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Quasimap Floer Cohomology for Varying Symplectic Quotients

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Abstract. We show that quasimap Floer cohomology for varying symplectic quotients resolves several puzzles regarding displaceability of toric moment fibers. For example, we present a compact Hamiltonian torus action containing an *open* subset of non-displaceable orbits and a codimension four singular set, partly answering a question of McDuff, and we determine displaceability for most of the moment fibers of a symplectic ellipsoid.

1 Introduction

Quasimap Floer cohomology, constructed in [16], is an obstruction to Hamiltonian displaceability of an invariant Lagrangian submanifold in the zero level set of a moment map for the action of a Lie group by an invariant time-dependent Hamiltonian. The differential for quasimap Floer cohomology counts orbits of the group on the space of holomorphic disks with boundary in the Lagrangian. Since holomorphic disks "upstairs" often have better properties than holomorphic disks in the symplectic quotient, quasimap Floer cohomology is a sometimes-better-behaved substitute for Floer cohomology in the quotient.

Here we restrict to the case of toric varieties. That is, the group $G \subset U(1)^N$ is a torus and the symplectic manifold $Y \cong \mathbb{C}^N$ is a Hermitian vector space. The group G acts in Hamiltonian fashion on Y with quadratic moment map $\Psi\colon Y \to \mathfrak{g}^\vee$. The symplectic quotient $X = Y /\!\!/ G := \Psi^{-1}(0)/G$ is a possibly singular toric manifold with action of the torus $T = U(1)^N/G$ and a moment map $\Phi\colon X \to \mathfrak{t}^\vee$ induced from that of $U(1)^N$ on Y. By definition, a smooth function on X is an equivalence class of smooth G-invariant functions on Y such that displaceability in X is equivalent to displaceability by a G-invariant Hamiltonian on Y. Non-displaceability results in the quotient X are provided by Floer-theoretic methods in Fukaya–Oh–Ohta–Ono [8,9]. Quasimap Floer cohomology gives the following result, which at first seems only slightly stronger. We denote by $v_1, \dots, v_N \in \mathfrak{t}$ the images of minus the standard basis vectors $e_1, \dots, e_N \in \mathbb{R}^N$. The moment polytope $\Phi(X)$ is the set of points satisfying linear inequalities

$$\Phi(X) = \{\lambda \in \mathfrak{t}^\vee \mid l_i(\lambda) \geq 0\}, \quad l_i(\lambda)/2\pi := \langle \lambda, v_i \rangle - \epsilon_i, \quad i = 1, \dots, N$$

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where $\langle \cdot , \cdot \rangle \colon \mathfrak{t}^\vee \times \mathfrak{t} \to \mathbb{R}$ is the canonical pairing and $\epsilon_1, \dots, \epsilon_N$ are constants given by the choice of moment map. This list of inequalities may not be minimal; that is, any particular inequality may or may not define a facet of $\Phi(X)$. Let Λ be the *universal Novikov field* consisting of possibly infinite sums of real powers of a formal variable q,

$$\Lambda = \Big\{ \sum_{n=0}^{\infty} c_n q^{d_n}, \quad c_n \in \mathbb{C}, d_n \in \mathbb{R}, \quad \lim_{n\to\infty} d_n = \infty \Big\}.$$

Let Λ_0 denote the subring consisting of sums with only non-negative powers. Any fiber $L_\lambda = \Phi^{-1}(\lambda)$ over an interior point $\lambda \in \operatorname{int}(\Phi(X))$ is a Lagrangian torus in X, namely a single free T-orbit, and has inverse image $\widetilde{L}_\lambda$ in $\Psi^{-1}(0)$ a $U(1)^N$-orbit in Y. We identify $H^1(L_\lambda, \Lambda_0) \cong H^1(T, \Lambda_0)^T \cong \mathfrak{t}^\vee \otimes \Lambda_0$. In particular, for any $v \in \mathfrak{t}$ and $b \in H^1(L_\lambda, \Lambda_0)$ we have a pairing $\langle v, b\rangle \in \Lambda_0$ and an exponential $e^{\langle v,b\rangle} \in \Lambda_0$. Choose $\delta = (\delta_1, \dots, \delta_N) \in \Lambda_0^N$. The *bulk-deformed potential* (introduced in [10, Theorem 3]) is

$$W_{\lambda,\delta} \colon H^1(L_\lambda, \Lambda_0) \to \Lambda_0, \quad b \mapsto \sum_{i=1}^{N} e^{\langle v_i, b\rangle - \delta_i} q^{l_i(\lambda)}. \tag{1.1}$$

Theorem 1.1 *For any $\lambda \in \operatorname{int}(\Phi(X))$, if there exists $\delta \in \Lambda_0^N$ such that $W_{\lambda,\delta}$ has a critical point, then L_λ is non-displaceable in X, or equivalently, $\widetilde{L}_\lambda$ is not displaceable by any G-invariant time-dependent Hamiltonian $H \in C^\infty([0,1] \times Y)^G$.*

This was proved in [16], but the possibility that the quotient is singular or that some of the inequalities do not define facets of the polytope was not included in the main result. Later we realized the importance of the more general result: even for understanding displaceability in open subsets of $\mathbb{C}^N$, the case of singular or "spurious" inequalities is highly relevant. The following example shows the importance of singular quotients.

