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A NOTE ON THE THEORY OF WELL ORDERS

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Abstract. We give a simple proof that the first-order theory of well orders is axiomatized by transfinite induction, and that it is decidable.

The first-order theory of the class $\mathcal{WO}$ of well-ordered sets $\langle L, < \rangle$ was developed by Tarski and Mostowski, and an in-depth analysis was finally published by Doner, Mostowski, and Tarski [1]: among other results, they provided an explicit axiomatization for the theory, and proved it decidable. Their main technical tool is a syntactic elimination of quantifiers, which however takes some work to establish, as various somewhat nontrivial properties of Cantor normal forms are definable in the theory after all. Alternatively, by way of hammering nails with a nuke, the decidability of $\mathrm{Th}(\mathcal{WO})$ follows from an interpretation of the MSO theory of countable linear orders in the MSO theory of two successors (S2S), which is decidable by a well-known difficult result of Rabin [5]. Our goal is to point out that basic properties of $\mathrm{Th}(\mathcal{WO})$ can be proved easily using ideas from Läuchli and Leonard's [4] proof of the decidability of the theory of linear orders. A similar technique was also used by Shelah [7].

Let $\mathcal{L}_<$ denote the set of sentences in the language $\{<\}$. The theory of (strict) linear orders is denoted LO; the $\mathcal{L}_<$-theory TI extends LO with the transfinite induction schema

$$\forall x \, \big(\forall y \, (y < x \to \varphi(y)) \to \varphi(x)\big) \to \forall x \, \varphi(x)$$

for all formulas φ (that may in principle include other free variables as parameters, though the parameter-free version is sufficient for our purposes). We will generally denote a linearly ordered set $\langle L, < \rangle$ as just L. Given linearly ordered sets I and L_i for $i \in I$, let $\sum_{i \in I} L_i$ denote the ordered sum with domain $\{\langle i, x \rangle : i \in I, x \in L_i\}$ and lexicographic order

$$\langle i, x \rangle < \langle j, y \rangle \iff i < j \text{ or } (i = j \text{ and } x < y).$$

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Put $L \cdot I = \sum_{i \in I} L$. We write $L \equiv_k L'$ if $L \vDash \varphi \iff L' \vDash \varphi$ for all $\varphi \in \mathcal{L}_<$ of quantifier rank $\mathrm{rk}(\varphi) \leq k$. It follows from the basic theory of Ehrenfeucht–Fraïssé games (see [3, 4]) that for each $k < \omega$, $\equiv_k$ has only finitely many equivalence classes as there are only finitely many formulas of rank $\leq k$ up to logical equivalence, and that $\equiv_k$ preserves sums and products.

Lemma 1. *If $L_i \equiv_k L'_i$ for each $i \in I$, then $\sum_{i \in I} L_i \equiv_k \sum_{i \in I} L'_i$. If $I \equiv_k I'$, then $L \cdot I \equiv_k L \cdot I'$.*

Proof Sketch. Duplicator can win the game $\mathrm{EF}_k\left(\sum_{i \in I} L_i, \sum_{i \in I} L'_i\right)$ by playing auxiliary games $\mathrm{EF}_k(L_i, L'_i)$ for each $i \in I$ on the side. When Spoiler plays an element in the ith summand, Duplicator simulates it in $\mathrm{EF}_k(L_i, L'_i)$, finds a response using a fixed winning strategy (given by the assumption $L_i \equiv_k L'_i$), and plays the corresponding element in the main game. The second assertion of the lemma is similar. ⊣

We come to the main theorem. It was originally proved in [1, Theorem 31, Corollaries 30, 32] by tedious quantifier elimination. (The equivalence of (i) and (ii) also follows from Ehrenfeucht [2], who proved by induction on k that ordinals congruent modulo ω^k are $\equiv_k$-equivalent.) Instead, we give a short argument inspired by the proof of [4, Theorem 2] that needs almost no ordinal arithmetic and no explicit EF game strategies.

Theorem 2. *The following are equivalent for all $\varphi \in \mathcal{L}_<$:*

(i) $\mathcal{WO} \vDash \varphi$.
(ii) $\alpha \vDash \varphi$ *for all* $\alpha < \omega^\omega$.
(iii) $\mathsf{TI} \vdash \varphi$.

