REPRESENTATION THEOREMS FOR THE WEIERSTRASS TRANSFORM

Z. DITZIAN

(Received 10 November 1970)

Communicated by J. B. Miller

1. Introduction

In this paper we shall be interested in the Weierstrass transform defined by

$$f(x) = \int_{-\infty}^{\infty} k(x-y,1)d\alpha(y) \tag{1.1}$$

converging (conditionally) for x in some interval, where

$$k(x,t) = (4\pi t)^{-\frac{1}{2}} e^{-x^2/4t}. \tag{1.2}$$

A representation theorem is a set of necessary and sufficient conditions on $f(x)$ so that $f(x)$ be represented by (1.1) with $\alpha(y)$ belonging to a certain class of functions. Representation theorems were discussed in [2], [3, Ch. VIII], [4], [5], [7] and [8]. In these papers conditions on $f(x)$ were given in order that $\alpha(y)$ would belong to one of the following classes:

(a) $\alpha(y)$ is increasing or decreasing, (see [8] and [3, p. 204]).

(b) $\alpha(y) \in B.V[-\infty,\infty]$, (see [7] and [3, p. 198]).

(c) $\alpha(y)$ satisfies $\int_{-\infty}^{\infty} k(x-y,1)|d\alpha(y)| < \infty$ for all $x \in (a,b)$ for some a, b $a < b$, (see [4, p. 37] and [2]).

(d) $\alpha(y) = \int^y \phi(u)du$ and $\phi \in L_p(-\infty,\infty)$ $1 < p \leqq \infty$ (see [3, p. 195]).

(e) $\alpha(y) = \int^y \phi(u)du$ and $e^{-(x-u)^2/4}\phi(u) \in L_p$ $1 < p \leqq \infty$ for $x \in (a,b)$ for some a and b, (see [4, p. 43] and [2]).

(f) Same as (e) for $p = 1$ (see [4, p. 48]).

(g) $|\phi(u)| \leqq Ne^{ay^2}$, $a < \frac{1}{4}$ and $-\infty < y < \infty$, (see [3, p. 207]).

Obviously there are functions $f(x)$ representable by (1.1) with determining functions $\alpha(y)$ that are not in any one of the classes (a) $\to$ (g). Our main result will be to find necessary and sufficient conditions on $f(x)$ so that there exist a function $\alpha(y)$ locally of bounded variation for which (1.1) converges conditionally in some interval (a,b) $a < b$. This obviously is the widest class of $f(x)$ for which the Weierstrass-Stieltjes transform (1.1) exists. We may also restrict ourselves to the transform $f(x)$

https://doi.org/10.1017/S144678870000968X Published online by Cambridge University Press

$$f(s) = \int_{-\infty}^{\infty} k(s-y,1)\phi(y)\,dy, \tag{1.3}$$

of locally Lebesgue integrable function $\phi(y)$. The widest class of $f(x)$ represented by (1.3) corresponds to the class of $\phi(y)$ for which (1.3) converges conditionally in a strip $a_1 < Re\ s < a_2$. Representation of this class is of special interest and will be the result of section 6. New representation theorems will be given for $f(s)$ satisfying (1.1) and (1.3) where the integral converges absolutely in sections 5 and 7 respectively. A representation theorem for $f(s)$ satisfying (1.3) where

$$|\phi(u)| \leqq \mathrm{M}e^{u^2/4}\min(e^{-au/2}, e^{-bu/2}) \tag{1.4}$$

will be given in section 3. This result generalizes a corresponding result of Hirschman and Widder [3, p. 207], it is also used in proof and for motivation in the rest of the paper.

2. A preliminary theorem for temperature functions

To prove our representation theorem for Weierstrass transforms of functions satisfying (1.4) we first have to obtain a result about functions satisfying the Heat equation which is interesting by itself. To state this result we have to define class H [3, p. 181].

Definition 2.1. A function $u(x,t)$ is said to belong to *class H in domain D* if $u_{xx}(x,t) = u_t(x,t)$ and $u(x,t) \in C^2$ in D.

Theorem 2.1. *The conditions*

$$u(x,t) \in H \quad \text{for } 0 < t < 1, \ -\infty < x < \infty \tag{1}$$

and

$$|u(x,t)| \leqq \frac{M}{\sqrt{1-t}} e^{\,{}^2/4(1-t)} \min_{i=1,2} \exp\left[-\frac{a_i x}{2(1-t)} + \frac{a_i^2 t}{4(1-t)}\right] \tag{2}$$

for $0 < t < 1 - \infty < x < \infty$ *and some* $a_1 < a_2$, *are necessary and sufficient that*

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)\phi(y)\,dy, \tag{2.1}$$

where the integral (2.1) *converges absolutely for* $0 < t < 1$, $-\infty < x < \infty$ *and* $\phi(y)$ *satisfies*

$$|\phi(y)| \leqq M e^{y^2/4} \min_{i=1,2} e^{-a_i y/2} \quad \text{for all } y. \tag{2.2}$$

To shorten some of the expressions we write

$$(2.3) \qquad R(a_i,x,t) = \exp\left[-\frac{a_i x}{2(1-t)} + \frac{a_i^2 t}{4(1-t)}\right].$$