Example 1.2 (Non-displaceability in $\mathbb{C}^2$ by $\mathbb{Z}_2$-invariant Hamiltonians) Let $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2_{>0}$, and $L = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 = \mu_1, |z_2|^2 = \mu_2\}$. The group $\mathbb{Z}_2 = \{\pm 1\}$ acts diagonally on $\mathbb{C}^2$. We claim that $L \subset \mathbb{C}^2$ is displaceable by a $\mathbb{Z}_2$-invariant Hamiltonian if and only if $\mu_1 \neq \mu_2$. Indeed, L is displaceable by a $\mathbb{Z}_2$-invariant time-dependent Hamiltonian if and only if $L/\mathbb{Z}_2$ is displaceable in the orbifold quotient $X = \mathbb{C}^2/\mathbb{Z}_2$. The latter admits the structure of a toric orbifold with moment polytope given by the span of the vectors $(1,1), (-1,1)$; see Example 3.6 and Figure 1. The quotient $L/\mathbb{Z}_2$ is the moment fiber over $(\lambda_1, \lambda_2) = (\mu_1 - \mu_2, \mu_1 + \mu_2)$. The probes of McDuff [14] (or the Hamiltonian action of $SU(2)$ on $\mathbb{C}^2$) show that if $\lambda_1 \neq 0$, then L is displaceable, since any such (λ_1, λ_2) is contained in a probe with direction $(0,1)$. Unfortunately quasimap Floer cohomology for $\mathbb{C}^2/\mathbb{Z}_2$ does not give any non-displaceable fibers, since the potential has no critical points. We apply the following trick: observe that $0 \in \mathbb{C}^2$ is fixed by the flow of any $\mathbb{Z}_2$-invariant Hamiltonian H, since $dH(t,0) = 0$ for all $t \in [0,1]$. Consider the singular symplectic quotient $\widehat{X}$ obtained from $X \times \mathbb{C}$ by symplectic quotient by S^1, acting on X with moment map $|z_1|^2 + |z_2|^2$, that is, by symplectic cut of X with respect to the diagonal circle. The space $\widehat{X}$ has the same moment polytope as X, but its realization as a symplectic quotient $Y /\!\!/ G$ involves a "spurious inequality" $\lambda_2 \geq 0$ as well as the inequalities for X given by $\lambda_1 + \lambda_2 \geq 0, -\lambda_1 + \lambda_2 \geq 0$. A toric moment fiber for X is displaceable if and only if the corresponding toric moment fiber for $\widehat{X}$ is displaceable, since after applying a cutoff function we may assume that H vanishes near the singular locus and X and $\widehat{X}$ are isomorphic away from the singular loci; see Proposition 2.2. The bulk-deformed potential $q^{\lambda_1+\lambda_2}e^{b_1+b_2} + q^{-\lambda_1+\lambda_2}e^{-b_1+b_2} + q^{\lambda_2}e^{b_2-\delta}$ has a critical point if and only if $\lambda_1 = b_1 = 0$ and $2e^{b_2} + e^{b_2-\delta} = 0$ or $e^{-\delta} = -2$. See Figure 1. A similar result for the deformation of $\mathbb{C}^2/\mathbb{Z}_2$ was studied in Fukaya et al. [7].

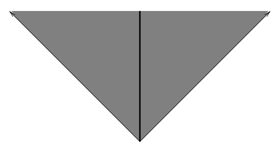

Figure 1: Displaceable and non-displaceable fibers for $\mathbb{C}^2/\mathbb{Z}_2$.

Below we give further examples of displaceability of torus orbits in open subsets of $\mathbb{C}^N$. Embedding such open subsets in singular symplectic quotients turns out to be quite useful for resolving displaceability. We use the same technique to partially answer a question of McDuff by giving an example of a compact toric orbifold with an open subset of non-displaceable fibers.

2 Displaceability in Orbifolds

In this section we review some basic facts about Hamiltonian displaceability in orbifolds. Recall that an *orbifold* is a Hausdorff second-countable topological space X equipped with an equivalence class of *orbifold structures*: a smooth proper étale groupoid $\widetilde{X}$ together with a homeomorphism from the space of isomorphism classes of objects in $\widetilde{X}$ to X; see *e.g.,* Adem–Klaus [3]. For any orbifold X and element $x \in X$, we denote by $\operatorname{Aut}(x)$ the group of automorphisms of any object $\widetilde{x}$ in $\widetilde{X}$ mapping to x, independent up to isomorphism of the choice of orbifold structure and choice of $\widetilde{x}$. Denote by $X^{\text{orb}} = \{x \in X \mid \# \operatorname{Aut}(x) > 1\}$ the subset of X consisting of points with more than one automorphism and by $X^{\text{mfd}} = X - X^{\text{orb}}$ the locus of points with only the identity morphism. Thus X^{mfd} is a smooth manifold and admits an open embedding into X.

Orbifolds typically arise as quotients of smooth manifolds by locally free actions of compact groups. The quotient Y/G of a smooth manifold Y by a compact group G has a canonical orbifold structure, given by taking local slices for the action. A G-space Y together with an orbifold equivalence $Y/G \to X$ is called a *global quotient presentation* of X. If the generic automorphism group of an orbifold X is trivial so

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that X^{mfd} is non-empty, then X admits a global quotient presentation, namely the orthogonal frame bundle of X by the action of the orthogonal group. The notion of an action of a Lie group on an orbifold X is somewhat complicated in general because of the various notions of an action of a Lie group on a category; see *e.g.*, [12]. In this paper all group actions will arise from global presentations, that is, from a G-equivariant action on Y where $X = Y/G$. The notion of symplectic form and Hamiltonian action have natural extensions to the orbifold case, which are somewhat simpler in the globally presented case: a symplectic form on a globally presented orbifold $X = Y/G$ is a closed G-basic form on Y that is non-degenerate on the normal bundles to the G-orbits.

