Glasgow Math. J. **61** (2019) 1–11. © Glasgow Mathematical Journal Trust 2018.
doi:10.1017/S0017089517000386.

EXAMPLE OF A GROUP WHOSE QUANTUM ISOMETRY GROUP DOES NOT DEPEND ON THE GENERATING SET

ARNAB MANDAL

School of Mathematical Sciences,
National Institute of Science Education and Research Bhubaneswar,
HBNI, Jatni 752050, India
e-mail: arnabmaths@gmail.com

(Received 22 June 2017; revised 6 November 2017; accepted 29 November 2017; first published online 28 January 2018)

Abstract. In this paper, we have shown that the quantum isometry group of $C_r^*(\mathbb{Z})$, denoted by $\mathbb{Q}(\mathbb{Z}, S)$ as in Goswami and Mandal, *Rev. Math. Phys.* **29**(3) (2017), 1750008, with respect to a symmetric generating set S does not depend on the generating set S. Moreover, we have proved that the result is no longer true if the group $\mathbb{Z}$ is replaced by $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ copies}}$ $\forall\, n > 1$.

1. Introduction. Quantum groups are very important mathematical entities which appear in several areas of mathematics and physics, often as some kind of generalised symmetry objects. Beginning from the pioneering work by Drinfeld, Jimbo, Manin, Woronowicz and others nearly three decades ago ([**14**, **18**, **24**] and references therein) there is now a vast literature on quantum groups both from algebraic and analytic (operator algebraic) viewpoints. Generalising group actions on spaces, notions of (co)actions of quantum groups on possibly non-commutative spaces have been formulated and studied by many mathematicians. In this context, S. Wang [**23**] introduced the definition of quantum automorphism groups of certain mathematical structures (typically finite sets, matrix algebras, etc.) and such quantum groups have been studied in depth since then. Later on, a number of mathematicians including Banica, Bichon and others ([**1**, **9**] and references therein) developed a theory of quantum automorphism groups of finite metric spaces and finite graphs. With a more geometric setup in 2009, Goswami [**15**] defined and proved existence of an analogue of the group of isometries of a Riemannian manifold, in the framework of the so-called compact quantum groups à la Woronowicz. In fact, he considered the more general setting of non-commutative manifold, given by spectral triples defined by Connes [**12**] and under some mild regularity conditions, he proved the existence of a universal compact quantum group (termed the quantum isometry group) acting on the C^*-algebra underlying the non-commutative manifold such that the action also commutes with a natural analogue of Laplacian of the spectral triple. Furthermore, Goswami and Bhowmick formulated in [**7**] the notion of a quantum group analogue of the group of orientation preserving isometries and its existence as the universal object in a suitable category was proved. After that, several authors studied quantum isometry groups of different spectral triples in recent years.

In literature, we have an interesting as well as important spectral triple on $C_r^*(\Gamma)$ [**11**], coming from the word length of a finitely generated discrete group Γ corresponding to a symmetric generating set, say S. There have been several papers already on computations and study of the quantum isometry groups of such spectral triples, e.g., [**3**, **5**, **8**, **13**, **16**, **17**, **21**, **22**] and references therein. We denote it by $\mathbb{Q}(\Gamma, S)$ as in [**16**]. It is already known that in general $\mathbb{Q}(\Gamma, S_1)$ and $\mathbb{Q}(\Gamma, S_2)$ are not isomorphic for different choices of S_1 and S_2. Indeed, they are drastically different for certain choices of generating sets. We give an example here. If we choose n such that $g.c.d(n, 4) = 1$, then the group $\mathbb{Z}_n \times \mathbb{Z}_4$ is isomorphic to $\mathbb{Z}_{4n}$. Consider the generating sets $S_1 = \{(\bar{1}, \bar{0}), (\bar{0}, \bar{1}), (-\bar{1}, \bar{0}), (\bar{0}, -\bar{1})\}$ and $S_2 = \{(\bar{1}, \bar{1}), (-\bar{1}, -\bar{1})\}$, respectively for $\mathbb{Z}_n \times \mathbb{Z}_4$. The underlying C^*-algebra of $\mathbb{Q}(\mathbb{Z}_n \times \mathbb{Z}_4, S_1)$ is non-commutative by Theorem 4.10 of [**16**]. On the other hand, $\mathbb{Q}(\mathbb{Z}_n \times \mathbb{Z}_4, S_2)$ is the doubling of $C^*(\mathbb{Z}_n \times \mathbb{Z}_4)$ corresponding to the automorphism given by $a \mapsto a^{-1}\ \forall\, a \in \mathbb{Z}_n \times \mathbb{Z}_4$ from [**8**]. Hence, its underlying C^*-algebra is commutative. They are non-isomorphic even in the vector space level. In this context, it is quite natural to find out the groups whose quantum isometry group does not depend on the generating set. Our main goal of this paper is to provide one such example.

The paper is organized as follows. In Section 2, we recall some definitions and necessary facts regarding to compact quantum groups, quantum isometry groups and the doubling procedure of a compact quantum group. Section 3 contains the main results of this paper. In Theorem 3.1, we have proved that the quantum isometry group of $C_r^*(\mathbb{Z})$ remains unchanged if we change the generating sets. Theorem 3.11 tells us that this is no longer true if $\mathbb{Z}$ is replaced by $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n\ copies}$ for $n > 1$.

