THE PSEUDO-ISOSCELES TRIANGLE.

(Two external Bisectors equal, triangle not isosceles).

Intermédiaire des Mathématiciens, 1894, pp. 70, 149 ; 1895, pp. 101, 169.

Mathesis, 1895, p. 261 ; 1900, pp. 129 ff (Emmerich) ; 1901, p. 24 ; 1902, p. 43 ; 1902, pp. 112, 114 (Delahaye) ; 1925, pp. 316 ff.

Bulletin des Sciences Math. et Phys. Elementaires, 1903-4, IX. p. 146 ; 1907-8, XII. p. 22 (Fontene).

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Crelle's Journal (Steiner), 1844, and

Philosophical Magazine (J. J. Sylvester), 1853.

The inextensible string

By A. G. WALKER.

An object to which we were all introduced at an early stage in mechanics is the inextensible string. This appears frequently without causing much trouble, but there is one type of problem which, in my opinion, stands apart from the rest, and which certainly caused me a lot of trouble. Such a problem is when impulses are given to a system which includes an inextensible string, as, for example, a system consisting of two rigid parts joined by a string. If an impulse is applied to one of these parts, an impulsive tension (T) may be set up in the string, which, in turn, gives an impulse to the other part. One new quantity, T, has appeared, and one equation in addition to the ordinary dynamical equations is thus required before the problem of finding the change in motion of the system can be solved. It is at this stage that opinions can differ, for this extra equation depends essentially upon what concept of an inextensible string is being adopted, and there is more than one. The usual procedure is to employ a "geometrical equation" based upon the argument that the two ends must have equal component velocities in the line of the string as long as the string is taut. This seems almost obvious when described in such general terms, and is followed by such eminent writers as Routh[1] and Loney,[2] amongst others. I suggest,

[1] See for example the worked exercise (170) on p. 149 of his *Elementary Rigid Dynamics* (1882).

[2] Loney devotes two sections to methods involving the geometrical equation in *Dynamics of a Particle and of Rigid Bodies* (1919), p. 180.

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however, that this method implies a concept which leads to results contrary to common-sense and to everyday experience. This is best illustrated by a simple example.

Two equal particles lie on a smooth table and are joined by an inextensible string. If one particle is projected away from the other with velocity V, *find the velocities immediately after the string becomes taut.*

Applying the geometrical condition, the momentum equation at once gives both velocities as $V/2$. In the first place, this means that 50 per cent. of the energy has been lost, presumably dispersed by the string which need be nothing more than a light thread! This, in my opinion, is against common-sense.[1] In the second place, it is common experience that, in such a situation, the string becomes slack immediately after the jerk, so that the particles would not be expected to procede with the same velocity. The theoretical result is thus contrary to experience.

It is, of course, realised that each concept in mechanics is an ideal, and that we cannot expect theoretical results to agree exactly with experience. It has usually been possible, however, to adopt concepts which lead to results approximating very closely to those actually observed; the inextensible string leading to the above results clearly does not belong to this class. Since we do in fact deal with strings which are very nearly inextensible in a general sense, the problem now is to idealise such a string so that the consequences are reasonable and in agreement with experience.

In my opinion the most important requirement is that energy shall be conserved, and since this usually means that the string must slacken, it appears that the string must be allowed to have elastic properties. This leads me to regard an inextensible string as a perfectly elastic string, with a very large modulus of elasticity. The energy is then conserved as far as the string is concerned, and for a system which includes only one such string, the energy equation is the one extra equation which enables the change in motion to be found without going back to the differential equations of motion. In such a case the actual value of the modulus does not appear. In general the string stretches and returns to its natural length in a very short time, and the calcu-

[1] Routh's exercise, cited above, gives a similar considerable loss of energy.

lated velocities are those existing just as the string slackens, for then only will the string provide no part of the total energy.

Returning to the problem of two equal particles joined by a string, we would now argue that the energy imparted to the system is $\frac{1}{2}mV^2$, and since momentum is still conserved, it follows that the velocities after the string has become taut are 0 and V, the projected particle being brought to a standstill. In this case, therefore, the string slackens as we desired. Many students would probably give this result for such a simple problem, but would return to the geometrical condition, *i.e.*, postulate the other quite different kind of string, in a less obvious problem. It should be noted that we are not talking of different methods of solving a problem, but of different problems, involving different concepts of an inextensible string.