Definition 2.1 Let X be a symplectic orbifold. A subset $L \subset X$ is *Hamiltonian displaceable* if and only if there exists a function $H \in C_c^\infty([0,1] \times X)$ with time t Hamiltonian flow $\phi_{H,t} \colon X \to X$ such that $\phi_{H,t}(L) \cap L = \varnothing$ for some t.

We collect a few elementary properties of displaceability in the following proposition.

Proposition 2.2 (i) *Suppose that $X_1 \subset X_2$ is an open set and $L \subset X_1$. If L is displaceable in X_1, then L is displaceable in X_2.*

(ii) *Suppose that either $L_1 \subset X_1$ or $L_2 \subset X_2$ are displaceable. Then $L_1 \times L_2$ is displaceable in $X_1 \times X_2$.*

(iii) *Suppose that X is a Hamiltonian G-orbifold and $X /\!\!/ G$ is its symplectic quotient. Then $L \subset X /\!\!/ G$ is displaceable if and only if the inverse image of L in X is displaceable by the flow of a G-invariant time-dependent Hamiltonian.*

(iv) *Suppose that $L_1, L_2 \subset X$ are disjoint subsets such that L_1 is displaceable by a flow $\phi_{t,H}$ with $\phi_{t,H}(L_2) = L_2$ for all $t \in [0,1]$. Then L_1 is displaceable by a flow ϕ_{t,H_2} equal to the identity in a neighborhood of L_2 for all $t \in [0,1]$.*

Proof (i) If $H_1 \in C_c^\infty(X_1)$ displaces L_1 in X_1, then the extension of H_1 by zero to $C_c^\infty(X_2)$ displaces L_1 in X_2. (ii) Suppose without loss of generality that H_1 displaces L_1. Then $\pi_1^* H_1$ displaces $L_1 \times L_2$ where $\pi_1 \colon X_1 \times X_2 \to X_1$ is the projection. (iii) If H displaces L then any invariant extension of $\pi^* H$, where $\pi \colon \Phi^{-1}(0) \to X /\!\!/ G$, to X, displaces the inverse image of L. The converse is similar. (iv) If $\phi_{t,H}$ displaces L_1 and maps L_2 to itself, then let $\rho \in C^\infty(X)$ be a function equal to 1 on the image of L_1 under the flow $\phi_{H,t}$, and zero on an open neighborhood of L_2. Then the flow of ρH displaces L_1 and is equal to the identity on a neighborhood of L_2. ∎

3 Displaceability of Toric Moment Fibers

We consider the following class of possibly non-compact Hamiltonian torus actions on orbifolds. Let T be a torus and let $\mathfrak{t}_{\mathbb{Z}} = \exp^{-1}(1)$ be the integral lattice.

Definition 3.1 X is an *open symplectic toric orbifold* for T if X is a connected Hamiltonian T-orbifold with moment map $\Phi \colon X \to \mathfrak{t}^\vee$ satisfying the following conditions:

(i) $\Phi(X)$ is a defined by a finite set of affine linear inequalities defined by vectors in $\mathfrak{t}_{\mathbb{Z}}$ and strict affine linear inequalities defined by vectors in $\mathfrak{t}$;

(ii) $\Phi\colon X \to \Phi(X)$ is proper;
(iii) the T-action is generically free;
(iv) $\dim(T) = \dim(X)/2$;

Remark 3.2 By (i) the image $\Phi(X)$ is given by

$$\Phi(X) = \left\{ \lambda \in \mathfrak{t}^\vee \,\middle|\, \begin{array}{ll} \langle \lambda, \nu_i \rangle \geq \epsilon_i & i = 1, \dots, k, \\ \langle \lambda, \nu_i \rangle > \epsilon_i & i = k+1, \dots, N \end{array} \right\} \tag{3.1}$$

for some vectors $\nu_i \in \mathfrak{t}_\mathbb{Z}, i = 1, \dots k$ and $\nu_i \in \mathfrak{t}, i = k+1, \dots, N$. By (ii), (iii), and the results of [13], the stabilizer of any point in the inverse image $\Phi^{-1}(F)$ of an open facet F is isomorphic to the cyclic group $\mathbb{Z}_{n(F)}$ for some integer $n(F) \geq 1$. We assume that ν_i is normalized to be the $n(F_i)$-th multiple of the primitive lattice vector pointing inward from the facet F_i corresponding to ν_i. In this way, in the compact case, the vectors ν_i are the data used in the *weighted fan* classification of toric orbifolds in [4, 13].

Example 3.3 We have that $X = \mathbb{C}^n$ itself is an open symplectic toric manifold with symplectic form $-2\sum_{i=1}^n dq_j \wedge dp_j$ where $z_j = q_j + ip_j$ and moment map

$$\Phi\colon X \to \mathfrak{t}^\vee \cong \mathbb{R}^n, (z_1, \dots, z_n) \mapsto (|z_1|^2, \dots, |z_n|^2).$$

The vectors $\nu_1, \dots, \nu_n$ are the standard basis vectors.