Proof. Clearly, (iii) → (i) → (ii). For (ii) → (iii), if $\mathsf{TI} \nvdash \varphi$, let L be a countable model of $\mathsf{TI} + \neg\varphi$, and $k = \mathrm{rk}(\varphi)$; it suffices to show that there exists $\alpha < \omega^\omega$ such that $L \equiv_k \alpha$. Put

$$S = \left\{c \in L : \forall a, b \in L \left(a < b \leq c \to \exists \alpha < \omega^\omega \, [a, b) \equiv_k \alpha\right)\right\}.$$

While the definition speaks of half-open intervals $[a, b)$, the conclusion also holds for $[a, b]$: if $[a, b) \equiv_k \alpha$, then $[a, b] \equiv_k \alpha + 1$ by Lemma 1. Clearly, S is an initial segment of L, and $0 \in S$, where $0 = \min(L)$ (which exists by $L \vDash \mathsf{TI}$).

Claim 2.1. S is definable in L. ⊣

Proof. Since there are only finitely many formulas of rank $\leq k$ up to equivalence, we can form $\theta_k = \bigwedge\{\theta \in \mathcal{L}_< : \mathrm{rk}(\theta) \leq k, \forall \alpha < \omega^\omega \, \alpha \vDash \theta\}$. Then for any linear order L', $L' \equiv_k \alpha$ for some $\alpha < \omega^\omega$ iff $L' \vDash \theta_k$. In particular, $c \in S$ iff $L \vDash \forall x, y \left(x < y \leq c \to \theta_k^{[x,y)}\right)$, where $\theta_k^{[x,y)}$ denotes θ_k with quantifiers relativized to $[x, y)$. ⊣

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First, assume that S has a largest element, say m. If $S = L$, then $L = [0,m] \equiv_k \alpha$ for some $\alpha < \omega^\omega$, and we are done. Otherwise, we will derive a contradiction by showing that the successor of m (which exists by TI), denoted c, belongs to S. Indeed, if $a < b \le c$, then either $b \le m$, in which case $[a,b) \equiv_k \alpha$ for some $\alpha < \omega^\omega$ as $m \in S$, or $b = c$, in which case $[a,b) = [a,m] \equiv_k \alpha$ for some $\alpha < \omega^\omega$ as well.

Consequently, we may assume that S has no largest element. Put $S_{\ge a} = \{b \in S : b \ge a\}$.

CLAIM 2.2. For every $a \in S$, there is $\alpha < \omega^\omega$ such that $S_{\ge a} \equiv_k \alpha$.

PROOF. We use the idea of [4, Lemma 8]. Let $a < a_0 < a_1 < a_2 < \cdots$ be a cofinal sequence in S. For each $n < m < \omega$, let $t(\{n,m\}) = \min\{\alpha < \omega^\omega : [a_n, a_m) \equiv_k \alpha\}$. Since $\equiv_k$ has only finitely many equivalence classes, t is a coloring of pairs of natural numbers by finitely many colors; by Ramsey's theorem, there is $\beta < \omega^\omega$ and an infinite $H \subseteq \omega$ such that $t(\{n,m\}) = \beta$ for all $n, m \in H$, $n \ne m$. Let $\{b_n : n < \omega\}$ be the increasing enumeration of $\{a_n : n \in H\}$, and $\alpha < \omega^\omega$ be such that $[a, b_0) \equiv_k \alpha$. Then, $S_{\ge a} = [a, b_0) + \sum_{n<\omega}[b_n, b_{n+1}) \equiv_k \alpha + \beta \cdot \omega < \omega^\omega$ by Lemma 1. ⊣

Now, if $S = L$, then $L = S_{\ge 0} \equiv_k \alpha$ for some $\alpha < \omega^\omega$ by Claim 2.2. Otherwise, there exists $c = \min(L \setminus S)$ by Claim 2.1 and $L \vDash$ TI. We again derive a contradiction by showing $c \in S$: if $a < b \le c$, then either $b < c$ and $[a,b) \equiv_k \alpha$ for some $\alpha < \omega^\omega$ as $b \in S$, or $b = c$ and $[a,b) = S_{\ge a} \equiv_k \alpha$ for some $\alpha < \omega^\omega$ by Claim 2.2. ⊣

We have so far not actually used any results of Läuchli and Leonard [4], only their methods. But we will do so now: in order to prove the decidability of $\mathrm{Th}(\mathcal{WO})$, we need the following lemma.