2. Preliminaries. First of all, we fix some notational convention. The algebraic tensor product and spatial (minimal) C^*-tensor product are denoted by $\otimes$ and $\hat{\otimes}$, respectively throughout the paper. We will use the leg-numbering notation. Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{K}(\mathcal{H})$ the C^*-algebra of compact operators on it, and $\mathcal{Q}$ a unital C^*-algebra. The multiplier algebra $\mathcal{M}(\mathcal{K}(\mathcal{H})\hat{\otimes}\mathcal{Q})$ has two natural embeddings into $\mathcal{M}(\mathcal{K}(\mathcal{H})\hat{\otimes}\mathcal{Q}\hat{\otimes}\mathcal{Q})$, one obtained by extending the map $x \mapsto x \otimes 1$ and the second one is obtained by composing this map with the flip on the last two factors. We will write ω^{12} and ω^{13} for the images of an element $\omega \in \mathcal{M}(\mathcal{K}(\mathcal{H})\hat{\otimes}\mathcal{Q})$ under these two maps, respectively. We will denote by $\mathcal{H}\bar{\otimes}\mathcal{Q}$ the Hilbert C^*-module obtained by completing $\mathcal{H} \otimes \mathcal{Q}$ with respect to the norm induced by the $\mathcal{Q}$ valued inner product $\langle\langle \xi \otimes q, \xi' \otimes q' \rangle\rangle := \langle \xi, \xi' \rangle q^* q'$, where $\xi, \xi' \in \mathcal{H}$ and $q, q' \in \mathcal{Q}$.

2.1. Compact quantum groups. In this subsection, we recall some standard definitions related to compact quantum groups. We recommend [**19**, **24**] for more details.

DEFINITION 2.1. A compact quantum group (CQG in short) is a pair $(\mathcal{Q}, \Delta)$, where $\mathcal{Q}$ is a unital C^*-algebra and $\Delta : \mathcal{Q} \to \mathcal{Q}\hat{\otimes}\mathcal{Q}$ is a unital $*$-homomorphism (called the comultiplication or coproduct), such that

(1) $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ as homomorphism $\mathcal{Q} \to \mathcal{Q}\hat{\otimes}\mathcal{Q}\hat{\otimes}\mathcal{Q}$ (coassociativity).
(2) The spaces $\Delta(\mathcal{Q})(1 \otimes \mathcal{Q}) = Span\{\Delta(b)(1 \otimes a) | a, b \in \mathcal{Q}\}$ and $\Delta(\mathcal{Q})(\mathcal{Q} \otimes 1) = Span\{\Delta(b)(a \otimes 1) | a, b \in \mathcal{Q}\}$ are dense in $\mathcal{Q}\hat{\otimes}\mathcal{Q}$.

Sometimes, we may denote the CQG $(\mathcal{Q}, \Delta)$ simply as $\mathcal{Q}$, if Δ is clear from the context.

DEFINITION 2.2. A CQG morphism from $(\mathcal{Q}_1, \Delta_1)$ to another $(\mathcal{Q}_2, \Delta_2)$ is a unital C^*-homomorphism $\pi : \mathcal{Q}_1 \mapsto \mathcal{Q}_2$ such that $(\pi \otimes \pi)\Delta_1 = \Delta_2\pi$.

DEFINITION 2.3. We say that a CQG $(\mathcal{Q}, \Delta)$ acts on a unital C^*-algebra $\mathcal{B}$ if there is a unital C^*-homomorphism (called action) $\alpha : \mathcal{B} \to \mathcal{B}\hat{\otimes}\mathcal{Q}$ satisfying the following:

(1) $(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$.
(2) Linear span of $\alpha(\mathcal{B})(1 \otimes \mathcal{Q})$ is norm-dense in $\mathcal{B}\hat{\otimes}\mathcal{Q}$.

DEFINITION 2.4. Let $(\mathcal{Q}, \Delta)$ be a CQG. A unitary representation of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$ is a $\mathbb{C}$-linear map U from $\mathcal{H}$ to the Hilbert module $\mathcal{H}\bar{\otimes}\mathcal{Q}$ such that

(1) $\langle\langle U(\xi), U(\eta)\rangle\rangle = \langle \xi, \eta\rangle 1_{\mathcal{Q}}$, where $\xi, \eta \in \mathcal{H}$.
(2) $(U \otimes id)U = (id \otimes \Delta)U$.
(3) *Span* $\{U(\mathcal{H})(1 \otimes \mathcal{Q})\}$ is dense in $\mathcal{H}\bar{\otimes}\mathcal{Q}$.

Given such a unitary representation, we have a unitary element $\tilde{U}$ belonging to $\mathcal{M}(\mathcal{K}(\mathcal{H})\hat{\otimes}\mathcal{Q})$ given by $\tilde{U}(\xi \otimes b) = U(\xi)b, (\xi \in \mathcal{H},\quad b \in \mathcal{Q})$ satisfying $(id \otimes \Delta)(\tilde{U}) = \tilde{U}^{12}\tilde{U}^{13}$. The linear span of matrix elements of finite dimensional unitary representations forms a dense Hopf $*$-algebra $\mathcal{Q}_0$ of $(\mathcal{Q}, \Delta)$, on which an antipode κ and co-unit ϵ are defined.

2.2. Quantum isometry groups. In [**15**] Goswami introduced the notion of quantum isometry group of a spectral triple satisfying certain regularity conditions. We refer to [**3**, **7**, **15**] for the original formulation of quantum isometry groups and its various avatars including the quantum isometry group for orthogonal filtrations.