Example 3.4 Any open subset of $\mathbb{C}^n$ defined by $a_1|z_1|^2 + \dots + a_n|z_n|^2 < 1$ for some $a_1, \dots, a_n \in \mathbb{Q}$ is an open symplectic toric manifold obtained from a weighted projective space by removing a divisor at infinity.

The following well-known lemma indicates how to read off the automorphism group $\mathrm{Aut}(x)$ of a point $x \in X$ from the facets of $\Phi(X)$ containing $\Phi(x)$.

Lemma 3.5 (see *e.g.*, [13, Lemma 6.2]) *Let X be an open symplectic toric orbifold, $x \in X$, and let $I(x) = \{i | \langle \Phi(x), \nu_i \rangle = \epsilon_i\}$ be the indices of normal vectors of facets containing $\Phi(x)$. Then*

$$\mathrm{Aut}(x) \cong \ker\left(U(1)^{\#I(x)} \to T, \exp\left(\sum_{i \in I(x)} c_i e_i \right) \mapsto \exp\left(\sum_{i \in I(x)} c_i \nu_i \right) \right).$$

In particular, since $\mathrm{Aut}(x)$ is finite, this lemma implies that $\Phi(X)$ is a *simple* polytope; that is, the normal vectors at any point are linearly independent.

Example 3.6 Let X be the quotient of $Y = \mathbb{C}^2$ by the diagonal action of $\mathbb{Z}_2 = \{\pm 1\}$ given by scalar multiplication. The action of $T' = U(1)^2$ on $\mathbb{C}^2$ descends to a generically free action of $T = T'/\mathbb{Z}_2$ on X. The integral lattice $\mathfrak{t}_\mathbb{Z}$ is the inverse image of $\mathbb{Z}_2$ under the exponential map for T', hence $\mathfrak{t}_\mathbb{Z}$ is generated by $(1/2, 1/2), (1/2, -1/2)$.

We identify $\mathfrak{t} \to \mathbb{R}^2$ by $(\xi_1, \xi_2) \mapsto (\xi_1 + \xi_2, \xi_1 - \xi_2)$, so the integral lattice becomes the standard one. The moment map for the T action on X is then

$$\Phi\colon X \to \mathbb{R}^2, \quad (z_1, z_2) \mapsto \left(|z_1|^2 + |z_2|^2, |z_1|^2 - |z_2|^2\right).$$

The integral vectors are $(-1, 1), (1, 1)$. The automorphism group

$$\mathrm{Aut}(0) = \ker\left(U(1)^2 \to U(1)^2, (z_1, z_2) \mapsto (z_1 z_2, z_1 z_2^{-1})\right) = \mathbb{Z}_2.$$

Theorem 3.7 *Any open symplectic toric orbifold can be obtained by symplectic reduction by an open subset of $T^*U(1)^k \times \mathbb{C}^l$ for some k, l by a subtorus of $U(1)^{k+l}$.*

Proof The compact case is a consequence of the results of [11] as discussed in [16]. Given an open symplectic toric orbifold X the symplectic cutting construction produces a symplectic toric orbifold X' with the same moment polytope as X. The uniqueness result of [11] implies that X is isomorphic to X' as a Hamiltonian T-orbifold. ∎

We wish to understand the Hamiltonian displaceability of *toric moment fibers*:

Definition 3.8 Let X be an open symplectic toric orbifold. A *toric moment fiber* is a Lagrangian torus given as a fiber $L_\lambda = \Phi^{-1}(\lambda)$ for some $\lambda \in \mathrm{int}(\Phi(X))$.

Denote by $ND(X) \subset \mathrm{int}(\Phi(X))$ (resp. $D(X)$) the set of points corresponding to non-displaceable (resp. displaceable) toric moment fibers.

Example 3.9 (Moser) Let X be the unit disk with moment polytope $[0, 1)$. Then $D(X) = [0, 1/2)$ and $ND(X) = [1/2, 1)$, because Moser [15] shows that the only invariant of a symplectic surface is its area. Hence a circle L in the disk X encloses less than half the area if and only if L is displaceable in X. Similarly, if $X = \mathbb{P}^1$ with moment polytope $\Phi(X) = [-1, 1]$, then $ND(X) = \{0\}$.

Example 3.10 (McDuff [14]) Let X be a compact symplectic toric orbifold with moment map Φ, and let F be an open facet of $\Phi(X)$ such that $\Phi^{-1}(F) \subset X^{\mathrm{mfd}}$. Let $\nu \in \mathfrak{t}_{\mathbb{Z}}^{\vee}$ be a vector such that ν can be completed to a lattice basis by vectors parallel to F. If $\lambda_0 \in F$ and λ lies less than half-way along $(\lambda_0 + \mathbb{R}_{\geq 0}\nu) \cap \Phi(X)$, then $\Phi^{-1}(\lambda)$ is displaceable. Let $T_0 \subset T$ be the torus whose Lie algebra is the annihilator of ν. Moser's argument shows that $\Phi^{-1}(\lambda)/T_0$ is displaceable in $X /\!\!/ T_0$, and then Proposition 2.2(iii) implies that $\Phi^{-1}(\lambda)$ is displaceable in X. See Abreu–Borman–McDuff [1] for improvements on this method.