It is seen that a problem with one string is straightforward if the energy equation can be used. Great difficulties generally occur, however, when there is more than one string, for it is then, as far as I know, necessary to solve the differential equations of motion, the strings being assumed elastic with large and, perhaps, different moduli. The final velocities in the system are attained when the strings return to their natural lengths, and these will usually involve the ratios of the different moduli, assumed comparable. It would thus be necessary to distinguish between different kinds of "inextensible-elastic strings."

Another large class of problems still remains, in which an impulse of given magnitude P is applied to a system which includes an inextensible-elastic string. There is now no energy equation to start with, and another method of attack must be found. Since the time during which the string is stretched is very small owing to the large modulus, it may be comparable with the time during which the impulsive force acts, and so provide complications. It is therefore safer to start with the complete equations of motion, the impulse being replaced by a large force F acting for a short time τ where $P = F\tau$.

Consider for example the problem in which two particles m, m' on a smooth table are joined by a string which is just taut, and an impulse P is applied to m along the line of the string away from m'. If l and λ are the natural length and modulus of the string, and if $M = m + m'$ and $P = F\tau$, then it can be verified that

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the velocities v of m and v' of m' when the string returns to its natural length (after time $>\tau$) are

$$v = \frac{P}{M}\left(1 - k\frac{m'}{m}\right), \qquad v' = \frac{P}{M}(1+k) \tag{1}$$

where $$k = \frac{|\sin\epsilon|}{\epsilon}, \qquad \epsilon = \tfrac{1}{2}\left(\frac{M}{lmm'}\right)^{1/2}\tau\lambda^{1/2}. \tag{2}$$

The energy imparted to the system is

$$E = \frac{P^2}{2M}\left(1 + k^2\frac{m'}{m}\right). \tag{3}$$

The initial motion thus depends upon k, *i.e.*, upon the product $\tau\lambda^{\frac{1}{2}}$ which can take any value between 0 *and* ∞. At one extremity we can have $\tau\lambda^{\frac{1}{2}} \to \infty$, in which case $k = 0$ and

$$v = v' = \frac{P}{M}, \qquad E = \frac{P^2}{2M}, \tag{4}$$

showing that the geometrical condition now holds. We may say that in this case the inextensibility of the string overweighs the impulsiveness of the force; there are many small oscillations while the force is acting, with the consequence that the particles settle down to the same velocity.

At the other extremity we can have $\tau\lambda^{\frac{1}{2}} \to 0$, in which case $k = 1$ and

$$v = \frac{P}{M}\left(1 - \frac{m'}{m}\right), \qquad v' = \frac{2P}{M}, \qquad E = \frac{P^2}{2m}. \tag{5}$$

In this case the string slackens after the jerk. We may here say that the impulsiveness of the force overweighs the inextensibility of the string; the force ceases to act while the string is still stretching and before any force is effectively applied to the second particle. This solution is similar to that obtained when one particle is projected with given velocity away from the other; the two in fact agree exactly if the velocity of projection is P/m, *i.e.*, the velocity calculated from the momentum equation when the impulse is applied only to m, the string being ignored.

Between the two extremities we have the range of possibilities given by values of k between 0 and 1, and the solution to be adopted in any particular problem should depend upon the conditions of the problem, *i.e.*, the kind of string and the nature of the impulse. If we are to make a choice without further information, then the second extreme case given by $k = 1$ seems to be the most satisfactory. It agrees fairly closely with experience and it gives at the same time a simple rule, at least when only one string is involved. It also gives the maximum total energy with which the given impulse can provide the system, the minimum being given by $k = 0$; this follows from (3). The rule may be stated thus :—

If impulsive forces are applied to a system which includes one inextensible string, first calculate the initial motion and the energy of the system under the same forces but ignoring the string. Now return to the given system with the string and calculate the motion with the given amount of energy reckoned above. This energy equation is the additional information required to account for an unknown impulsive tension in the string.

It is evident that the problem becomes complicated algebraically when more than one string is involved. In principle, the results will depend upon the ratios of the (large) moduli and upon the values of products such as $\tau\lambda^{\frac{1}{2}}$. If such products are taken to be zero, as suggested above, then the total energy can be calculated as before, but it would still be necessary in general to consider the differential equations of motion in order to solve the problem completely. There is a need for more rules to deal adequately with such problems.

A proof of the "Theorem of the Means."

By C. E. WALSH.

Numerous proofs have been given of this familiar theorem,[1] which states that if $a_1, a_2, \ldots, a_n$ are positive, and not all equal, then

$$a_1^n + a_2^n + \ldots\ldots + a_n^n > na_1 a_2 \ldots\ldots a_n.$$

The following is an elementary proof by induction, which I

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[1] See *e.g.* Hardy, Littlewood & Pölya *Inequalities*, where many references will be found.