DEFINITION 2.5. Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple of compact type (à la Connes). Consider the category $Q(\mathcal{D}) \equiv Q(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ whose objects are $(\mathcal{Q}, \Delta, U)$, where $(\mathcal{Q}, \Delta)$ is a CQG having a unitary representation U on the Hilbert space $\mathcal{H}$ satisfying the following:

(1) $\tilde{U}$ commutes with $(\mathcal{D} \otimes 1_{\mathcal{Q}})$.
(2) $(id \otimes \phi) \circ ad_{\tilde{U}}(a) \in (\mathcal{A}^\infty)''$ for all $a \in \mathcal{A}^\infty$ and ϕ is any state on $\mathcal{Q}$, where $ad_{\tilde{U}}(x) := \tilde{U}(x \otimes 1)\tilde{U}^*$ for $x \in \mathcal{B}(\mathcal{H})$.

A morphism between two such objects $(\mathcal{Q}, \Delta, U)$ and $(\mathcal{Q}', \Delta', U')$ is a CQG morphism $\psi : \mathcal{Q} \to \mathcal{Q}'$ such that $U' = (id \otimes \psi)U$. If a universal (initial) object exists in $Q(\mathcal{D})$, then we denote it by $\widetilde{QISO^+}(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ and the corresponding largest Woronowicz subalgebra for which $ad_{\tilde{U}_0}$ is faithful, where U_0 is the unitary representation of $\widetilde{QISO^+}(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$, is called the quantum group of orientation preserving isometries and denoted by $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

Let us state Theorem 2.23 of [**7**] which gives a sufficient condition for the existence of $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

THEOREM 2.6. *Let $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ be a spectral triple of compact type. Assume that $\mathcal{D}$ has one-dimensional kernel spanned by a vector $\xi \in \mathcal{H}$, which is cyclic and separating for $\mathcal{A}^\infty$ and each eigenvector of $\mathcal{D}$ belongs to $\mathcal{A}^\infty\xi$. Then, $QISO^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$ exists.*

https://doi.org/10.1017/S0017089517000386 Published online by Cambridge University Press

Here, we briefly discuss a specific case of interest for us. For more details, see Section 2.2 of [16]. Let Γ be a finitely generated discrete group with a symmetric generating set S not containing the identitity of Γ (symmetric means $g \in S$ if and only if $g^{-1} \in S$) and let l be the corresponding word length function. We define an operator D_Γ by $D_\Gamma(\delta_g) = l(g)\delta_g$, where δ_g denotes the vector in $l^2(\Gamma)$ which takes value 1 at the point g and 0 at all other points. Observe that $\{\delta_g\}_{g\in\Gamma}$ forms an orthonormal basis of $l^2(\Gamma)$. Let τ be the canonical trace on the reduced group C^*-algebra given by $\tau(\sum_{g\in\Gamma} c_g\lambda_g) = c_e$, where e is the identity element of the group Γ. Connes first considered this spectral triple $(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ in [11]. It is easy to check that $(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ is a spectral triple using Lemma 1.1 of [20]. Moreover, $QISO^+(\mathbb{C}\Gamma, l^2(\Gamma), D_\Gamma)$ exists by Theorem 2.6, taking $\xi = \delta_e$ as the cyclic separating vector for $\mathbb{C}\Gamma$. It is denoted by $\mathbb{Q}(\Gamma, S)$. Note that its action α on $C_r^*(\Gamma)$ is determined by (see Section 2 of [8])

$$\alpha(\lambda_\gamma) = \sum_{\gamma'\in S} \lambda_{\gamma'} \otimes q_{\gamma,\gamma'},$$

where the matrix $[q_{\gamma,\gamma'}]_{\gamma,\gamma'\in S} \in M_{card(S)}(\mathbb{Q}(\Gamma, S))$. From now on, we will call this matrix the“fundamental unitary" of $\mathbb{Q}(\Gamma, S)$.

2.3. Doubling of a CQG. We briefly recall the doubling procedure of a compact quantum group from [13], [21]. Let $(\mathcal{Q}, \Delta)$ be a CQG with a CQG-automorphism θ such that $\theta^2 = id$. The doubling of this CQG, say $(\mathcal{D}_\theta(\mathcal{Q}), \tilde{\Delta})$, is given by $\mathcal{D}_\theta(\mathcal{Q}) := \mathcal{Q} \oplus \mathcal{Q}$ (direct sum as a C^*-algebra), and the coproduct is defined by the following, where we have denoted the injections of $\mathcal{Q}$ onto the first and second coordinate in $\mathcal{D}_\theta(\mathcal{Q})$ by ξ and η, respectively, i.e., $\xi(a) = (a, 0)$, $\eta(a) = (0, a)$, $(a \in \mathcal{Q})$.

$$\begin{aligned}\tilde{\Delta} \circ \xi &= (\xi \otimes \xi + \eta \otimes [\eta \circ \theta]) \circ \Delta,\\ \tilde{\Delta} \circ \eta &= (\xi \otimes \eta + \eta \otimes [\xi \circ \theta]) \circ \Delta.\end{aligned}$$

3. Main results. Before going to the main theorem, we make one convention. Inverse of any element $x \in \mathbb{Z}$ is denoted by $-x$. We will also follow the same convention for $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n\ copies}$, where $n > 1$.