Naive application of Theorem 1.1 (that is, without spurious inequalities) does not come close to resolving the questions of non-displaceability of toric fibers even for simple examples and after including bulk deformations in [9]. For example, for a weighted projective space the naive method gives a single non-displaceable fiber over $\lambda = (5/3, 5/3)$, while McDuff's method shows displaceability for only some of the other fibers. See Example 4.11.

4 Potentials for Varying Quotients

As explained in the introduction, open symplectic toric manifolds have various realizations as symplectic quotients, some singular, and the quasimap Floer cohomology for each realization can give additional information about displaceability. We combine the potentials for the different compactifications into a potential involving infinitely many variables as follows.

Definition 4.1 An affine linear function $\ell\colon \mathfrak{t}^\vee \to \mathbb{R}$ is *semipositive* on $\Phi(X)$ if and only if ℓ is positive on $\Phi(X^{\mathrm{mfd}})$ and non-negative on $\Phi(X^{\mathrm{orb}})$.

Remark 4.2 Any affine linear function ℓ on $\mathfrak{t}^\vee$ is given by $\ell(\lambda) = \langle v, \lambda\rangle - \epsilon$ for some $v \in \mathfrak{t}, \epsilon \in \mathbb{R}$. If the function corresponding to λ, ϵ is semipositive, then so is the function corresponding to λ, ϵ' for any $\epsilon' \leq \epsilon$.

Example 4.3 Let $X = \mathbb{P}(1, 1, 2)$ denote the weighted projective plane with moment map the convex hull of $(0, 0), (1, 0)$, and $(0, 2)$, with the orbifold singularity with automorphism group $\mathbb{Z}_2$ mapping to $(1, 0)$. Then the linear function $\langle(-1, 0), \cdot\,\rangle - \epsilon$ is semipositive for $\epsilon \leq -1$, while the linear function $\langle(0, -1), \cdot\,\rangle - \epsilon$ is semipositive for $\epsilon < -2$.

Definition 4.4 Denote by $C(\mathfrak{t}_{\mathbb{Z}}, \overline{\mathbb{R}})_+$ the set of maps $\epsilon\colon \mathfrak{t}_{\mathbb{Z}} \to \mathbb{R} \cup \{-\infty\}$ such that

(i) only finitely many values of ϵ are finite;
(ii) if v defines a facet of $\Phi(X)$ in the sense of (3.1), then $\epsilon(v) = \min_{\lambda\in\Phi(X)}\langle v, \lambda\rangle$;
(iii) if v does not define a facet, then $\langle v, \cdot\,\rangle - \epsilon(v)$ is semipositive on $\Phi(X)$.

Definition 4.5 The *potential* for $\lambda \in \operatorname{int}(\Phi(X)),\ \epsilon \in C(\mathfrak{t}_{\mathbb{Z}}, \overline{\mathbb{R}})_+,\ \delta \in C(\mathfrak{t}_{\mathbb{Z}}, \Lambda_0)$ is the function

$$W_{\lambda,\epsilon,\delta}\colon H^1(T, \Lambda_0) \to \Lambda_0, \quad b \mapsto \sum_{v\in\mathfrak{t}_{\mathbb{Z}}} q^{\langle v,\lambda\rangle-\epsilon(v)} e^{\langle v,b\rangle-\delta(v)}$$

where by convention $q^\infty = 0$.

Example 4.6 (Symplectic balls) Let $X = \{z \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 < 1\}$ be the unit ball in $\mathbb{C}^n$. Consider the coweight $v = (-1, \dots, -1)$ and let $\epsilon(v') = -c$ if $v = v'$, $\epsilon(e_i) = 0$ for all $1 \leq i \leq n$ (where e_i denotes the standard basis vector) and $\epsilon(v') = -\infty$ otherwise, and $\delta(e_i) = 0$. Then

$$W_{\lambda,\epsilon,\delta}(b) = \sum_{i=1}^n q^{\lambda_i} e^{b_i} + q^{-\lambda_1-\dots-\lambda_n+c} e^{-b_1-\dots-b_n}$$

for $c \geq 1$.

Theorem 4.7 *Suppose that $\lambda \in \operatorname{int}(\Phi(X))$ is such that for some ϵ, δ, $W_{\lambda,\epsilon,\delta}$ has a critical point. Then $\Phi^{-1}(\lambda) \subset X$ is non-displaceable.*

Before we give the proof, we present several examples showing how this theorem improves on that of [16].

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Example 4.8 (Symplectic balls continued) Continuing Example 4.6, $W_{\lambda,\epsilon,\delta}$ has a critical point if and only if $\lambda = (c,\dots,c)/(n+1)$ for $c \geq 1$ if and only if $\Phi^{-1}(\lambda)$ is non-displaceable, which is well known from the works of Cho [5] and Entov-Polterovich [6]. McDuff's method implies that the remaining toric fibers are displaceable. See Figure 2.

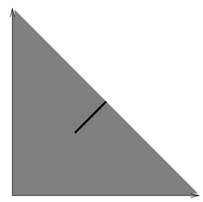

Figure 2: Displaceable and non-displaceable fibers for the symplectic 4-ball.

