On the continued fractions which represent the functions of Hermite and other functions defined by differential equations.

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§ 1. *Introduction.*

The functions of Hermite, which are the same as the functions associated with the parabolic cylinder in harmonic analysis, may be defined* by the differential equation which they satisfy, namely,

$$\frac{d^2y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where n denotes any constant.

The standard solution of the equation, which is denoted by $D_n(z)$, may be represented by the asymptotic expansion

$$D_n(z) = e^{-\frac{1}{4}z^2} z^n \left\{ 1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot z^4} - \ldots \right\}; \quad \ldots\ldots(2)$$

and a second independent solution of the differential equation is $D_{-n-1}(iz)$, so that the complete solution is of the form

$$y = aD_n(z) + bD_{-n-1}(iz)$$

where a and b are arbitrary constants.

In § 2 of the present paper it is shown that each of these solutions $D_n(z)$ and $D_{-n-1}(iz)$ can be calculated by means of a continued fraction. In § 3 it is shown that the two continued

* *Cf.* Hermite, *Comptes Rendus 58* (1864), pp. 93, 266.
Whittaker, *Proc. Lond. Math. Soc. 35* (1903), p. 417.
Myller-Lebedeff, *Math. Ann. 64* (1907), p. 388.
Watson, *Proc. Lond. Math. Soc.* (2) *8* (1910), p. 393.
Curzon, *Proc. Lond. Math. Soc.* (2) *12* (1912), p. 236.

fractions thus introduced bear to each other a relation similar to that between the two C.F.'s which represent the roots of a quadratic equation. In § 4 it is shown that there exist four other continued fractions connected with the Hermite functions, and that these are suited for calculation when z is small, the former C.F.'s being suited for calculation when z is large. In § 5 it is shown that other properties of the Hermite functions can be derived readily from the continued fractions. In § 6 it is explained why the C.F.'s represent certain particular solutions of the differential equation rather than others: and finally, in § 7 the results are extended to a much more general class of functions, which are defined by linear differential equations of the second order, and include many of the functions required in the applications of analysis.

§ 2. *Two continued fractions associated with the Hermite functions.*

In the differential equation (1) write

$$y = e^{-\frac{1}{4}z^2} u\,;$$

it becomes

$$u'' - zu' + nu = 0.$$

Differentiating this, we obtain

$$u''' - zu'' + (n-1)u' = 0.$$

Differentiating again, we obtain

$$u'''' - zu''' + (n-2)u'' = 0,$$

and so on; this series of recurrence-equations at once gives formally the continued fraction

$$\frac{u'}{u} = \frac{n}{z-}\ \frac{n-1}{z-}\ \frac{n-2}{z-}\ \frac{n-3}{z-}\ \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}.$$

If, on the other hand, we integrate the differential equation instead of differentiating it, we have (denoting $\int u dx$ by u_1, $\int u_1 dx$ by u_2, etc.)

$$u' - zu + (n+1)u_1 = 0,$$
$$u - zu_1 + (n+2)u_2 = 0,$$

and so on; this series of recurrence-equations gives formally the continued fraction

$$\frac{u'}{u}=z-\frac{n+1}{z-}\;\frac{n+2}{z-}\;\frac{n+3}{z-}\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

This method of solving differential equations is due to Euler. Its subsequent neglect is not easily explicable, for its simplicity and directness are very attractive; the methods of solution by power-series and by definite integrals appear forced and artificial in comparison.