THEOREM 3.1. *For any symmetric generating set S, the quantum isometry group $\mathbb{Q}(\mathbb{Z}, S)$ is isomorphic to $\mathcal{D}_\theta(C^*(\mathbb{Z}))$ with respect to the automorphism θ given by $\theta(x) = -x \;\forall\; x \in \mathbb{Z}$.*

Proof. Let us assume that S is any generating set for $\mathbb{Z}$, i.e., $S = \{a_1, -a_1, \ldots, a_k, -a_k\}$. Without loss of generality, we can assume that $a_i > 0 \;\forall\; i$ and $a_1 < a_2 < \ldots < a_k$. We would like to mention here that the largest number a_k and the smallest number $-a_k$ of the generating set S will play a crucial role in the proof. For each $i = 1, \ldots, k-1$, there exists positive integers c_i, d_i such that $c_ia_i = d_ia_k \;\forall\; i = 1, \ldots, k-1$. Moreover, $c_i > d_i$ as $a_i < a_k \forall\; i = 1, \ldots, k-1$. Now

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the action α of $\mathbb{Q}(\mathbb{Z}, S)$ on $C_r^*(\mathbb{Z})$ is defined as

$$\begin{aligned}
\alpha(\lambda_{a_1}) &= \lambda_{a_1} \otimes A_{11} + \lambda_{-a_1} \otimes A_{12} + \lambda_{a_2} \otimes A_{13} + \lambda_{-a_2} \otimes A_{14} + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{1(2k-1)} + \lambda_{-a_k} \otimes A_{1(2k)}, \\
\alpha(\lambda_{-a_1}) &= \lambda_{a_1} \otimes A_{12}^* + \lambda_{-a_1} \otimes A_{11}^* + \lambda_{a_2} \otimes A_{14}^* + \lambda_{-a_2} \otimes A_{13}^* + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{1(2k)}^* + \lambda_{-a_k} \otimes A_{1(2k-1)}^*, \\
\alpha(\lambda_{a_2}) &= \lambda_{a_1} \otimes A_{21} + \lambda_{-a_1} \otimes A_{22} + \lambda_{a_2} \otimes A_{23} + \lambda_{-a_2} \otimes A_{24} + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{2(2k-1)} + \lambda_{-a_k} \otimes A_{2(2k)}, \\
\alpha(\lambda_{-a_2}) &= \lambda_{a_1} \otimes A_{22}^* + \lambda_{-a_1} \otimes A_{21}^* + \lambda_{a_2} \otimes A_{24}^* + \lambda_{-a_2} \otimes A_{23}^* + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{2(2k)}^* + \lambda_{-a_k} \otimes A_{2(2k-1)}^*, \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\alpha(\lambda_{a_k}) &= \lambda_{a_1} \otimes A_{k1} + \lambda_{-a_1} \otimes A_{k2} + \lambda_{a_2} \otimes A_{k3} + \lambda_{-a_2} \otimes A_{k4} + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{k(2k-1)} + \lambda_{-a_k} \otimes A_{k(2k)}, \\
\alpha(\lambda_{-a_k}) &= \lambda_{a_1} \otimes A_{k2}^* + \lambda_{-a_1} \otimes A_{k1}^* + \lambda_{a_2} \otimes A_{k4}^* + \lambda_{-a_2} \otimes A_{k3}^* + \cdots + \\
&\quad \lambda_{a_k} \otimes A_{k(2k)}^* + \lambda_{-a_k} \otimes A_{k(2k-1)}^*.
\end{aligned}$$

The fundamental unitary is of the following form:

$$U = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1(2k-1)} & A_{1(2k)} \\
A_{12}^* & A_{11}^* & A_{14}^* & A_{13}^* & \cdots & A_{1(2k)}^* & A_{1(2k-1)}^* \\
A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2(2k-1)} & A_{2(2k)} \\
A_{22}^* & A_{21}^* & A_{24}^* & A_{23}^* & \cdots & A_{2(2k)}^* & A_{2(2k-1)}^* \\
\vdots & & \vdots & & & & \\
A_{k1} & A_{k2} & A_{k3} & A_{k4} & \cdots & A_{k(2k-1)} & A_{k(2k)} \\
A_{k2}^* & A_{k1}^* & A_{k4}^* & A_{k3}^* & \cdots & A_{k(2k)}^* & A_{k(2k-1)}^*
\end{pmatrix}.$$

Note that the antipode κ is defined by $\kappa(A_{i(2j-1)}) = A_{j(2i-1)}^*, \kappa(A_{i(2j)}) = A_{j(2i)}^*$, $\kappa(A_{i(2j-1)}^*) = A_{j(2i-1)}, \kappa(A_{i(2j)}^*) = A_{j(2i)}\ \forall\ i, j = 1, 2, \ldots, k$ using the unitarity condition of U. Our aim is to show that it is of the following form:

$$\begin{pmatrix}
A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\
A_{12}^* & A_{11}^* & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & A_{23} & A_{24} & \cdots & 0 & 0 \\
0 & 0 & A_{24}^* & A_{23}^* & \cdots & 0 & 0 \\
\vdots & & \vdots & & & & \\
0 & 0 & 0 & 0 & \cdots & A_{k(2k-1)} & A_{k(2k)} \\
0 & 0 & 0 & 0 & \cdots & A_{k(2k)}^* & A_{k(2k-1)}^*
\end{pmatrix}, \tag{1}$$

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i.e., only the diagonal (2×2) block survives and others are zero. First, we will show that it is of the following form:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots & 0 & 0 \\ A_{12}^* & A_{11}^* & A_{14}^* & A_{13}^* & \cdots & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots & 0 & 0 \\ A_{22}^* & A_{21}^* & A_{24}^* & A_{23}^* & \cdots & 0 & 0 \\ \vdots & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & A_{k(2k-1)} & A_{k(2k)} \\ 0 & 0 & 0 & 0 & \cdots & A_{k(2k)}^* & A_{k(2k-1)}^* \end{pmatrix},$$

i.e., $A_{i(2k-1)} = 0$, $A_{i(2k)} = 0 \; \forall \; i = 1, \ldots, k-1$. Using the antipode $A_{k(2j-1)} = A_{k(2j)} = 0 \; \forall \; j = 1, \ldots, k-1$. We break the proof into a number of lemmas. □

LEMMA 3.2. $A^{c_i}_{i(2k-1)} = A^{c_i}_{i(2k)} = 0 \; \forall \; i = 1, \ldots, k-1$.

