ON SOME SOLUTIONS OF SECOND ORDER HYPERBOLIC DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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1. If we seek solutions of the hyperbolic differential equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + k^2 u - \frac{\partial^2 u}{\partial t^2} = 0 \qquad (k \geqslant 0) \tag{1}$$

which depend only on the variables t and $r = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}$, we see that these solutions must be even in r and satisfy the differential equation

$$T_n[u(r, t)] = \frac{\partial^2 u}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial u}{\partial r} + k^2 u - \frac{\partial^2 u}{\partial t^2} = 0. \tag{2}$$

The object of this paper is to show that some recent results in the fractional calculus can be used to prove the following theorem.

THEOREM. *For odd values of $n \geqslant 3$ and arbitrary functions ϕ with continuous derivatives up to the order $n-1$, the functions*

$$u(r, t) = T_n^{(n-3)/2}\left[\int_{-1}^{1} J_0\{kr\sqrt{(1-\xi^2)}\}\phi(t+\xi r)\, d\xi\right] \tag{3}$$

are solutions of the differential equation

$$T_n[u(r, t)] = 0. \tag{4}$$

A corresponding result for the n-dimensional wave equation with rotational symmetry (i.e. equation (2) with $k = 0$) is given in [**1**].

2. In what follows we shall make use of the generalized Erdélyi–Kober operator of fractional integration $\mathfrak{I}_k(\eta, \alpha)$ which is defined in [**2**] by

$$\mathfrak{I}_k(\eta, \alpha) f(r) = 2^\alpha k^{1-\alpha} r^{-2(\alpha+\eta)} \int_0^r x^{2\eta+1}(r^2 - x^2)^{(\alpha-1)/2} J_{\alpha-1}\{k\sqrt{(r^2-x^2)}\} f(x)\, dx, \tag{5}$$

where $r > 0$, $\alpha > 0$, $k \geqslant 0$ and $J_{\alpha-1}$ is the Bessel function of the first kind.

A useful result connecting the above operator with the singular differential operator

$$L_\eta = \frac{\partial^2}{\partial r^2} + \frac{(2\eta+1)}{r}\frac{\partial}{\partial r} \tag{6}$$

is contained in the following lemma [**2**].

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LEMMA. *If $\alpha>0$, $f(r)\in C^2(0,b)$ for some $b>0$, $r^{2\eta+1}f(r)$ is integrable at the origin and $r^{2\eta+1}f'(r)\to 0$ as $r\to 0+$, then*

$$\mathfrak{I}_k(\eta,\alpha)L_\eta f(r)=(L_{\eta+\alpha}+k^2)\mathfrak{I}_k(\eta,\alpha)f(r). \tag{7}$$

3. Adopting the notation of (6) we see that the one-dimensional wave equation

$$L_{-1/2}w-\frac{\partial^2 w}{\partial t^2}=0 \tag{8}$$

with the conditions

$$w(0,t)=2\phi(t),\qquad \frac{\partial}{\partial r}w(0,t)=0 \tag{9}$$

has the solution

$$w(r,t)=\phi(t+r)+\phi(t-r), \tag{10}$$

for arbitrary differentiable functions ϕ.

We now introduce the function

$$w_\alpha(r,t)=\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})}\mathfrak{I}_k(-\tfrac{1}{2},\alpha)w(r,t)\qquad(\alpha>0) \tag{11}$$

and apply the operator $[\Gamma(\frac{1}{2})]^{-1}\Gamma(\alpha+\frac{1}{2})\mathfrak{I}_k(-\frac{1}{2},\alpha)$ to equations (8), (9) and (10). In this way, on using the result (7) of the lemma, we find that the solution of the differential equation

$$T_{2\alpha+1}[w_\alpha(r,t)]=0\qquad(\alpha>0) \tag{12}$$

with the conditions

$$w_\alpha(0,t)=2\phi(t),\qquad \frac{\partial}{\partial r}w_\alpha(0,t)=0 \tag{13}$$

is given by

$$\begin{aligned} w_\alpha(r,t)&=\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})}\mathfrak{I}_k(-\tfrac{1}{2},\alpha)[\phi(t+r)+\phi(t-r)]\\ &=2^\alpha\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})}(kr)^{1-\alpha}\int_{-1}^{1}(1-\xi^2)^{(\alpha-1)/2}J_{\alpha-1}(\rho)\phi(t+\xi r)\,d\xi, \end{aligned} \tag{14}$$

where $\rho=kr\surd(1-\xi^2)$.