PROOF. We first prove the necessity of conditions (1) and (2). Condition (1) is implied by (2.1). Combining (2.1) and (2.2) we write:

$$\begin{aligned}
|u(x,t)| &\leqq \int_{-\infty}^{\infty} k(x-y,t)|\phi(y)|\,dy \\
&\leqq M \int_{-\infty}^{\infty} k(x-y,t)e^{y^2/4}\left\{\min_{i=1,2} e^{-a_i y/2}\right\}dy \\
&\leqq M \min_{i=1,2} \int_{-\infty}^{\infty} k(x-y,t)\exp[\tfrac{1}{4}y^2 - \tfrac{1}{2}a_i y]\,dy \\
&= M \min_{i=1,2} \frac{1}{\sqrt{4\pi t}} \exp\left[\frac{x^2}{4(1-t)} - \frac{xa_i}{2(1-t)} + \frac{a_i^2 t}{4(1-t)}\right] \\
&\quad \cdot \int_{-\infty}^{\infty} \exp\left[-\frac{1-t}{4t}\left(y - \frac{x}{1-t} + \frac{a_i t}{1-t}\right)^2\right] dy \\
&\leqq \frac{M}{\sqrt{1-t}} \exp\left(\frac{x^2}{4(1-t)}\right) \min_{i=1,2} \exp\left[-\frac{xa_i}{2(1-t)} + \frac{a_i^2 t}{4(1-t)}\right]
\end{aligned}$$

which completes the proof of necessity of condition (2).

We shall prove now the sufficiency of conditions (1) and (2). Define $V(x,t)$ by

$$(2.4) \quad V(x,t) = \int_{-\infty}^{\infty} k(x-y,t)e^{y^2/4}\left\{\min_{i=1,2} e^{-a_i y/2}\right\}dy \equiv \int_{-\infty}^{\infty} k(x-y,t)d\beta(y).$$

(Choosing $\beta(0) = 0$ a normalized $\beta(y)$ is unique). Recalling [1, p. 146 (21)] that

$$(2.5) \qquad \int_0^{\infty} e^{-u^2/4\alpha} e^{-su}\,du = \pi^{\frac{1}{2}} \alpha^{\frac{1}{2}} e^{\alpha s^2} Erfc(\alpha^{\frac{1}{2}} s)$$

where $Erfc(x) = 2\pi^{-\frac{1}{2}} \int_x^{\infty} e^{-t^2} dt$ we calculate $V(x,t)$ and obtain

$$\begin{aligned}
V(x,t) = \frac{1}{2\sqrt{1-t}} \exp\left(\frac{x^2}{4(1-t)}\right) &\left\{R(a_1,x,t) \cdot Erfc\left[\frac{x}{2(t-t^2)^{\frac{1}{2}}} - \frac{a_1 t^{\frac{1}{2}}}{2(1-t)^{\frac{1}{2}}}\right]\right. \\
&\left. + R(a_2,x,t)\, Erfc\left[-\frac{x}{2(t-t^2)^{\frac{1}{2}}} + \frac{a_2 t^{\frac{1}{2}}}{2(1-t)^{\frac{1}{2}}}\right]\right\}.
\end{aligned}$$

Obviously the necessity of (1) and (2) implies

$$(2.6) \qquad V(x,t) \leqq \frac{1}{\sqrt{1-t}} \exp\left(\frac{x^2}{4(1-t)}\right) \min_{i=1,2} R(a_i,x,t) \equiv H(x,t).$$

We shall need in our proof that for every fixed a_1 and a_2 and $\varepsilon > 0$ $0 < t < \delta(\varepsilon)$

https://doi.org/10.1017/S144678870000968X Published online by Cambridge University Press

$$V(x,t) \geqq (1-\varepsilon)H(x,t). \tag{2.7}$$

We can choose $\eta_1(\varepsilon)$ so that for $|y| \leqq \eta_1(\varepsilon)$ $e^{y^2/4} \min_{i=1,2} e^{-a_i y/2} \geqq 1-\varepsilon/3$ and then using (2.3) we obtain for $|x| \leqq \frac{1}{2}\eta_1(\varepsilon)$ and $0 < t < \delta_1(\varepsilon)$ $V(x,t) \geqq 1 - 2\varepsilon/3$. Since $H(x,t)$ is continuous at a neighbourhood of $(0,0)$ and $H(0,0) = 1$ we have for $|x| \leqq \eta_2$ and $0 < t < \delta_2$ $V(x,t) \geqq (1-\varepsilon)H(x,t)$. For $|x| \geqq \eta_2$ we can choose $\delta_3 < \delta_2$ such that for $t < \delta_3$

$$\min_{i=1,2} R(a_i,x,t) = \begin{cases} R(a_1,x,t) & x \leqq -\eta_2 \\ R(a_2,x,t) & x \geqq \eta_2. \end{cases}$$

To prove (2.7) it is enough now to show for $|t| < \delta \leqq \delta_3$ and $x \geqq \eta_2$ that

(a) $$Erfc\left[\frac{x}{2(t-t^2)^{\frac{1}{2}}} - \frac{a_1 t^{\frac{1}{2}}}{2(1-t)^{\frac{1}{2}}}\right] < \frac{\varepsilon}{2}\exp\left[\frac{(a_2^2-a_1^2)t}{4(1-t)} + \frac{(a_1-a_2)x}{2(1-t)}\right] < \frac{\varepsilon}{2}\exp\left[\frac{(a_2^2-a_1^2)t}{4(1-t)}\right],$$