Proof. Consider the term $\alpha(\lambda_{c_i a_i}) = \alpha(\lambda_{d_i a_k})$ for all $i = 1, \ldots, k-1$. Comparing the coefficients of $\lambda_{c_i a_k}$ and $\lambda_{c_i(-a_k)}$ on both sides of the relation $\alpha(\lambda_{c_i a_i}) = \alpha(\lambda_{d_i a_k})$, we obtain $A^{c_i}_{i(2k-1)} = A^{c_i}_{i(2k)} = 0$ as the right hand side of the equation does not contain any terms with coefficients $\lambda_{c_i a_k}$ and $\lambda_{c_i(-a_k)}$ as well. □

Our goal is to show that $A_{i(2k-1)}$ and $A_{i(2k)}$ are normal $\forall \; i = 1, \ldots, k-1$. Then, by Lemma 3.2, one can conclude $A_{i(2k-1)} = A_{i(2k)} = 0$.

LEMMA 3.3. *If $a_p + a_q = a_l + a_m$ for some $p, q, l, m \in \mathbb{N}$, then* $A_{k(2p-1)}A_{k(2q-1)} = A_{k(2l-1)}A_{k(2m-1)}$.

Proof. Using the relation $\alpha(\lambda_{a_p+a_q}) = \alpha(\lambda_{a_l+a_m})$, comparing the coefficient of λ_{2a_k} on both sides, we have $A_{p(2k-1)}A_{q(2k-1)} = A_{l(2k-1)}A_{m(2k-1)}$. Applying the antipode we obtain $A^*_{k(2q-1)}A^*_{k(2p-1)} = A^*_{k(2m-1)}A^*_{k(2l-1)}$. This implies $A_{k(2p-1)}A_{k(2q-1)} = A_{k(2l-1)}A_{k(2m-1)}$. □

We state three auxiliary lemmas (Lemmas 3.4–3.6) whose proof will follow by exactly the same arguments used in Lemma 3.3. We omit the proofs.

LEMMA 3.4. *If $a_p + a_q = -a_l + a_m$, then* $A_{k(2p-1)}A_{k(2q-1)} = A_{k(2l)}A_{k(2m-1)}$.

LEMMA 3.5. *If $a_p - a_q = -a_l + a_m$, then* $A_{k(2p-1)}A_{k(2q)} = A_{k(2l)}A_{k(2m-1)}$.

LEMMA 3.6. *If $a_p - a_q = -a_l - a_m$, then* $A_{k(2p-1)}A_{k(2q)} = A_{k(2l)}A_{k(2m)}$.

LEMMA 3.7. $A_{k(2i-1)}A_{k(2i)} = A_{k(2i)}A_{k(2i-1)} = 0 \; \forall \; i = 1, \ldots, k$.

Proof. Comparing the coefficients of λ_{2a_k} and λ_{-2a_k} on both sides from the relation $\alpha(\lambda_{a_i}).\alpha(\lambda_{-a_i}) = \lambda_e \otimes 1$ one can get $A_{i(2k-1)}A^*_{i(2k)} = A_{i(2k)}A^*_{i(2k-1)} = 0$. Applying the antipode, we have $A^*_{k(2i)}A^*_{k(2i-1)} = 0$ which implies $A_{k(2i-1)}A_{k(2i)} = 0$. Similarly, we can get $A_{k(2i)}A_{k(2i-1)} = 0$ from the relation $\alpha(\lambda_{-a_i}).\alpha(\lambda_{a_i}) = \lambda_e \otimes 1$. □

LEMMA 3.8. $A_{ki}A_{kj} = 0 \; \forall \; i, j$ *with $i \neq j$.*

Proof. We will show that $A_{k(2j)}A_{k1} = A_{k(2j-1)}A_{k1} = 0$ for all $j = 2, \ldots, k$. By Lemma 3.7, we have $A_{k2}A_{k1} = 0$. Then, $A_{ki}A_{k1} = 0$ for all i with $i \neq 1$ will be proved. Other relations will follow by repeating the same line of arguments. For some fixed $j \neq 1$, we

