On the evaluation of Planck's integral

By R. L. Goodstein.

The object of this note is to outline a *rigorous* evaluation of Planck's integral by methods which presuppose no more than an elementary knowledge of the Calculus. The proof has been divided into four theorems each of which is of some interest in itself.

The only properties of the logarithmic and exponential functions which the proof assumes are

(i) $\log x = \int_1^x (1/t)\,dt$, $x > 0$, (ii) e^x is the unique inverse of $\log x$.

Theorem 1. If $0 < t < 1$, then $\int_t^1 \frac{\log(1+u)}{u}\,du = \sum_{r=1}^{2n} \frac{(-1)^{r-1}}{r^2}$ with an error less than $t + \{1/(2n+1)^2\}$.

For all t, except $t = -1$,

$$\frac{1}{1+t} = 1 - t + t^2 - \ldots - t^{2n-1} + \frac{t^{2n}}{1+t},$$

which integrated from 0 to t, $t > -1$, gives

$$\log(1+t) = \sum_{r=1}^{2n}\{-(-t)^r/r\} + R(t) \text{ where } R(t) = \int_0^t \{u^{2n}/(1+u)\}\,du;$$

and so, for $t > 0$,

$$\{\log(1+t)\}/t = \sum_{r=1}^{2n}\{(-t)^{r-1}/r\} + R(t)/t. \qquad (1)$$

For $t > 0$,

$$0 < \int_0^t \{u^{2n}/(1+u)\}\,du < \int_0^t u^{2n}\,du = t^{2n+1}/(2n+1);$$

and so, if $0 < t < 1$, $0 < \int_t^1 \{R(u)/u\}\,du < 1/(2n+1)^2$. Since

$$\sum_{r=1}^{n}\left\{\frac{t^{2r-1}}{(2r-1)^2} - \frac{t^{2r}}{(2r)^2}\right\} = \sum_{r=1}^{2n}\left\{-\frac{(-t)^r}{r^2}\right\}$$
$$= t - \sum_{r=1}^{n-1}\left\{\frac{t^{2r}}{(2r)^2} - \frac{t^{2r+1}}{(2r+1)^2}\right\} - \frac{t^{2n}}{(2n)^2},$$

therefore, when $0 < t < 1$, $0 < \sum_{r=1}^{2n}\{-(-t)^r/r^2\} < t$.

Integrating equation (1) from t to 1 we obtain Theorem 1.

https://doi.org/10.1017/S095018430000286X Published online by Cambridge University Press

THEOREM 2. $$\sum_{r=1}^{2n}\{(-1)^{r-1}/r^2\} = \tfrac{1}{12}\pi^2$$

with an error less than $8/(8n+1)$.

If x is not an odd multiple of π,

$$\tfrac{1}{2} + \sum_{r=1}^{4n}(-1)^r \cos rx = \{\cos(4n+\tfrac{1}{2})x\}/2\cos\tfrac{1}{2}x.$$

Integrating from 0 to x, $-\pi < x < \pi$, and then from 0 to $\tfrac{1}{2}\pi$, we find

$$\tfrac{1}{2}x + \sum_{r=1}^{4n}\frac{(-1)^r}{r}\sin rx = \tfrac{1}{2}\int_0^x \frac{\cos(4n+\tfrac{1}{2})x}{\cos\tfrac{1}{2}x}\,dx = T(x), \text{ say,}$$

$$\frac{\pi^2}{16} + \sum_{r=1}^{4n}\frac{(-1)^r}{r^2} - \tfrac{1}{4}\sum_{r=1}^{2n}\frac{(-1)^r}{r^2} = \int_0^{\frac{1}{2}\pi} T(x)\,dx.$$