Example 4.9 (A weighted projective plane with a measure zero set of non-displaceable fibers) Suppose that $X = \mathbb{P}(1,1,2)$ is the weighted projective plane with moment polytope $(0,0),(1,0),(0,2)$. We write

$$\Phi(X) = \left\{ (\lambda_1,\lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \geq 0, 2\lambda_1 + \lambda_2 \leq 2, \lambda_1 \leq 1 \right\}.$$

The corresponding potential is

$$W_{\lambda,\epsilon,\delta}(b) = q^{\lambda_1} e^{b_1} + q^{\lambda_2} e^{b_2} + q^{-2\lambda_1-\lambda_2+2} e^{-2b_1-b_2} + q^{-\lambda_1-\epsilon} e^{-b_1-\delta_1}.$$

For $\epsilon = 1$ we obtain a critical point if and only if $\lambda = (1,0) + \zeta(-1,1)$ where $\zeta \leq 1/2$. See Figure 3.

Example 4.10 (A symplectic ellipsoid) Suppose that

$$X = \left\{ (z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2/2 < 1 \right\}.$$

We write

$$\Phi(X) = \left\{ (\lambda_1,\lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \geq 0, 2\lambda_1 + \lambda_2 < 2, \lambda_1 < 1 \right\}.$$

For $\epsilon_1, \epsilon_2 \geq 0$, the corresponding potential is

$$W_{\lambda,\epsilon,\delta}(b) = q^{\lambda_1} e^{b_1} + q^{\lambda_2} e^{b_2} + q^{-2\lambda_1-\lambda_2-\epsilon_1} e^{-2b_1-b_2-\delta_1} + q^{-\lambda_1-\epsilon_2} e^{-b_1-\delta_2}.$$

Then $W_{\lambda,\epsilon,\delta}$ has a critical point for some ϵ,δ if and only if $\lambda_1 + \lambda_2 \geq 1, \lambda_1 \geq 1/2$, namely $\epsilon_1 = 2\epsilon_2$ and $\epsilon_2 = -\lambda_1 - \lambda_2$. Most of the remaining fibers are displaceable by probes, although we did not manage to resolve the question completely; see Figure 4.

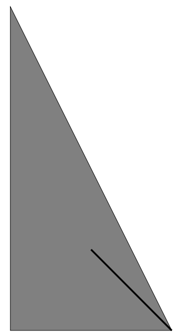

Figure 3: Displaceable and non-displaceable fibers for $\mathbb{P}(1,1,2)$.

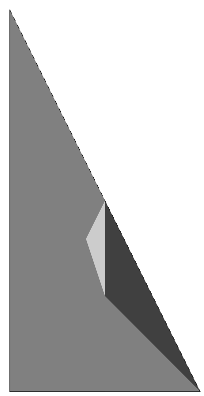

Figure 4: Displaceable and non-displaceable fibers for the ellipsoid

Example 4.11 (A weighted projective plane with a positive measure subset of non-displaceable fibers) We show that the toric orbifold $X = \mathbb{P}(1,3,5)$ contains an open subset of non-displaceable moment fibers. This partly answers a question of McDuff who asked whether there is an example of such an action, presumably thinking of the smooth case. The moment polytope is the convex hull $(0,0)$, $(3,0)$, $(0,5)$, and can be defined by the inequalities

$$\Phi(X) = \left\{ \lambda \in \mathbb{R}^2, \lambda_1 \geq 0, \lambda_2 \geq 0, 5\lambda_1 + 3\lambda_2 \leq 15, -\lambda_1 \geq -3, -2\lambda_1 - \lambda_2 \geq -6 \right\}.$$

Consider the potential

$$W_{\lambda,\epsilon_1,\epsilon_2,\delta_1,\delta_2}(b) = q^{\lambda_1} e^{b_1} + q^{\lambda_2} e^{b_2} + q^{-5\lambda_1 - 3\lambda_2 + 15} e^{-5b_1 - 3b_2} + q^{-\lambda_1 + 3 - \epsilon_1} e^{-b_1 - \delta_1} + q^{-2\lambda_1 - \lambda_2 + 6 - \epsilon_2} e^{-2b_1 - b_2 - \delta_2}.$$

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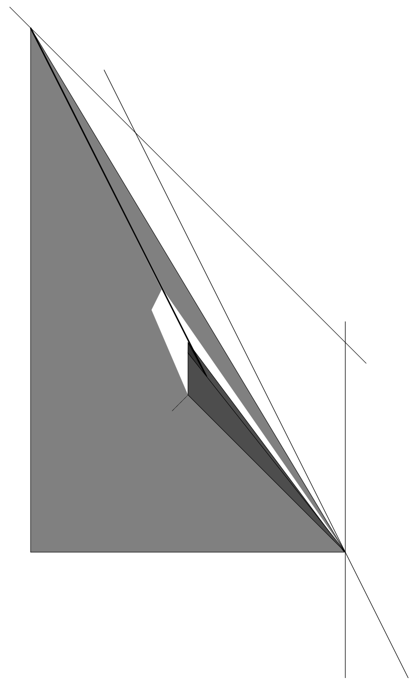

Figure 5: More non-displaceable fibers for $\mathbb{P}(1,3,5)$ using spurious inequalities