(b) $$Erfc\left[-\frac{x}{2(t-t^2)^{\frac{1}{2}}} + \frac{a_2 t^{\frac{1}{2}}}{2(1-t)^{\frac{1}{2}}}\right] > 2-\varepsilon$$

and corresponding results for $x \leqq -\eta_2$. Using the estimate

$$\int_x^\infty e^{-y^2}dy \leqq \int_x^\infty ye^{-y^2}dy = \tfrac{1}{2}e^{-x^2}$$

for $x \geqq 1$ and straightforward computation we can prove (a) and (b) and therefore (2.7).

We recall now that (see Th. 12.2 of [3, p. 202]) necessary and sufficient conditions for $u(x,t)$ to be written as

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)d\alpha(y) \quad \text{in } 0 < t < \delta,\ -\infty < x < \infty$$

with $\alpha(y)$ nondecreasing is

$$u(x,t) \geqq 0 \text{ and } u(x,t) \in H \quad \text{for } 0 < t < \delta,\ -\infty < x < \infty.$$

Using (2.7) we have

$$-M(1-\varepsilon)^{-1}V(x,t) \leqq u(x,t) \leqq M(1-\varepsilon)^{-1}V(x,t) \ \ 0 < t < \delta(\varepsilon),$$

and this implies the existence of $\gamma_i(t)$ $i = 1,2$, both nondecreasing and unique after normalization, such that

$$M(1-\varepsilon)^{-1}V(x,t) + (-1)^i u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)d\gamma_i(y) \quad 0 < t < \delta \ \ i = 1,2. \tag{2.8}$$

Recalling (2.4) there exists $\alpha(y)$ locally of bounded variation such that

$$u(x,t) = \int_{-\infty}^{\infty} k(x-y,t)d\alpha(y) \tag{2.9}$$

for $0 < t < \delta(\varepsilon)$ and $\gamma_i(y) = M(1-\varepsilon)^{-1}\beta(y) + (-1)^i\alpha(y)$. Following now arguments in [7] and [3, p. 207] we get $\alpha(y) = \int^y \phi(x)dx$ and

$$|\phi(y)| \leqq M(1-\varepsilon)^{-1}e^{y^2/4} \min_{i=1,2} e^{-a_iy/2}. \tag{2.10}$$

The function $\alpha(y)$ is independent of ε, in spite of the dependence of $\gamma_i(y)$ on ε, since $\alpha(y)$ satisfying (2.9) in $0 < t < \delta$ is unique. $\phi(y)$ now satisfies (2.10) for all ε and therefore (2.2) but then (2.1) converges absolutely in $0 < t < 1$.

REMARK 2.1.a. In condition (2) of Theorem 2.1 we replace $0 < t < 1$ by $0 < t < \delta$ and call it (2)*. Conditions (1) of Theorem 2.1 and (2)* can replace (1) and (2) as necessary and sufficient for (2.1) and (2.2). The necessity is obvious while sufficiency follows the proof of Theorem 2.1.

3. The asymptotic representation theorem

In this section a representation theorem for the Weierstrass transform of ϕ satisfying (1.3) will be obtained. This result will be used in the motivation and proof of the following theorems of this paper. For our theorem we define first, class $A[a,b]$.

DEFINITION 3.1. A function $f(z)$ analytic in $a < Re\, z < b$ *belongs to class* $A[a,b]$ if $f(x+iy) = 0\,(e^{y^2/4})$ uniformly for x in every closed subinterval of (a,b).

Define also (see [3]) $K(s,t)$ by

$$K(s,t) = \left(\frac{\pi}{t}\right)^{\frac{1}{2}} e^{s^2/4t} = 2\pi k(is,t). \tag{3.1}$$

THEOREM 3.1. *The conditions* (1) $f(z) \in A[a_1,a_2]$ *and*

$$\left|\frac{1}{2\pi}\int_{d-i\infty}^{d+i\infty} K(s-x,t)f(s)ds\right| \leqq Mt^{-\frac{1}{2}}e^{x^2/4t} \min_{i=1,2} R(a_i,x,1-t) \tag{2}$$

(where $R(a_i,x,t)$ *was defined by* (2.3)) *for some* d, $a_1 < d < a_2$ *and* $0 < r < 1$ *are necessary and sufficient that*

$$f(x) = \int_{-\infty}^{\infty} k(x-y,1)\phi(y)dy \tag{3.2}$$

converges absolutely for $a_1 < x < a_2$ *and*

$$|\phi(y)| \leqq Me^{y^2/4} \min_{i=1,2} e^{-a_iy/2}. \tag{3.3}$$

PROOF. To prove necessity of (1) and (2) we observe that (3.3) implies $\|\exp[-(x-y)^2/4]\phi(y)\|_1 < \infty$ for $a_1 < x < a_2$ and therefore using Lemma 1 of [4, p. 32] (3.3) implies condition (1). Using Theorem 7.3 of [3, pp. 189–191] we obtain for $a_1 < d < a_2$

$$(3.4) \qquad \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} K(z-x,t)f(z)\,dz = \int_{-\infty}^{\infty} k(x-u,1-t)\phi(u)\,du.$$

The necessity of condition (2) follows now the corresponding part of Theorem 2.1 replacing t by $1-t$.