Since $|2T(x)| = \left|\dfrac{2}{8n+1}\,\dfrac{\sin(4n+\tfrac{1}{2})x}{\cos\tfrac{1}{2}x} - \dfrac{1}{8n+1}\displaystyle\int_0^x \frac{\sin(4n+\tfrac{1}{2})x\sin\tfrac{1}{2}x}{\cos^2\tfrac{1}{2}x}\,dx\right|$

$$\leqq (\pi + 2\sqrt{2})/(8n+1), \text{ for } 0 \leqq x \leqq \tfrac{1}{2}\pi,$$

therefore $\left|\displaystyle\int_0^{\frac{1}{2}\pi} T(x)dx\right| \leqq \tfrac{1}{4}\pi(\pi+2\sqrt{2})/(8n+1) < 5/(8n+1)$.

Since $\displaystyle\sum_{r=2n+1}^{4n}\{(-1)^{r-1}/r^2\}$ may be expressed in either of the forms

$$\sum_{r=1}^{n}\left\{\frac{1}{(2n+2r-1)^2} - \frac{2}{(2n+2r)^2}\right\},$$

$$\frac{1}{(2n+1)^2} - \sum_{r=1}^{n-1}\left\{\frac{1}{(2n+2r)^2} - \frac{1}{(2n+2r+1)^2}\right\} - \frac{1}{(4n)^2},$$

it is positive and less than $1/(2n+1)^2$, and so

$$\sum_{r=1}^{2n}\{(-1)^{r-1}/r^2\} = \tfrac{1}{12}\pi^2$$

with an error less than

$$\tfrac{4}{3}\left\{\frac{1}{(2n+1)^2} + \frac{5}{8n+1}\right\} \leqq \frac{8}{8n+1}.$$

The error term in Theorem 2 is by no means the best possible, but it suffices for our purpose. It may readily be deduced from the result stated that the error is in fact less than $1/(2n+1)^2$.

It follows from Theorem 1 and 2 that

$$\left|\int_t^1\{\log(1+u)/u\}\,du - \tfrac{1}{12}\pi^2\right| < t + 1/(2n+1)^2 + 8/(8n+1)$$

for $0 < t < 1$; and so, since the left hand side is independent of n,

$$\left|\int_t^1\{\log(1+u)/u\}\,du - \tfrac{1}{12}\pi^2\right| \leqq t,$$

https://doi.org/10.1017/S095018430000286X Published online by Cambridge University Press

whence

THEOREM 3. $\int_0^1 \{\log(1+u)/u\}\,du = \lim_{t\to 0}\int_t^1 \{\log(1+u)/u\}\,du$ *exists*, and has the value $\frac{1}{12}\pi^2$.

THEOREM 4. $$\int_0^\infty \frac{x}{e^x-1}\,dx = 2\int_0^1 \frac{\log(1+x)}{x}\,dx.$$

We prove first some simple inequalities.

If $0 < x < 1$ then $(1-x)^{-1} < (1-x)^{-3/2}$, with equality at $x=0$, and so

$$\text{(i)} \quad -\log(1-x) < 2(1-x)^{-\frac{1}{2}};$$

if $0 \leqq x < \frac{1}{2}$ then $1 < (1-x)^{-1} < 2$, and so for $0 < x < \frac{1}{2}$,

$$\text{(ii)} \quad x < -\log(1-x) < 2x,$$

and therefore (iii) $\frac{1}{2}x < 1 - e^{-x} < x$;

if $x > 0$ then $1 > (1+x)^{-1}$ and so $x > \log(1+x)$, whence $e^{-x} < 1/(1+x)$

and so $\quad$ (iv) $\quad x/(1+x) < 1 - e^{-x} < 1$;