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are defining the sets

$$T_1^{(1,j)} = \{(l,m)|\ l,m \in \{1,\ldots,k\}\ such\ that\ a_1 + a_j = a_l - a_m\},$$

$$T_2^{(1,j)} = \{(g,t)|\ g,t \in \{2,\ldots,j-1,j+1,\ldots,k\}\ with\ g < t\ such\ that\ a_1 + a_j = a_g + a_t\},$$

$$T_3^{(1,j)} = \{s|\ s \in \{1,\ldots,k\}\ such\ that\ a_1 + a_j = 2a_s\}.$$

The sets $T_1^{(1,j)}$, $T_2^{(1,j)}$, $T_3^{(1,j)}$ may be empty depending on the choice of the generating set S. Observe that the sets $T_1^{(1,j)}$, $T_2^{(1,j)}$ are not necessarily singleton but finite. $T_3^{(1,j)}$ is always singleton if it is non-empty. Now from the condition $\alpha(\lambda_{a_k}).\alpha(\lambda_{-a_k}) = \lambda_e \otimes 1$ comparing the coefficient of $\lambda_{a_1+a_j}$ on both sides, one can deduce

$$\begin{aligned} &A_{k1}A^*_{k(2j)} + A_{k(2j-1)}A^*_{k2} + \sum_{(g,t)\in T_2^{(1,j)}} [A_{k(2g-1)}A^*_{k(2t)} + A_{k(2t-1)}A^*_{k(2g)}] \\ &+ \sum_{(l,m)\in T_1^{(1,j)}} [A_{k(2l-1)}A^*_{k(2m-1)} + A_{k(2m)}A^*_{k(2l)}] + A_{k(2s-1)}A^*_{k(2s)} = 0. \end{aligned} \tag{2}$$

Multiplying $A_{k(2j)}$ and A^*_{k1} on the left hand side and right hand side, respectively of the equation (2), we get

$$\begin{aligned} &A_{k(2j)}A_{k1}A^*_{k(2j)}A^*_{k1} + A_{k(2j)}A_{k(2j-1)}A^*_{k2}A^*_{k1} + \sum_{(g,t)\in T_2^{(1,j)}} A_{k(2j)}A_{k(2g-1)}A^*_{k(2t)}A^*_{k1} \\ &+ \sum_{(g,t)\in T_2^{(1,j)}} A_{k(2j)}A_{k(2t-1)}A^*_{k(2g)}A^*_{k1} + \sum_{(l,m)\in T_1^{(1,j)}} A_{k(2j)}A_{k(2l-1)}A^*_{k(2m-1)}A^*_{k1} \\ &+ \sum_{(l,m)\in T_1^{(1,j)}} A_{k(2j)}A_{k(2m)}A^*_{k(2l)}A^*_{k1} + A_{k(2j)}A_{k(2s-1)}A^*_{k(2s)}A^*_{k1} = 0. \end{aligned} \tag{3}$$

Now the relation $a_1 + a_j = a_j + a_1$ gives us $a_1 - a_j = -a_j + a_1$. By Lemma 3.5, we have $A_{k1}A_{k(2j)} = A_{k(2j)}A_{k1}$. Similarly, $A_{k(2j)}A_{k(2g-1)} = A_{k1}A_{k(2t)}$ as $a_1 - a_t = -a_j + a_g$ from the assumed condition $a_1 + a_j = a_g + a_t$. Moreover, by Lemmas 3.4–3.6, we get

$$A_{k(2j)}A_{k(2t-1)} = A_{k1}A_{k(2g)},\ A_{k(2j)}A_{k(2l-1)} = A_{k1}A_{k(2m-1)},$$

$$A_{k(2j)}A_{k(2s-1)} = A_{k1}A_{k(2s)},\ A_{k(2j)}A_{k(2m)} = A_{k1}A_{k(2l)},$$

as $-a_j + a_t = a_1 - a_g$, $-a_j + a_l = a_1 + a_m$, $-a_j + a_s = a_1 - a_s$ and $-a_j - a_m = a_1 - a_l$, respectively. Using these relations and Lemma 3.7, the equation (3) reduces to

$$\begin{aligned} &A_{k(2j)}A_{k1}(A_{k(2j)}A_{k1})^* + \sum_g A_{k(2j)}A_{k(2g-1)}(A_{k(2j)}A_{k(2g-1)})^* + \\ &\sum_t A_{k(2j)}A_{k(2t-1)}(A_{k(2j)}A_{k(2t-1)})^* + \sum_l A_{k(2j)}A_{k(2l-1)}(A_{k(2j)}A_{k(2l-1)})^* \\ &+ \sum_m A_{k(2j)}A_{k(2m)}(A_{k(2j)}A_{k(2m)})^* + A_{k(2j)}A_{k(2s-1)}(A_{k(2j)}A_{k(2s-1)})^* = 0. \end{aligned} \tag{4}$$

This shows that $A_{k(2j)}A_{k1} = 0$ as the left hand side of the equation (4) is the sum of some positive elements of a C^*-algebra. Using the relation $\alpha(\lambda_{a_k}).\alpha(\lambda_{-a_k}) = \lambda_e \otimes 1$ comparing the coefficient of $\lambda_{a_1-a_j}$ on both sides, we get $A_{k(2j-1)}A_{k1} = 0$ as well. $\square$

LEMMA 3.9. *$A_{i(2k)}$ and $A_{i(2k-1)}$ are normal* $\forall\ i = 1, \ldots, k$.

