-Weyl theorem to embed it in a product of compact Lie groups).

Now let R be the group of positive real numbers under multiplication. Define $G = R \times K$. If K is semi-simple (and this is the case for the K constructed above) R is the radical of G. In the general case the radical of G certainly contains R. Since G/R is compact, $G/(\textit{radical of } G)$ is compact. Now let r be a fixed element of R different from the identity. An element of N may be always written as $t^k m$ where m is in M. We make correspond to $t^k m$ the element $(r^k, \pi(t^k m))$. Then it is easily seen that in this way G contains a closed subgroup isomorphic to N. This completes our construction.

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