The potential for $(\lambda_1, \lambda_2) = (3,0) + c_1(-1,1) + c_2(-3,4)$ has a critical point if $c_1, c_2 \geq 0$, but $\langle(\lambda_1,\lambda_2),(-1,0)\rangle + 3 \leq \langle(\lambda_1,\lambda_2),(1,0)\rangle$. This is the condition that the terms defined by the facets with normal vectors $(-1,0),(0,1),(-2,-1)$ have leading order terms that of equal order and lower order than the terms arising from the remaining facets. As in the previous examples, this means that the potential arising from these terms has a non-degenerate critical point. The additional terms do not affect the existence of a critical point, by [8, Lemma 10.16] (which is a version of the implicit function theorem for formal functions with values in the Novikov ring). Note that [8, Lemma 10.16] was written for integral polytopes (polytopes corresponding to smooth toric varieties), but the technique works equally well for arbitrary potentials, since integrality of the basis given by the normal vectors at a vertex is never used in the proof. It follows that $\mathbb{P}(1,3,5)$ has an open subset of non-displaceable fibers. An additional line segment of non-displaceable fibers is determined by the equality of powers in the leading order of terms from the facets with normal vectors $(1,0),(-5,-3)$ and a "spurious" facet with normal vector $(-1,-1)$. Additional open region of non-displaceable torus fibers in $\mathbb{P}(1,3,5)$ are determined by the leading order terms of (i) the facet with normal vector $(0,1)$ and the spurious facets with normal vectors $(-1,-1)$ and $(-1,0)$, and (ii) the facet with normal vector $(-5,-3)$ and the spurious facets with normal vectors $(-1,0),(-2,-1)$ These

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were pointed out to us by M. S. Borman; see [1]. See Figure 5, where the regions displaceable by McDuff's probes are shaded in lighter grey. Note that the projective line $\mathbb{P}(1,2)$ also has an open subset of non-displaceable fibers as explained in [16], but this is somewhat more expected, since displaceability in $\mathbb{P}(1,2)$ is equivalent to dispaceability in the disk and has singularities in codimension 2, not 4.

Proof of Theorem 4.7 First suppose that X is a compact manifold, so that all of the additional affine linear functions are strictly positive on $\Phi(X)$. Suppose that $W_{\lambda,\epsilon,\delta}$ has a critical point for some $\epsilon = (\epsilon(\nu))$. Then X is a symplectic quotient of the representation Y by a torus given as the kernel G of the homomorphism $U(1)^N \to T$ defined by the matrix formed by the vectors $\nu \in \mathfrak{t}_{\mathbb{Z}}$ where $\epsilon(\nu) \neq -\infty$. The theorem then follows from [16, Theorem 7.1].

Next consider the case where X is a compact orbifold. Suppose that $W_{\lambda,\epsilon,\delta}$ has a critical point for some $\epsilon = (\epsilon(\nu))$. Let $Y /\!\!/ G$ be the symplectic quotient of the representation Y as in the previous paragraph. Then $Y /\!\!/ G$ is a Hamiltonian T-orbifold on the locus where G acts freely, and the singular set of $Y /\!\!/ G$ (which can be worse than orbifold) is contained in the singular set of X. The proof of [16, Theorem 7.1] shows that there is no G-invariant Hamiltonian on Y displacing the inverse image of L_λ in Y. On the other hand, suppose that L_λ is displaceable in X by some Hamiltonian H. Necessarily, the flow of H preserves X^{orb}, so if $\phi_H(L_\lambda) := \cup_{t\in[0,1]}\phi_{H,t}(L_\lambda)$ is the flow-out, then $\phi_H(L_\lambda)$ is disjiont from X^{orb}. Choose a cutoff function $\rho \in C^\infty(X)$ such that ρ is equal to 1 on an open neighborhood of $\phi_H(L_\lambda)$, and support contained in X^{mfd}. Then the flow of ρH also displaces L_λ. Since ρH vanishes in a neighborhood of the singular set, ρH lifts to a smooth function on Y that displaces the inverse image of L_λ.

Finally consider the case where X is non-compact. Then X^{mfd} is an open subset of the space $Y /\!\!/ G$ defined in the previous paragraph. Suppose that L_λ is displaced by the flow of some $H \in C_c^\infty([0,1] \times X)$, and $W_{\lambda,\epsilon,\delta}$ has a critical point for some $\epsilon = (\epsilon(\nu))$. Then after choosing a cutoff function ρ as in the previous paragraph, ρH lifts to a smooth invariant function on Y displacing the inverse image of L_λ, which is a contradiction. ∎

5 Functoriality of the Quasimap Mirror

According to the philosophy of mirror symmetry, the *mirror* of a symplectic orbifold X should be a complex space with potential function $W\colon X^\vee \to \Lambda_0$, so that the Fukaya category of X is equivalent to the derived category of matrix factorizations. As explained in Fukaya et al. [8], the mirror of a toric variety is a quantum correction of a potential obtained by Givental (1.1). In this section, we describe the quasimap mirror construction (which is a somewhat naive version of the mirror but perhaps more useful for determining displaceability) as a contravariant functor that behaves well with respect to inclusions, which clarifies various aspects of the displaceability problem.

Definition 5.1 Let X be a (possibly) open toric orbifold in the sense of Definition 3.1. The *quasimap mirror* for X is the space

$$X^\vee := \mathfrak{t}^\vee \times C(\mathfrak{t}_\mathbb{Z}, \overline{\mathbb{R}})_+ \times C(\mathfrak{t}_\mathbb{Z}, \Lambda_0) \times H^1(T, \Lambda_0)$$

equipped with the potential $W\colon X^\vee \to \Lambda_0$, $(\lambda, \epsilon, \delta, b) \mapsto W_{\lambda,\epsilon,\delta}(b)$.