To prove (1) and (2) are sufficient we define

$$(3.5) \qquad u(x,1-t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} K(s-x,t)f(s)\,ds.$$

Using Cauchy's theorem and the asymptotic behaviour of both $K(s,t)$ and $f(s)$ it follows that (3.5) is independent of d, provided d satisfies $a_1 < d < a_2$. Recalling that $(\partial/\partial x)^2 K(s-x,t) = -(\partial/\partial t)K(s-x,t)$ and differentiating under the integral sign in (3.5), which is easily justified, we obtain

$$(3.6) \qquad \left(\frac{\partial}{\partial x}\right)^2 u(x,1-t) = -\frac{\partial}{\partial t}u(x,1-t) \text{ for } 0 < t < 1 \text{ and } -\infty < x < \infty.$$

The sufficiency part of Theorem 2.1 implies now

$$(3.7) \qquad u(x,1-t) = \int_{-\infty}^{\infty} k(x-y,1-t)\phi(y)\,dy \text{ for } 0 < t < 1 \quad -\infty < x < \infty$$

where $\phi(y)$ satisfies (3.3). For such $\phi(y)$

$$(3.8) \qquad f_*(x) \equiv \int_{-\infty}^{\infty} k(x-y,1)\phi(y)\,dy$$

converges absolutely for $a_1 < x < a_2$. To complete the proof it will be sufficient to show $f_*(x) = f(x)$ on $a_1 < x < a_2$. Using the Lebesgue convergence theorem we obtain

$$(3.9) \qquad f_*(x) = \lim_{t\to 0+} u(x,1-t) = \lim_{t\to 0+}\int_{-\infty}^{\infty} k(x-y,1-t)\phi(y)\,dy.$$

Combining (3.5) and (3.9) we have

$$\begin{aligned} f_*(x) &= \lim_{t\to 0+}\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} K(s-x,t)f(s)\,ds \\ &= \lim_{t\to 0+}\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} K(s-x,t)f(s)\,ds \\ &= \lim_{t\to 0+}\frac{1}{2\pi}\int_{-\infty}^{\infty} K(iy,t)f(x+iy)\,dy = \lim_{t\to 0+}\int_{-\infty}^{\infty} k(y,1)f(x+iy\sqrt{t})\,dy. \end{aligned}$$

Since for $x \in [A_1, A_2]$, $a_1 < A_1 < A_2 < a_2$ and $0 < t \leqq 1-\delta$ $\int_{-\infty}^{\infty} k(y,1)$ $|f(x+iy\sqrt{t})|\,dy \leqq M \int_{-\infty}^{\infty} e^{-y^2/4} e^{y^2(1-\rho)/4} dy < \infty$ we can use again Lebesgue convergence theorem now to show that $f(x) = \lim_{t \to 0+} \int_{-\infty}^{\infty} k(y,1) f(x+iy\sqrt{t})\,dy$ which completes the proof.

We conclude this section with a few remarks. We shall define first, class $B(a,b)$.

DEFINITION 3.2. A function $f(z)$ analytic in $a < Re\, z < b$ *belongs to class* $B[a,b]$ *iff* $f(x+iy) = 0(ye^{y^2/4})\,|y| \to \infty$ uniformly for every closed subinterval of (a,b).

REMARK 3.1.a. In a related result of Hirschman and Widder [3, p. 207] where the Weierstrass transform of $\phi(y)$ satisfying $|\phi(y)| \leqq Me^{ay^2}$ $0 < a < \frac{1}{4}$ is represented the condition $f(z) \in B[a,b]$ is required. Using Nessel's result [4, p. 31] in the theorem above [3, p. 207] we can assume there $f(z) \in A[a,b]$ instead of $f(z) \in B[a,b]$.

REMARK 3.1.b. If we follow carefully the sufficiency proof of Theorem 3.1 we can see that $f(z) \in B[a,b]$ can replace $f(z) \in A[a,b]$ there. (The necessity parts is easier then).

REMARK 3.1.c. In fact, in both theorems $f(z) = 0(|y|^n e^{y^2/4})$ $y \to \infty$ uniformly in any closed subinterval of (a,b) can replace $A[a,b]$ and $B[a,b]$. But we do use only $B[a,b]$ for theorems that will be proved later in this paper.

REMARK 3.1.d. In theorem 3.1 in condition (2) $0 < t < 1$ could be replaced by $1-\delta < t < 1$. This follows from Remark 2.1.a since Theorem 3.1 uses for its sufficiency part, the sufficiency part of Theorem 2.1 with $1-t$ replacing t.

4. Functions of locally bounded variation whose Weierstrass transform converges conditionally

In this section the most general class of functions $\alpha(y)$ for which Weierstrass-Stieltjes transform is defined will be treated.