It follows from (i) that if $\frac{1}{2} < p^2 < p < 1$ then

$$\text{(v)} \quad \int_{p^2}^{p} -\frac{\log(1-x)}{x}\,dx < 8\left[-(1-x)^{\frac{1}{2}}\right]_{p^2}^{p} < 8(1-p^2)^{\frac{1}{2}},$$

and from (ii) that if $0 < p < \frac{1}{2}$ then

$$\text{(vi)} \quad \int_{p^2}^{p} -\frac{\log(1-x)}{x}\,dx < 2(p-p^2) < 2p.$$

Let $0 < a < b$ and $\alpha = 1 - e^{-a}$, $\beta = 1 - e^{-b}$, so that, by (iii), $\alpha \to 0$ when $a \to 0$, and, by (iv), $\beta \to 1$ when $b \to \infty$ (*i.e.* $1/b \to 0$):

Then $$\int_a^b \frac{x}{e^x-1}\,dx = \int_a^b \frac{xe^{-x}}{1-e^{-x}}\,dx$$

$$= \int_\alpha^\beta -\frac{\log(1-t)}{t}\,dt, \qquad 1 - e^{-x} = t,$$

$$= 2\int_{\sqrt{\alpha}}^{\sqrt{\beta}} -\frac{\log(1-u^2)}{u}\,du, \qquad u = +\sqrt{t},$$

$$= 2\int_{\sqrt{\alpha}}^{\sqrt{\beta}} -\frac{\log(1-u)}{u}\,du - 2\int_{\sqrt{\alpha}}^{\sqrt{\beta}} \frac{\log(1+u)}{u}\,du,$$

$$= 2\left\{\int_\alpha^\beta - \int_\alpha^{\sqrt{\alpha}} + \int_\beta^{\sqrt{\beta}} -\frac{\log(1-u)}{u}\,du\right\}$$

$$- 2\int_{\sqrt{\beta}}^{\sqrt{\beta}} \frac{\log(1+u)}{u}\,du,$$

https://doi.org/10.1017/S095018430000286X Published online by Cambridge University Press

whence

$$\int_a^b \frac{x}{e^x-1}\,dx = 2\int_{\sqrt{a}}^{\sqrt{\beta}} \frac{\log(1+u)}{u}\,du + 2\left\{\int_a^{\sqrt{a}} - \int_\beta^{\sqrt{\beta}} \frac{-\log(1-u)}{u}\,du\right\}.$$

Let $a \to 0$ and $b \to \infty$, then $\sqrt{a} \to 0$, $\sqrt{\beta} \to 1$ and so (the limit existing by Theorem 3) $\int_{\sqrt{a}}^{\sqrt{\beta}} \frac{\log(1+u)}{u}\,du \to \int_0^1 \frac{\log(1+u)}{u}\,du$; furthermore, by (v) and (vi), $\int_a^{\sqrt{a}} \frac{-\log(1-u)}{u}\,du \to 0$ and $\int_\beta^{\sqrt{\beta}} \frac{-\log(1-u)}{u}\,du \to 0$, whence $\int_0^\infty \frac{x}{e^x-1}\,dx$ exists and equals $2\int_0^1 \frac{\log(1+u)}{u}\,du$.

Combining theorems 3 and 4 we obtain Planck's integral

$$\int_0^\infty \frac{x}{e^x-1}\,dx = \tfrac{1}{6}\pi^2.$$

University College,
Leicester.

Some series for π

By C. E. Walsh.

Consider three sequences a_n, D_n, k_n $(n = 1, 2, 3, \ldots)$, such that $D_n a_n \to 0$ and, for $n > 1$,

$$(1) \qquad a_n + D_n a_n = D_{n-1} a_{n-1} + k_n a_n$$

Then $\sum_1^m a_n + D_m a_m = a_1(1 + D_1 - k_1) + \sum_1^m k_n a_n$. Hence, writing Σ for $\sum_1^\infty$,

$$(2) \qquad \Sigma a_n = a_1(1 + D_1 - k_1) + \Sigma k_n a_n$$

if either series converges. This will be applied to derive various series for π from the two known results [1]

[1] Knopp, *Infinite Series*, p. 269, Ex. 110 (a), and p. 246, Ex. 2.