The question now is, which of the recognised solutions of the differential equation in power-series correspond to the continued fractions (A) and (B). This question is settled by applying the usual method of converting continued fractions into series; we thus obtain from (A)

$$\frac{u'}{u}=\frac{n}{z}+\frac{n(n-1)}{z^3}+\frac{n(n-1)(2n-3)}{z^5}+\ldots\ldots\ldots\ldots\text{(A}'\text{)}$$

while from (B) we obtain

$$\frac{u'}{u}=z-\frac{n+1}{z}-\frac{(n+1)(n+2)}{z^3}-\ldots\ldots\ldots\ldots\ldots\ldots\text{(B}'\text{)}$$

Integrating, we obtain from (A′)

$$u=z^n\left\{1-\frac{n(n-1)}{2z^2}+\ldots\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A}''\text{)}$$

while from (B′) we obtain

$$u=e^{\frac{1}{2}z^2}z^{-n-1}\left\{1+\frac{(n+1)(n+2)}{2z^2}+\ldots\right\}\ldots\ldots\ldots\ldots\text{(B}''\text{)}$$

Comparing (A″) and (B″) with equation (2), we see that (A″) may be written

$$u=e^{\frac{1}{4}z^2}\mathrm{D}_n(z)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A}'''\text{)}$$

while (B″) may be written

$$u=i^{n+1}e^{\frac{1}{4}z^2}\mathrm{D}_{-n-1}(iz).\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B}'''\text{)}$$

We see therefore that *the two continued fractions* (A) *and* (B), *which are associated with the differential equation* (1) *correspond respectively to the two independent solutions* $\mathrm{D}_n(z)$ *and* $\mathrm{D}_{-n-1}(iz)$.

These continued fractions are well suited to numerical computation when z is large, and can be used for this purpose directly, without the reduction to series-form and the subsequent integration. They have, moreover, the advantage over series that the approximation is more rapid—a well-known property of continued fractions, depending ultimately on the fact that in a C.F. both the numerators and the denominators of the convergents are helping on the approximation, whereas a series is (so to speak) all numerator, and so is deprived of the assistance which might be afforded by a denominator; for this reason a C.F. convergent which includes only the n^{th} power of a variable in its numerator and denominator will generally give as good an approximation as a series which is continued as far as the terms involving the $2n^{\text{th}}$ power of the variable. In our case the continued fraction (A) truncated at its third term

$$\frac{n}{z-}\;\frac{n-1}{z-}\;\frac{n-2}{z}$$

is equivalent to a series carried as far as the term in $1/z^6$.

§ 3. *Analogy with the solution of a quadratic equation.*

The above result presents an analogy with the solution of an ordinary quadratic equation by continued fractions; if, for example, we consider the equation

$$x^2 - 3x + 2 = 0,$$

we can derive from it the two continued fractions

$$x = 3 - \tfrac{2}{3}\,-\,\tfrac{2}{3}\,-\,\tfrac{2}{3}\,-\,\cdots$$

and

$$x = \tfrac{2}{3}\,-\,\tfrac{2}{3}\,-\,\tfrac{2}{3}\,-\,\cdots$$

of which the first has the value 2 and the second has the value 1; and these are the two roots of the quadratic. The resemblance with the work above becomes evident if we reflect that these are precisely the two C.F.'s which we should obtain if we were to solve the differential equation

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0$$

by the method of § 2.

§ 4. *Four other continued fractions associated with the Hermite functions.*

The expansion (2) for the function $D_n(z)$ is an asymptotic expansion proceeding in *descending* powers of z, and therefore useful for calculation when z is large. If, on the other hand, z is not large, we require series which proceed in *ascending* powers of z. It is easy to show that one solution of the differential equation (1) is the series

$$E_n(z) = e^{-\frac{1}{4}z^2}\left\{1 - \frac{n}{2!}z^2 + \frac{n(n-2)}{4!}z^4 - \ldots\right\} \ldots\ldots\ldots\ldots\ldots(3)$$

or

$$E_n(z) = e^{\frac{1}{4}z^2}\left\{1 - \frac{n+1}{2!}z^2 + \frac{(n+1)(n+3)}{4!}z^4 - \ldots\right\} \ldots\ldots(4)$$

and another solution is

$$O_n(z) = e^{-\frac{1}{4}z^2}\left\{z - \frac{n-1}{3!}z^3 + \frac{n-1\,.\,n-3}{5!}z^5 - \ldots\right\} \ldots\ldots\ldots(5)$$

$$= e^{\frac{1}{4}z^2}\left\{z - \frac{n+2}{3!}z^3 + \frac{n+2\,.\,n+4}{5!}z^5 - \ldots\right\} \ldots\ldots\ldots\ldots(6)$$