Note that the work of Fukaya et al. [8, 9] shows the existence of a particular deformation of the naive potential that has the properties predicted by mirror symmetry, such as the correct number of critical points that the definition above lacks. However, as we saw in the previous section, the above formulation is more useful for detecting displaceability. The quasimap mirror also has good functoriality properties, parallel to the properties of displaceable fibers listed in Proposition 2.2. The following definition will be used in the theorem to relate the mirror of an action with the mirror for a quotient.

Definition 5.2 For any sub-torus $T_0 \subset T$, define $\pi\colon \mathfrak{t} \to \mathfrak{t}/\mathfrak{t}_0$ to be the projection and

$$(5.1) \qquad (\pi_*\epsilon)(\nu_0) = \min_{\pi(\nu)=\nu_0} \epsilon(\nu), \quad (\pi_*\delta)(\nu_0) = \sum_{\pi(\nu)=\nu_0, \epsilon(\nu)=(\pi_*\epsilon)(\nu_0)} \delta(\nu).$$

Theorem 5.3 (Functorial properties of quasimap mirrors) (i) *If $X_1 \to X_2$ is an open embedding of toric orbifolds then $X_2^\vee$ embeds canonically in $X_1^\vee$. If $X_1 \to X_2 \to X_3$ are open embeddings then $X_3^\vee \to X_1^\vee$ is the composition of $X_3^\vee \to X_2^\vee$ and $X_2^\vee \to X_1^\vee$.*

(ii) *If X_1, X_2 are open toric sub-orbifolds of a toric orbifold X, then $(X_1 \cup X_2)^\vee = X_1^\vee \cap X_2^\vee$ and $(X_1 \cap X_2)^\vee = X_1^\vee \cup X_2^\vee$.*

(iii) $(X_1 \times X_2)^\vee = X_1^\vee \times X_2^\vee$.

(iv) *If $T_0 \subset T$ is a subtorus, then considering $X/\!\!/T_0$ as a toric T/T_0 orbifold then the space obtained from $X^\vee$ by composition with $H^1(T/T_0, \Lambda_0) \to H^1(T, \Lambda_0)$ and composition with π_* from* (5.1) *embeds into $(X/\!\!/T_0)^\vee$.*

The proof is immediate from the definition of semipositivity; in particular, in the setting of (i) if ℓ is semipositive on $\Phi(X_2)$, then ℓ is automatically semipositive on $\Phi(X_1)$. Part (i) says that the quasimap mirror construction gives a contravariant functor. Part (iv) is only an embedding, because some of the true facets of $\Phi(X)$ will not define facets of $\Phi(X/\!\!/T_0)$, so the mirror of $X/\!\!/T_0$ is in general larger than that obtained from X.

The functorial properties of the quasimap mirror construction translate into the following functorial properties of the corresponding non-displaceable moment fibers. Say that $L_\lambda \subset X$ is *(quasimap) Floer non-displaceable* if $W_{\lambda,\epsilon,\delta}$ has a critical point for some ϵ, δ. Let $\mathrm{FND}(X)$ denote the set of Floer non-displaceable Lagrangians. The following is a consequence of Theorem 5.3.

Corollary 5.4 (Functorial properties of Floer non-displaceable sets)

(i) *For any open embedding $X_1 \to X_2$,* $\mathrm{FND}(X_1) \supset \mathrm{FND}(X_2)$.

(ii) *If X_1, X_2 are open subsets of a toric orbifold X, then*

$$\mathrm{FND}(X_1 \cup X_2) \subset \mathrm{FND}(X_1) \cap \mathrm{FND}(X_2) \text{ and } \mathrm{FND}(X_1 \cap X_2) \supset \mathrm{FND}(X_1) \cup \mathrm{FND}(X_2).$$

(iii) $\mathrm{FND}(X_1 \times X_2) = \mathrm{FND}(X_1) \times \mathrm{FND}(X_2)$.

(iv) *If $T_0 \subset T$ is a subtorus, then* $\mathrm{FND}(X /\!\!/ T_0)$ *contains the intersection of* $\mathrm{FND}(X)$ *with the fiber over* 0 *under the map* $\mathfrak{t}^\vee \to (\mathfrak{t}/\mathfrak{t}_0)^\vee$.

The importance of the last item was pointed out to us by Abreu–Macarini [2].

Obviously one would like to know whether one can obtain the non-displaceable set from a cover.

Proposition 5.5 *Any compact symplectic toric orbifold has a canonical open cover indexed by the fixed point set X^T, given as follows: for each $x \in X^T$, let $X(x)$ be the open symplectic toric orbifold obtained from X by removing all divisors not containing x. Then $X = \cup_{x \in X^T} X(x)$.*

Because the quasimap mirror construction is contravariant, one cannot expect to "recover" the symplectic topology of a toric orbifold from the symplectic topology of its canonical cover. Rather, the symplectic topology of each open subset already "knows" about the symplectic topology of the compactification. Still one would like to know the relationship between displaceability in X and displaceability in the open cover. The following question is, as far as we know, open.

Question 5.6 *Is $ND(X) = \bigcap_{x \in X^T} ND(X(x)), \quad D(X) = \bigcup_{x \in X^T} D(X(x))$?*

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