LEMMA 3. *The relation* $\{\langle \alpha, \varphi \rangle \in \omega^\omega \times \mathcal{L}_< : \alpha \vDash \varphi\}$ *is recursively enumerable* (*whence decidable*).

Here, we assume $\alpha < \omega^\omega$ is represented by a finite string describing its Cantor normal form (CNF) in a natural way. Lemma 3 is a special case of [4, Theorem 1]: more generally, Läuchli and Leonard prove uniform decidability of linear order types described by "terms" using a constant 1, a binary function $+$, unary functions $x \cdot \omega$ and $x \cdot \omega^*$, and a certain variable-arity shuffle operation. It is easy to see that the CNF of an ordinal $\alpha < \omega^\omega$ can be transformed to such a term using 1, $+$, and $x \cdot \omega$, hence Lemma 3 follows.

We include a proof of Lemma 3 to make the paper more self-contained. It turns out that for well orders, it is more convenient to consider terms using 1, $+$, and $\omega \cdot x$ rather than $x \cdot \omega$: then we can directly axiomatize the theory by a finite sentence without expanding the language with extra predicates as in [4]. This argument can be found, e.g., in Rosenstein [6].

Proof of Lemma 3. Given $\alpha < \omega^\omega$, we can compute an $\mathcal{L}_<$-sentence T_α such that $\mathrm{Th}(\alpha, <) = \mathsf{LO} + T_\alpha$ by induction on α as follows. We can take $\forall x, y\, x = y$ for T_1. It is well known that T_ω can be defined by axioms postulating that a least element exists, every element has a successor, and every non-minimal element has a predecessor. If T_α and T_β are already constructed, let $T_{\alpha+\beta}$ be $\exists x\, \big(T_\alpha^{<x} \wedge T_\beta^{\geq x}\big)$, where $T_\alpha^{<x}$ denotes the sentence T_α with all quantifiers relativized to $(-\infty, x) = \{y : y < x\}$, and similarly for $T_\beta^{\geq x}$: in particular, any $L \vDash \mathsf{LO} + T_{\alpha+\beta}$ contains an element $a \in L$ such that $(-\infty, a) \equiv \alpha$ and $[a, \infty) \equiv \beta$, which implies $L \equiv \alpha + \beta$ by Lemma 1.

Finally, we consider $\omega \cdot \alpha$ for a limit α. Let $\lambda(x)$ denote the formula $\forall y < x\, \exists z < x\, y < x$, meaning "$x$ is not a successor." We define $T_{\omega\cdot\alpha}$ as the conjunction of T_α^λ and axioms postulating that for each x, $\max\{y \leq x : \lambda(y)\}$ and $\min\{y > x : \lambda(y)\}$ exist. Clearly, $\omega \cdot \alpha \vDash T_{\omega\cdot\alpha}$. Conversely, if $L \vDash \mathsf{LO} + T_{\omega\cdot\alpha}$, then $L^\lambda := \{x \in L : L \vDash \lambda(x)\} \vDash T_\alpha$, thus $L^\lambda \equiv \alpha$, and $L = \sum_{x \in L^\lambda} [x, x^+)$, where $x^+ = \min\{y > x : y \in L^\lambda\}$. It is easy to see that $[x, x^+) \vDash T_\omega$ for each x, thus $L \equiv \sum_{x\in L^\lambda} \omega = \omega \cdot L^\lambda \equiv \omega \cdot \alpha$ by Lemma 1. ⊣

The following consequence is Theorem 33 of [1].

Theorem 4. *The theory* $\mathrm{Th}(\mathcal{WO}) = \mathsf{TI}$ *is decidable.*

Proof. $\{\varphi \in \mathcal{L}_< : \mathsf{TI} \vdash \varphi\}$ is recursively enumerable as TI is recursively axiomatized; by Theorem 2 and Lemma 3, $\{\varphi \in \mathcal{L}_< : \mathsf{TI} \nvdash \varphi\} = \{\varphi \in \mathcal{L}_< : \exists \alpha < \omega^\omega\, \alpha \vDash \neg\varphi\}$ is also recursively enumerable. ⊣

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