The standard solution $D_n(z)$ may be expressed in terms of these by the equation

$$D_n(z) = \frac{2^{\frac{n}{2}}\pi^{\frac{1}{2}}}{\Gamma\left(-\frac{n-1}{2}\right)}E_n(z) - \frac{2^{\frac{n+1}{2}}\pi^{\frac{1}{2}}}{\Gamma\left(-\frac{n}{2}\right)}O_n(z).$$

We shall now find continued fractions which correspond to the solutions $E_n(z)$ and $O_n(z)$.

If in (1) we write

$$z^2 = x, \quad y = ue^{-\frac{1}{4}x},$$

the equation becomes

$$4x\frac{d^2u}{dx^2} + (-2x+2)\frac{du}{dx} + nu = 0,$$

and obtaining a continued fraction by successive differentiation of this equation, we have

$$\frac{u'}{u} = \frac{n}{2x-2-}\;\frac{4(n-2)x}{2x-6-}\;\frac{4(n-4)x}{2x-10-}\;; \ldots\ldots\ldots\ldots\ldots(C)$$

if, on the other hand, we obtain a continued fraction by successive integration of the same equation, it is found to be

$$2x\frac{u'}{u} = 1 + x - \frac{(n+2)x}{3+x-}\;\frac{(n+4)x}{5+x-} \ldots\ldots\ldots\ldots\ldots\ldots(D)$$

Again, if in (1) we write $z^2 = x$, $y = we^{\frac{1}{4}x}$, the equation becomes

$$4x\frac{d^2w}{dx^2} + (2x+2)\frac{dw}{dx} + (n+1)w = 0.$$

By successive differentiation of this equation we obtain the continued fraction

$$\frac{w'}{w} = -\frac{n+1}{2x+2-}\;\frac{4(n+3)x}{2x+6-}\;\frac{4(n+5)x}{2x+10-} \ldots\ldots\ldots\ldots\ldots\ldots(E)$$

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while by successive integration of the equation we obtain

$$2x\frac{w'}{w} = 1 - x - \frac{(n-1)x}{3-x-}\ \frac{(n-3)x}{5-x-} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(F)}$$

It may easily be shown by the methods of § 2 that *the continued fractions* (C), (D), (E), (F) *correspond respectively to the series* (3), (4), (5), (6); thus whether z be large or small, the computation of the parabolic-cylinder functions can be performed by means of continued fractions.

Taking as an example $n = \frac{1}{2}$, $z = \frac{1}{2}$, the continued fraction (C) furnishes the value of u'/u correctly to the first, second, third, or fourth place of decimals according as the first, second, third, or fourth convergent is used. Mr A. Milne has kindly calculated for me the following table:

z	$\frac{1}{u}\frac{du}{dx}$	$\frac{1}{u}\frac{du}{dz}$
0	$-0{\cdot}2500$	$-0{\cdot}0000$
0·1	$-0{\cdot}2513$	$-0{\cdot}0503$
0·2	$-0{\cdot}2551$	$-0{\cdot}1020$
0·3	$-0{\cdot}2617$	$-0{\cdot}1570$
0·4	$-0{\cdot}2714$	$-0{\cdot}2171$
0·5	$-0{\cdot}2847$	$-0{\cdot}2847$
0·6	$-0{\cdot}3025$	$-0{\cdot}3630$
0·7	$-0{\cdot}3260$	$-0{\cdot}4564$
0·8	$-0{\cdot}3569$	$-0{\cdot}5710$
0·9	$-0{\cdot}3983$	$-0{\cdot}7169$
1·0	$-0{\cdot}4548$	$-0{\cdot}9097$

The second column gives the value of $\frac{1}{u}\frac{du}{dx}$, computed from the continued fraction (C) ; and the third column gives the corresponding value of $\frac{1}{u}\frac{du}{dz}$, which is obtained by multiplying the number in the second column by $2z$. From this we can derive $\Delta \log u$, then $\log u$, and finally u itself, if this be the form in which the function is required.