With the above results we can write

$$\begin{aligned} T_n[w_\alpha(r,t)]&=T_{2\alpha+1}[w_\alpha(r,t)]+\frac{(n-2\alpha-1)}{r}\frac{\partial}{\partial r}w_\alpha\\ &=\frac{(n-2\alpha-1)}{r}\frac{\partial}{\partial r}w_\alpha \end{aligned} \tag{15}$$

and from equations (14) and (15) we find that

$$T_n[w_\alpha(r,t)] = 2^\alpha \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{(n-2\alpha-1)}{r}\left\{-k(kr)^{1-\alpha}\int_{-1}^{1}(1-\xi^2)^{\alpha/2}J_\alpha(\rho)\phi(t+\xi r)\,d\xi\right.$$
$$\left. + (kr)^{1-\alpha}\int_{-1}^{1}\xi(1-\xi^2)^{(\alpha-1)/2}J_{\alpha-1}(\rho)\phi'(t+\xi r)\,d\xi\right\}. \tag{16}$$

On performing an integration by parts on the last integral in the above equation we get

$$T_n[w_\alpha(r,t)] = 2^\alpha \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{(n-2\alpha-1)}{(kr)^\alpha}\int_{-1}^{1}(1-\xi^2)^{\alpha/2}J_\alpha(\rho)[\phi''(t+\xi r)-k^2\phi(t+\xi r)]\,d\xi \tag{17}$$

and with the aid of this result we can now prove the theorem.

4. Proof of the theorem. When $\alpha=1$ the solution of equations (12) and (13) is given by

$$w_1(r,t) = \int_{-1}^{1} J_0(\rho)\phi(t+\xi r)\,d\xi, \tag{18}$$

where $\rho = kr\sqrt{(1-\xi^2)}$.

Using the result (17) we have

$$T_n[w_1(r,t)] = \frac{(n-3)}{kr}\int_{-1}^{1}(1-\xi^2)^{1/2}J_1(\rho)[\phi''(t+\xi r)-k^2\phi(t+\xi r)]\,d\xi \tag{19}$$

and repeated applications of the formula (17) yield the expression

$$T_n^m[w_1(r,t)] = \frac{(n-3)(n-5)\dots(n-2m-1)}{(kr)^m}\int_{-1}^{1}(1-\xi^2)^{m/2}J_m(\rho)\Phi_m(t+\xi r)\,d\xi \tag{20}$$

when $n \geqslant 2m+1$,

$$\Phi_m(t+\xi r) = \sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}k^{2(m-s)}\phi^{(2s)}(t+\xi r) \tag{21}$$

and ϕ is any function with continuous derivatives up to order $2m$.

In this way we find that, for odd values of $n \geqslant 3$,

$$T_n^{(n-1)/2}[w_1(r,t)] = T_n\{T_n^{(n-3)/2}[w_1(r,t)]\} = 0 \tag{22}$$

and this proves the theorem.

5. In order to construct a simple example we take $\phi(t)=e^{i\beta t}$ and in this case we see that equation (18) gives

$$\begin{aligned} w_1(r,t) &= \int_{-1}^{1} J_0\{kr\sqrt{(1-\xi^2)}\}e^{i\beta(t+\xi r)}\,d\xi \\ &= 2e^{i\beta t}\int_0^1 J_0\{kr\sqrt{(1-\xi^2)}\}\cos(\xi\beta r)\,d\xi \\ &= 2e^{i\beta t}\frac{\sin(ar)}{ar}, \end{aligned} \tag{23}$$

where $a=\sqrt{(\beta^2+k^2)}$ and the integral has been evaluated by a result given in [**3**].

Using the theorem we have that, for odd values of $n\geqslant 3$, the functions

$$v_n(r,t)=T_n^{(n-3)/2}\left[2e^{i\beta t}\frac{\sin(ar)}{ar}\right] \tag{24}$$

satisfy the differential equation

$$T_n[v_n(r,t)]=0. \tag{25}$$

As two special cases it can easily be shown that when $n=5$,

$$v_5(r,t)=4e^{i\beta t}\left[\frac{\cos(ar)}{r^2}-\frac{\sin(ar)}{ar^3}\right]$$

and when $n=7$,

$$v_7(r,t)=16e^{i\beta t}\left[\frac{3\sin(ar)}{ar^5}-\frac{3\cos(ar)}{r^4}-\frac{a\sin(ar)}{r^3}\right],$$

which are even functions of the variable r.

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