THEOREM 4.1. *The conditions*
(1) $f(z) \in B[a_1, a_2]$ $a_1 < a_2$ (*Def.* 3.2).
(2) *For some* d $a_1 < d < a_2$

$$\left| \int_0^x \left\{ \int_{d-i\infty}^{d+i\infty} K(s-\xi,t) f(s)\,ds \right\} d\xi \right| \leqq M(\alpha_1,\alpha_2)\, t^{-\frac{1}{2}} e^{x^2/4t} \min_{i=1,2} R(\alpha_i, x, 1-t)$$

for all α_i *satisfying* $a_1 < \alpha_1 < \alpha_2 < a_2$, $0 < t < 1$ *and* $-\infty < x < \infty$; *and*

$$(3)\qquad \int_a^b \left| \int_{d-i\infty}^{d+i\infty} K(s-\xi,t)f(s)ds \right| d\xi \leqq L(a,b) \qquad 1-\delta \leqq t < 1$$

for any (a,b) $-\infty < a < b < \infty$; *are necessary and sufficient that*; $f(x)$ *will be represented as* $f(x) = \int_{-\infty}^{\infty} k(x-y,1)d\alpha(t)$ *where the integral converges conditionally for* $a_1 < x < a_2$.

REMARK 4.1.a. Actually we shall prove that (1) (2) for a fixed pair $(\alpha_1\alpha_2)$ and (3) will imply the conditional convergence of (1.1) for $x \in (\beta_1, \beta_2)$ $\alpha_1 < \beta_1 < \beta_2 < \alpha_2$ and that will imply (1), (2) and (3) with γ_i instead of α_i where $\beta_1 < \gamma_1 < \gamma_2 < \beta_2$.

PROOF. We shall show (1), (2) and (3) are necessary first. The necessity of (1) follows [3, p. 180]. The conditional convergence of (1.1) in (a_1, a_2) implies (see [3, p. 190]).

$$(4.1)\qquad |\alpha(y)| \leqq M(\alpha_1,\alpha_2)e^{y^2/4} \min_{i=1,2} e^{-\alpha_i y/2}$$

for any (α_1, α_2) satisfying $a_1 < \alpha_1 < \alpha_2 < a_2$, and also for $a_1 < d < a_2$

$$(4.2)\qquad \frac{1}{2\pi i}\int_{d+i\infty}^{d+i\infty} K(s-x,t)f(s)ds = \int_{-\infty}^{\infty} k(x-u,1-t)d\alpha(u).$$

Writing now

$$\begin{aligned}\int_0^x \left\{\int_{-\infty}^{\infty} k(\xi-u,1-t)d\alpha(u)\right\}d\xi &= \int_0^x \left\{\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial u}k(\xi-u,1-t)\right]\alpha(u)du\right\}d\xi \\ &= -\int_{-\infty}^{\infty}\int_0^x \frac{\partial}{\partial \xi}k(\xi-u,1-t)\alpha(u)d\xi du \\ &= -\int_{-\infty}^{\infty} k(x-u,1-t)\alpha(u)du \\ &\quad + \int_{-\infty}^{\infty} k(-u,1-t)\alpha(u)du \equiv I_1 + I_2.\end{aligned}$$

The interchange of order of integration above is justified by Fubini theorem using (4.1). Theorem 3.1 used on both I_1 and I_2 implies condition (2). Recalling that $\alpha(y)$ satisfies $\int_{a-1}^{b+1}|d\alpha(y)| \leqq A_*(a,b)$ ($\alpha(y)$ is locally of bounded variation), we have

$$\begin{aligned}\int_a^b \left|\int_{-\infty}^{\infty} K(\xi-u,1-t)d\alpha(u)\right| d\xi &\leqq 2\pi\left\{\int_a^b\left|\int_{-\infty}^{a-1} k(\xi-u,1-t)d\alpha(u)\right|d\xi\right. \\ &\quad + \int_a^b\left|\int_{a-1}^{b+1} k(\xi-u,1-t)d\alpha(u)\right|d\xi \\ &\quad \left. + \int_a^b\left|\int_{b+1}^{\infty} k(\xi-u,1-t)d\alpha(u)\right|d\xi\right\} \\ &\equiv 2\pi\{J_1+J_2+J_3\}.\end{aligned}$$

It is easy to see that $J_2 \leqq A_*(a,b)$. To estimate J_1 (treatment of J_3 is similar) we write

$$J_1 \leqq \int_a^b \left| \int_{-\infty}^{a-1} \frac{\partial}{\partial \xi} k(\xi - u, 1-t)\alpha(u)du \right| d\xi + \int_a^b |\alpha(a-1)| k(\xi - u, 1-t)d\xi$$
$$= J^*_{1,1} + J^*_{1,2}.$$

Obviously $J^*_{1,2}$ is bounded independently of t (we choose $\alpha(0) = 0$). For $\xi \in (a,b)$ and $u \in (-\infty, a-1)$ $\partial/\partial\xi\, k(\xi - u, 1-t) > 0$

$$\begin{aligned} J^*_{1,1} &\leqq \int_a^b \left\{ \int_{-\infty}^{a-1} \frac{\partial}{\partial \xi} k(\xi - u, 1-t) |\alpha(u)| du \right\} d\xi \\ &\leqq \int_{-\infty}^{a-1} k(a-u, 1-t)|\alpha(u)|du + \int_{-\infty}^{a-1} k(b-u,1-t)|\alpha(u)|du \\ &= 0(1)\ t \to 1- \end{aligned}$$

Therefore J_2 is bounded for $1-\delta < t < 1$ which completes the proof of condition (3).