§5. *Deduction of the recurrence-formulae for the Hermite functions.*

It is easy to deduce from the continued fractions the characteristic properties of the Hermite functions. For example, we see at once that the continued fraction

$$\frac{n-1}{z-}\;\frac{n-2}{z-}\;\frac{n-3}{z-}\cdots$$

represents both $\frac{u'_{n-1}}{u_{n-1}}$ and $\frac{u_n''}{u_n'}$, where u_{n-1} denotes the same function of the parameter $(n-1)$ that u_n is of n ; and therefore we must have

$$\frac{u_n''}{u_n'}=\frac{u'_{n-1}}{u_{n-1}}$$

or $$u_n'=Cu_{n-1}$$

where C is independent of z. Making z very large in (A″), we see that the constant C has the value n ; so we have

$$u_n'=nu_{n-1},$$

which can be written

$$\frac{dD_n(z)}{dz}+\tfrac{1}{2}z\,D_n(z)-n\,D_{n-1}(z)=0.$$

This is one of the recurrence-formulae satisfied by the Hermite function $D_n(z)$.

§6. *The selection of particular solutions from the general solution, in the continued-fraction method.*

We are naturally led to enquire how the method of solving a differential equation by a continued fraction leads in every case to some particular solution of the equation instead of to the general solution; for it does not at first sight seem evident where the restriction to particularity is introduced in the process of solution.

To discuss this question, let us consider a simple equation such as

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0,$$

and suppose the process of solution arrested at the r^{th} step, so that we obtain the finite continued fraction

$$\tfrac{2}{3} - \tfrac{2}{3} - \tfrac{2}{3} - \cdots - \tfrac{2}{3} - \frac{y^{(r+1)}}{y^{(r)}}.$$

This continued fraction represents y'/y if y is *any solution whatever* of the equation; so that if for $y^{(r+1)}/y^{(r)}$ in the final quotient we substitute the value 1 (corresponding to the solution $y = e^z$), we get the value 1 for the entire continued fraction; and if for $y^{(r+1)}/y^{(r)}$ we substitute the value 2 (corresponding to the solution $y = e^{2z}$), we get the value 2 for the entire continued fraction. The question now arises, what value has the entire C.F. when we assign to $y^{(r+1)}/y^{(r)}$ in the final quotient a value which is neither 1 nor 2. To determine this, first take $y^{(r+1)}/y^{(r)} = 1 + \epsilon$, where ϵ is supposed small, so that its square can be neglected, then the entire continued fraction has the value $1 + 2^{-r}\epsilon$; so that if to the last quotient we assign a value near to unity, the entire C.F. will have a value nearer still to unity. On the other hand, if we take $y^{(r+1)}/y^{(r)} = 2 + \epsilon$, where ϵ is small, then the entire C.F. has the value $2 + 2^r\epsilon$; so that if to the last quotient we assign a value near to 2, the entire C.F. will have a value much more remote from 2.

The two values 1 and 2 of the entire C.F. may therefore, by analogy with the theory of orbits in dynamics, be called the *stable* and *unstable* values respectively; and it is in consequence of its instability that the value 2 drops out.

§ 7. *Extension of the preceding results to more general functions.*

The functions of Hermite are included* in an extensive class

* *Cf.* a paper by the present writer in *Bull. Amer. Math. Soc.* (2) *10* (1903), p. 125.

of functions denoted by the symbol $W_{k,m}(z)$; this class of functions comprehends also the error-function, the incomplete-gamma-function, the logarithm-integral, the cosine-integral, and the Bessel functions.