Proof. By the unitarity condition of U, we have

$$\sum_i A_{k(2i-1)}A^*_{k(2i-1)} + \sum_i A_{k(2i)}A^*_{k(2i)} = 1,$$

$$\sum_i A^*_{k(2i-1)}A_{k(2i-1)} + \sum_i A^*_{k(2i)}A_{k(2i)} = 1.$$

Thus, $A^2_{k(2i-1)}A^*_{k(2i-1)} = A_{k(2i-1)}$ and $A^*_{k(2i-1)}A^2_{k(2i-1)} = A_{k(2i-1)}$ by using Lemma 3.8. Hence, $A_{k(2i-1)}A^*_{k(2i-1)} = A^*_{k(2i-1)}A^2_{k(2i-1)}A^*_{k(2i-1)} = A^*_{k(2i-1)}A_{k(2i-1)}$. Applying the antipode, we get that $A_{i(2k-1)}$ is normal. Similarly, it can be shown that $A_{i(2k)}$ is normal. $\square$

By Lemmas 3.2 and 3.9, we have $A_{i(2k)} = A_{i(2k-1)} = 0\ \forall\ i = 1, \ldots, k-1$. Repeating the same arguments using from Lemma 3.2–3.9, we can conclude that the fundamental unitary finally reduces to the form as in (1), i.e.,

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 & \cdots & 0 & 0 \\ A^*_{12} & A^*_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} & \cdots & 0 & 0 \\ 0 & 0 & A^*_{24} & A^*_{23} & \cdots & 0 & 0 \\ \vdots & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & A_{k(2k-1)} & A_{k(2k)} \\ 0 & 0 & 0 & 0 & \cdots & A^*_{k(2k)} & A^*_{k(2k-1)} \end{pmatrix}.$$

Note that $A_{i(2i-1)}A_{i(2i)} = A_{i(2i)}A_{i(2i-1)} = 0\ \forall\ i = 1, \ldots, k$ and all the entries of the fundamental unitary are normal. Moreover, for each i, $A_{i(2i-1)}A^*_{i(2i-1)}$ and $A_{i(2i)}A^*_{i(2i)}$ are projections. We also have $A^2_{i(2i-1)}A^*_{i(2i-1)} = A_{i(2i-1)}$ and $A^2_{i(2i)}A^*_{i(2i)} = A_{i(2i)}$. For every a_i and a_j, there exists positive integers $p(i, j)$ and $q(i, j)$ depending on i, j such that $p(i, j)a_i = q(i, j)a_j$. Using this condition, one can easily get $A^{p(i,j)}_{i(2i)} = A^{q(i,j)}_{j(2j)}$ and $A^{p(i,j)}_{i(2i-1)} = A^{q(i,j)}_{j(2j-1)}\ \forall\ i, j = 1, \ldots, k$. This gives us $A_{i(2i-1)}A^{q(i,j)}_{j(2j)}(A^{q(i,j)}_{j(2j)})^* = 0$ as $A_{i(2i-1)}A_{i(2i)} = 0$. Thus, $A_{i(2i-1)}A_{j(2j)}A^*_{j(2j)} = 0$ by using that $A_{j(2j)}A^*_{j(2j)}$ is a projection and $A_{j(2j)}$ is normal. Finally, we get that $A_{i(2i-1)}A_{j(2j)} = 0$ as $A^2_{j(2j)}A^*_{j(2j)} = A_{j(2j)}$ and $A_{j(2j)}$ is normal. Similarly, one can deduce that $A_{i(2i)}A_{j(2j-1)} = 0\ \forall\ i, j = 1, \ldots, k$.

We can define the map from $\mathbb{Q}(\mathbb{Z}, S)$ to $\mathcal{D}_\theta(C^*(\mathbb{Z}))$ by

$$A_{i(2i-1)} \mapsto (\lambda_{a_i}, 0),$$

$$A_{i(2i)} \mapsto (0, \lambda_{-a_i}) = (0, \lambda_{\theta(a_i)}),$$

$\forall\ i = 1, \ldots, k$. Clearly, this gives an isomorphism between the two CQG's.

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REMARK 3.10. From now on, the quantum isometry group $\mathbb{Q}(\mathbb{Z}, S)$ can be written simply as $\mathbb{Q}(\mathbb{Z})$.

Now we are going to show that the quantum isometry group of $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n\ copies}$ for $n > 1$ depends on the generating set. We present the case $n = 2$ for simplicity of the exposition. The proof for any n can be adapted similarly from the proof of Theorem 3.11. Let $S' = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ and $S'' = \{(1, 0), (0, 1), (-1, 0), (0, -1), (2, 0), (-2, 0)\}$ be the two different generating sets for $\mathbb{Z} \times \mathbb{Z}$.

THEOREM 3.11. *$\mathbb{Q}(\mathbb{Z} \times \mathbb{Z}, S')$ and $\mathbb{Q}(\mathbb{Z} \times \mathbb{Z}, S'')$ are not isomorphic to each other.*

Proof. First of all, note that by Proposition 2.29 and Theorem 4.1 of **[16]** we get that $\mathbb{Q}(\mathbb{Z} \times \mathbb{Z}, S') \cong C((\mathbb{T} \times \mathbb{T}) \rtimes (\mathbb{Z}_2^2 \rtimes S_2))$. We will show that $\mathbb{Q}(\mathbb{Z} \times \mathbb{Z}, S'')$ is different from $C((\mathbb{T} \times \mathbb{T}) \rtimes (\mathbb{Z}_2^2 \rtimes S_2))$. Let us assume that $a = (1, 0)$, $b = (0, 1)$ and $c = (2, 0)$. The action α of $\mathbb{Q}(\mathbb{Z} \times \mathbb{Z}, S'')$ on $C_r^*(\mathbb{Z} \times \mathbb{Z})$ is given by

$$\alpha(\lambda_a) = \lambda_a \otimes A + \lambda_{-a} \otimes B + \lambda_b \otimes C + \lambda_{-b} \otimes D + \lambda_c \otimes E + \lambda_{-c} \otimes F,$$

$$\alpha(\lambda_{-a}) = \lambda_a \otimes B^* + \lambda_{-a} \otimes A^* + \lambda_b \otimes D^* + \lambda_{-b} \otimes C^* + \lambda_c \otimes F^* + \lambda_{-c} \otimes E^*,$$