To prove that conditions (1), (2) and (3) are sufficient our first step will be to show for a fixed t $0 < t < 1$ and for $x \in (a_1, a_2)$

$$\int_{-\infty}^{\infty} k(x-\xi, t) \left\{ \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} K(s-\xi, t) f(s) ds \right\} d\xi = f(x). \tag{4.3}$$

Condition (2) implies the convergence of the integral in (4.3) (conditional convergence). Condition (1) combined with Cauchy Theorem implies for $a_1 < d_1$, $d < a_2$

$$\int_{d-i\infty}^{d+i\infty} K(s-\xi, t) f(s) ds = \int_{d_1-i\infty}^{d_1+i\infty} K(s-\xi,t)f(s)ds. \tag{4.4}$$

Straightforward computation yields for $0 < \tau < t < 1$

$$\int_{-\infty}^{\infty} k(x-\xi, \tau) \left\{ \int_{d-i\infty}^{d+i\infty} |K(s-\xi, t)f(s)| ds \right\} d\xi < \infty. \tag{4.5}$$

Therefore using (4.4) and (4.5) for $a_1 < x < a_2$ and [3, p. 177, (1)] we have

$$\begin{aligned} &\int_{-\infty}^{\infty} k(x-\xi,\tau) \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} K(s-\xi,t)f(s)ds\} d\xi \\ &\qquad = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} f(s) \left\{ \int_{-\infty}^{\infty} k(x-\xi,\tau) K(s-\xi,t) d\xi \right\} ds \\ &\qquad = \int_{-\infty}^{\infty} f(x+iy)dy \int_{-\infty}^{\infty} k(\xi,\tau)k(i\xi - u, t) d\xi \\ &\qquad = \int_{-\infty}^{\infty} f(x+iy)k(y, t-\tau)dy. \end{aligned}$$

Obviously

$$\lim_{\tau\to t-}\int_{-\infty}^{\infty} f(x+iy)k(y,t-\tau)dy = f(x)$$

and therefore to prove (4.3) it is enough to show

$$(4.6)\quad \left|\int_{-\infty}^{\infty}(k(x-\xi,\tau)-k(x-\xi,t))\left\{\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}K(s-\xi,t)f(s)ds\right\}d\xi\right| = o(1) \quad \tau\to t-$$

which we can obtain applying condition (2) again.

Our next step will be to determine $\alpha(y)$. We define $\alpha_t(y)$ by

$$(4.7)\qquad \alpha^t(y) = \int_0^y\left\{\int_{d-i\infty}^{d+i\infty}K(s-\xi,t)f(s)\,ds\right\}d\xi.$$

Using condition (3) and Helly-Bray's Theorem [5, p. 31] there exist a sequence t_n and a function $\alpha(y)$, $y\in[a,b]$ such that $\lim_{n\to\infty}\int_a^b f(y)d\alpha_{t_n}(y) = \int_a^b g(y)d\alpha(y)$ for all $g(y)\in[a,b]$ where $\int_a^b|d\alpha(y)| \leqq L(a,b)$ and $\alpha_{t_n}(y)$ tend to $\alpha(y)$ at all points of continuity of $\alpha(y)$. We take the sequence $\alpha_{t_{n(1)}}(y)$ to correspond to $[-1,1]$ (for $[a.b]$) and a subsequence of $\alpha_{t_{(n1)}}y$, $\alpha_{t_{(n2)}}(y)$ to correspond to $[-2,\ 2]$ etc. Define now $\alpha_{t(m)}(y)$ by Cantor diagonal selection principle. It seems as if we have different functions $\alpha(n,y)$ for each interval $[-n,n]$ but normalizing the $\alpha(n,y)$ and recalling that $\alpha_{t_{n(k)}}(y)$ is a subsequence of $\alpha_{t_{n(k-1)}}(y)$ we observe that a unique function $\alpha(y)$ exists, is locally of bounded variation, satisfies

$$\lim_{m\to\infty}\int_{-n}^{n} g(y)d\alpha_{t(m)}(y) = \int_{-n}^{n} g(y)d\alpha(y)\ \forall n$$

and $\lim_{m\to\infty}\alpha_{t(m)}(y) = \alpha(y)$ at all points of continuity of $\alpha(y)$ (that is at all but a countable set of points). Therefore, for any $(\alpha_1\alpha_2)$ $a_1<\alpha_1<\alpha_2<a_2$

$$|\alpha(y)| \leqq \lim_{t(n)\to 1-}|\alpha_{t(n)}(y)| \leqq M(\alpha_1\alpha_2)\lim_{t(n)\to 1-}|t(n)|^{-\frac{1}{2}}e^{y^2/4t(n)}\cdot$$

$$\cdot\ \min_{i=1,2}R(\alpha_i,y,1-t(n)) \leqq M(\alpha_1,\alpha_2)e^{y^2/4}\min_{i=1,2}e^{-\alpha_i y/2}$$