It will now be shown that *the results of* § 4 *can be extended to these functions* $W_{k,m}(z)$.

For these functions satisfy the differential equation

$$\frac{d^2u}{dz^2}+\left(-\tfrac{1}{4}+\frac{k}{z}+\frac{\tfrac{1}{4}-m^2}{z^2}\right)u=0. \qquad \text{(7)}$$

This equation has the solutions

$$M(z)=z^{\frac{1}{2}+m}e^{\frac{1}{2}z}\left\{1-\frac{\tfrac{1}{2}+m+k}{1\,.\,2m+1}z+\frac{\tfrac{1}{2}+m+k\,.\,\tfrac{3}{2}+m+k}{1\,.\,2\,.\,2m+1\,.\,2m+2}z^2-\dots\right\} \qquad \text{(8)}$$

$$=z^{\frac{1}{2}+m}e^{-\frac{1}{2}z}\left\{1+\frac{\tfrac{1}{2}+m-k}{1\,.\,2m+1}z+\frac{\tfrac{1}{2}+m-k\,.\,\tfrac{3}{2}+m-k}{1\,.\,2\,.\,2m+1\,.\,2m+2}z^2+\dots\right\} \qquad \text{(9)}$$

and

$$P(z)=z^{\frac{1}{2}-m}e^{-\frac{1}{2}z}\left\{1+\frac{\tfrac{1}{2}-m-k}{1\,.\,-2m+1}z+\frac{\tfrac{1}{2}-m-k\,.\,\tfrac{3}{2}-m-k}{1\,.\,2\,.\,-2m+1\,.\,-2m+2}z^2+\dots\right\} \qquad \text{(10)}$$

$$=z^{\frac{1}{2}-m}e^{\frac{1}{2}z}\left\{1-\frac{\tfrac{1}{2}-m+k}{1\,.\,-2m+1}z+\frac{\tfrac{1}{2}-m+k\,.\,\tfrac{3}{2}-m+k}{1\,.\,2\,.\,-2m+1\,.\,-2m+2}z^2-\dots,\right\} \qquad \text{(11)}$$

and the standard solution $W_{k,m}(z)$ is expressible in terms of these two solutions by the formula

$$W_{k,m}(z)=\frac{\Gamma(-2m)}{\Gamma(\tfrac{1}{2}-m-k)}M(z)+\frac{\Gamma(2m)}{\Gamma(\tfrac{1}{2}+m-k)}P(z).$$

Now writing $u=yz^{m+\frac{1}{2}}e^{\frac{1}{2}z}$, the differential equation (7) becomes

$$y\frac{d^2y}{dz^2}+(2m+1+z)\frac{dy}{dz}+(k+m+\tfrac{1}{2})y=0.$$

By successive differentiation of this, as in § 2, we obtain the continued fraction

$$\frac{y'}{y}=-\frac{k+m+\tfrac{1}{2}}{2m+1+z}\,-\,\frac{(k+m+\tfrac{3}{2})z}{2m+2+z}\,-\,\frac{(k+m+\tfrac{5}{2})z}{2m+3+z}-\dots$$

Moreover, since the differential equation (7) is unaltered when m is changed to $-m$, another continued fraction (corresponding to a second solution of the differential equation) can be obtained by changing the sign of m in this; and since the equation (7) is also unaltered when the signs of k and z are reversed simultaneously, we can obtain two more continued fractions from these by reversing the signs of k and z. These C.F.'s correspond respectively to the series (8), (9), (10), (11) above.

Thus we obtain altogether 4 continued fractions associated with the function $W_{k,m}(z)$; these correspond exactly to the 4 C.F.'s obtained in § 4 for the Hermite functions. By taking special values for k and m, we can deduce various continued fractions for the error-function, cosine-integral, etc.