$$\alpha(\lambda_b) = \lambda_a \otimes G + \lambda_{-a} \otimes H + \lambda_b \otimes I + \lambda_{-b} \otimes J + \lambda_c \otimes K + \lambda_{-c} \otimes L,$$

$$\alpha(\lambda_{-b}) = \lambda_a \otimes H^* + \lambda_{-a} \otimes G^* + \lambda_b \otimes J^* + \lambda_{-b} \otimes I^* + \lambda_c \otimes L^* + \lambda_{-c} \otimes K^*,$$

$$\alpha(\lambda_c) = \lambda_a \otimes M + \lambda_{-a} \otimes N + \lambda_b \otimes O + \lambda_{-b} \otimes P + \lambda_c \otimes Q + \lambda_{-c} \otimes R,$$

$$\alpha(\lambda_{-c}) = \lambda_a \otimes N^* + \lambda_{-a} \otimes M^* + \lambda_b \otimes P^* + \lambda_{-b} \otimes O^* + \lambda_c \otimes R^* + \lambda_{-c} \otimes Q^*.$$

The fundamental unitary is of the following form:

$$\begin{pmatrix} A & B & C & D & E & F \\ B^* & A^* & D^* & C^* & F^* & E^* \\ G & H & I & J & K & L \\ H^* & G^* & J^* & I^* & L^* & K^* \\ M & N & O & P & Q & R \\ N^* & M^* & P^* & O^* & R^* & Q^* \end{pmatrix}. \tag{5}$$

Note that the product of any two different elements of each row of the fundamental unitary is zero by the arguments similar to those in the proof of Lemma 3.8. Hence, all the entries of matrix (5) are normal following the line of arguments of Lemma 3.9. Using the relation $\alpha(\lambda_{2a}) = \alpha(\lambda_c)$, comparing the coefficients of λ_b and λ_{-b} on both sides, we have $O = P = 0$. Applying the antipode and involution, we get $K = L = 0$. Similarly, comparing the coefficients of λ_{2b}, λ_{-2b}, λ_{2c} and λ_{-2c} from the same condition, we have $C^2 = D^2 = E^2 = F^2 = 0$ as well. This gives us $C = D = E = F = 0$ as they are normal. Using the antipode and the involution, we obtain $G = H = M = N = 0$.

https://doi.org/10.1017/S0017089517000386 Published online by Cambridge University Press

Thus, the fundamental unitary is of the following form:

$$\begin{pmatrix} A & B & 0 & 0 & 0 & 0 \\ B^* & A^* & 0 & 0 & 0 & 0 \\ 0 & 0 & I & J & 0 & 0 \\ 0 & 0 & J^* & I^* & 0 & 0 \\ 0 & 0 & 0 & 0 & Q & R \\ 0 & 0 & 0 & 0 & R^* & Q^* \end{pmatrix}.$$

Observe that $A^2 = Q$, $B^2 = R$, comparing the coefficients of λ_c and λ_{-c} from the condition $\alpha(\lambda_{2a}) = \alpha(\lambda_c)$. Moreover, $AR = BQ = 0$ as $AB = QR = 0$. The underlying C^*-algebra of $\mathbb{Q}(\mathbb{Z} \times \mathbb{Z}, S'')$ is generated by the elements A, B, I and J. We also get that $AI = IA, AJ = JA, BI = IB$ and $BJ = JB$ comparing the coefficients of $\lambda_{a+b}, \lambda_{-a+b}, \lambda_{a-b}$ and λ_{-a-b} on both sides from the relation $\alpha(\lambda_{a+b}) = \alpha(\lambda_{b+a})$. Clearly, the CQG $\mathbb{Q}(\mathbb{Z} \times \mathbb{Z}, S'')$ is identified with $\mathbb{Q}(\mathbb{Z})\hat{\otimes}\mathbb{Q}(\mathbb{Z})$. Note that the underlying C^*-algebra of $\mathbb{Q}(\mathbb{Z})\hat{\otimes}\mathbb{Q}(\mathbb{Z})$ is isomorphic to $[C^*(\mathbb{Z}) \oplus C^*(\mathbb{Z})]\hat{\otimes}[C^*(\mathbb{Z}) \oplus C^*(\mathbb{Z})]$. The isomorphism is defined as follows:

$$\begin{aligned} A &\mapsto (\lambda_1, 0) \otimes 1, \\ B &\mapsto (0, \lambda_{-1}) \otimes 1, \\ I &\mapsto 1 \otimes (\lambda_1, 0), \\ J &\mapsto 1 \otimes (0, \lambda_{-1}), \end{aligned}$$

where $\{1, -1\}$ is the standard minimal generating set for $\mathbb{Z}$. Thus, $\mathbb{Q}(\mathbb{Z} \times \mathbb{Z}, S'') \cong C((\mathbb{T} \times \mathbb{T}) \rtimes \mathbb{Z}_2^2)$ as $\mathbb{Q}(\mathbb{Z})$ is isomorphic to $C(\mathbb{T} \rtimes \mathbb{Z}_2)$. It is clearly not isomorphic with $C((\mathbb{T} \times \mathbb{T}) \rtimes (\mathbb{Z}_2^2 \rtimes S_2))$, hence we are done. $\square$

ACKNOWLEDGEMENTS. I would like to thank the anonymous referee for pointing out some mistakes and valuable comments in the previous version of the paper.

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