This implies

$$(4.8)\qquad f_*(x) \equiv \int_{-\infty}^{\infty} k(x-y,1)d\alpha(y)$$

converges conditionally in $a_1<x<a_2$. The above means that for $A\leqq A_1$ and $B\geqq B_1$ for a fixed x, $x\in(a_1,a_2)$

$$(4.9)\qquad \left|\int_A^B k(x-y,1)d\alpha(y) - f_*(x)\right| < \varepsilon.$$

Using condition (2) one can show recalling (4.3) that for $A\leqq A_2<A_1$ and $B\geqq B_2>B_1$ and t, $t_0\leqq t<1$

$$\left| \int_A^B k(x-y,t)d\alpha_t(y) - f(x) \right| < \varepsilon \tag{4.10}$$

(where A_2 and B_2 are independent of t). Choose $A = -N$ $B = N$. For t satisfying $t_0 < t_* < t < 1$ we have

$$\left| \int_{-N}^{N} (k(x-y,1) - k(x-y,t))d\alpha_t(y) \right| < \varepsilon. \tag{4.11}$$

Choosing $t(m) > t(m_0) > t_*$ we have

$$\left| \int_{-N}^{N} k(x-y,1)d\alpha_{t(m)}(y) - \int_{-N}^{N} k(x-y,1)d\alpha(y) \right| < \varepsilon. \tag{4.12}$$

Combining (4.9), (4.10), (4.11) and (4.12) we have $|f(x) - f_*(x)| < 4\varepsilon$. But both $f(x)$ and $f_*(x)$ are independent of N and t and therefore $f(x) = f_*(x)$. The above being true for $a_1 < x < a_2$ we have

$$f(x) = \int_{-\infty}^{\infty} k(x-y,1)d\alpha(y) \qquad a_1 < x < a_2$$

which completes the proof of our theorem.

5. Absolute convergence

In this section necessary and sufficient conditions on $f(x)$ to be represented as absolutely convergent Weierstrass-Stieltjes transform will be achieved. We shall need the following definition:

DEFINITION 5.1. A function $f(z)$ analytic in the strip $a < Re\, z < b$ and satisfying $f(x+iy) = 0(e^{y^2/4})$ uniformly in any closed subinterval *belongs to class* $C[a,b]$.

THEOREM 5.1. *The conditions*

$$f(x) \in C[a_1, a_2],\ a_1 < a_2 \tag{1}$$

and

$$\int_0^x \left| \int_{d-i\infty}^{d+i\infty} K(s-\xi,t)f(s)ds \right| d\xi \leqq M(\alpha_1,\alpha_2)t^{-\frac{1}{2}}e^{x^2/4t} \min_{i=1,2} R(\alpha_i, x, 1-t) \tag{2}$$

where $a_1 < d < a_2$, $0 < t < 1$, $-\infty < x < \infty$ *and* α_1, α_2 *are any pair satisfying* $a_1 < \alpha_1 < \alpha_2 < a_2$; *are necessary and sufficient that*; $f(z) = \int_{-\infty}^{\infty} k(z-y,1)d\alpha(y)$ *and the integral will converge absolutely for* $a_1 < Re\, z < a_2$.

PROOF. The necessity proof of (2) is computational and that of (1) follows [4, p. 32].

https://doi.org/10.1017/S144678870000968X Published online by Cambridge University Press

To prove the sufficiency of (1) and (2) we observe that these conditions imply conditions (1), (2) and (3) of Theorem 4.1 and therefore the conditional convergence of $f(x) = \int_{-\infty}^{\infty} k(x-y,1)d\alpha(y)$. We can complete the proof if we show that $f_*(x) = \int_{-\infty}^{\infty} k(x-y,1)|d\alpha(y)| < \infty$ for $a_1 < x < a_2$. We recall that

$$\int_0^u |d\alpha(y)| \leqq \lim_{n\to\infty} \int_0^u |d\alpha_{t_n}(y)| \tag{5.1}$$

where $\alpha_t(y)$ was defined in (4.7). Condition (2) implies now for $\alpha_1 < x < \alpha_2$

$$\begin{aligned}\int_0^u |d\alpha(y)| &\leqq M \lim_{t\to 1-} e^{u^2/4t} \min_{i=1,2} R(\alpha_i, u, 1-t) \\ &\leqq M e^{u^2/4} \min_{i=1,2} e^{-\alpha_i u/2}.\end{aligned}$$

The last estimate establishes the absolute convergence of $\int_{-\infty}^{\infty} k(x-y,1)d\alpha(y)$.

REMARK 5.1.a. One can observe that the class of functions $\alpha(t)$ is the same as treated by Nessel [4, p. 37]; the conditions are different however. Condition (2) here replaces Nessel's condition

(5.2)

$$\left\| \exp[-(t-x)^2/4] \cdot \frac{1}{\sqrt{4\pi i}} \int_{x-iT}^{x+iT} \left(1 - \frac{|y|}{T}\right) \exp[(s-x)^2/4] f(s) ds \right\|_{L_1} = o(1)$$

for all T. Also here most of the proof follows as a corollary of the representation of the more general class.

6. Weierstrass transform of locally Lebesgue integrable functions

Representation theorem for Weierstrass transform

$$f(x) = \int_{-\infty}^{\infty} k(x-y,1)\phi(y)dy \tag{6.1}$$

where $\phi(y)$ is locally Lebesgue integrable and (6.1) converges conditionally in some strip would be obtained as follows:

THEOREM 6.1. *Conditions* (1) *and* (2) *of Theorem* 4.1 *and*

$$(3)^* \quad \int_a^b \left| \int_{d-i\infty}^{d+i\infty} [K(s-\xi,t_1) - K(s-\xi,t_2)] f(s) ds \right| d\xi = o(1) \;\; t_i \to 1-,\; 1-\delta < t_i < 1$$

for any a, b $-\infty < a < b < \infty$ (but the rate at which the double integral tends to zero depends on (a,b)), are necessary and sufficient for $f(x)$ to be represented by (6.1) *converging conditionally in $a_1 < x < a_2$ and $\phi(y) \in L_1(a,b)$ for all $-\infty < a < b < \infty$.*

Proof. Condition (1) and (2) are necessary since they were necessary already for Theorem 4.1. To prove (3) we write

$$\int_a^b \left| \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} [K(s-\xi,t_1) - K(s-\xi,t_2)]f(s)ds \right| d\xi$$
$$= \int_a^b \left| \int_{-\infty}^{\infty} [k(\xi - y, 1-t_1) - k(\xi - y, 1-t_2)]\phi(y)dy \right| d\xi$$
$$\leqq \sum_{i=1}^{2} \int_a^b \left| \int_{-\infty}^{\infty} k(\xi - y, 1-t_i)\phi(y)dy - \phi(\xi) \right| d\xi \equiv \sum_{i=1}^{2} I_i.$$

To estimate I_1 (I_2 is estimated similarly) we follow the proof of Theorem 4.1 and write

$$I_1 = \int_a^b \left| \int_{a-1}^{b+1} k(\xi - y, 1-t_1)\phi(y)dy - \phi(\xi) \right| d\xi + o(1) \qquad t \to 1-.$$

For a fixed ε there exist N such that $\int_{-N}^{N} k(x,1)dx \geqq 1-\varepsilon$ and therefore for $(1-t_1)N < 1$ we write

$$I_1 = \int_a^b \left| \int_{-N}^{N} k(v,1)[\phi(\xi + \sqrt{1-t_1}v) - \phi(\xi)]dv \right| d\xi + \varepsilon \int_a^b |\phi(\xi)| d\xi$$
$$+ \varepsilon \int_{a-1}^{b+1} |\phi(y)| dy + 0(1) \qquad t \to 1-.$$

We now use Fubini's Theorem to write

$$I_1 \leqq \int_{-N}^{N} k(v,1) \left\{ \int_a^b |\phi(\xi + \sqrt{1-t_1}v) - \phi(\xi)| d\xi \right\} dv + 2\varepsilon \int_{a-1}^{b+1} |\phi(\xi)| d\xi.$$

Recalling that $\tau(h) = \int_a^b |\phi(\xi + h) - \phi(\xi)| d\xi$ satisfies $\tau(h) = o(1)$ $h \to 0+$ we complete the proof of condition (3).

To prove sufficiency we recall that conditions (1) (2) and (3)* imply the corresponding conditions of Theorem 4.1 and therefore

$$f(x) = \int_{-\infty}^{\infty} k(x-y,1)d\alpha(y).$$

Condition (3)* implies $\alpha(y) = \int^y \phi(u)du$ and this completes the proof of our theorem.

7. Absolutely convergent Weierstrass transform

The following theorem corresponding to those of former section can be obtained.

Theorem 7.1. *The condition* (1) $f(z) \in A[a_1, a_2]$, (2) *condition* (2) *of Theorem* 5.1, *and* (3) *condition* (3)* *of Theorem* 6.1, *are necessary and sufficient for* $f(x)$

https://doi.org/10.1017/S144678870000968X Published online by Cambridge University Press

to be written as $f(x) = \int_{-\infty}^{\infty} k(x-y, 1)\phi(y)dy$ *the integral converging absolutely for* $a_1 < x < a_2$.

The proof is similar to proof of former theorems in this paper and would not be given here.

The same class of functions has also a different representation theorem [4, p. 48, Satz 3].

References

[1] Erdélyi (and others), *Tables of integral transforms, Vol. I* (McGraw Hill, 1954).

[2] H. P. Heinig, 'Representation of functions as Weierstrass-transforms', *Canadian Mathematical Bulletin* 10 (1967), 711–722.

[3] I. I. Hirschman and D. V. Widder, *The Convolution Transform* (Princeton Univ. Press, 1955).

[4] R. J. Nessel, 'Ueber die Darstellung holomorpher Funktionen durch Weierstrass and Weierstrass-Stieltjes Integrale', *Journal fur die reine und angewandte Mathematik* (1965), 31–50.

[5] H. Pollard, 'Representation as Gaussian integral', *Duke Math. Jour.* 10 (1943), 59–65.

[6] D. V. Widder, *The Laplace transform.* (Princeton Univ. Press, 1946).

[7] D. V. Widder, 'Necessary and sufficient conditions for representation of a function by a Weierstrass transform', *Trans. Amer. Math. Soc.* 71 (1951), 430–439.

[8] D. V. Widder, 'Weierstrass transforms positive functions', *Proc. of Nat. Acad. of Science* 37 (1951), 315–317.

Department of Mathematics
University of Alberta
Canada

https://doi.org/10.1017/S144678870000968X Published online by Cambridge University Press