ARTINIAN QUOTIENT RINGS OF GROUP RINGS

Dedicated to the memory of Hanna Neumann

IAN HUGHES

(Received February 6 1973)

Communicated by M. F. Newman

1. Introduction

Smith [6, Theorem 2.18] proved that if A is a ring which has a right artinian right quotient ring and G is a poly- (cyclic-or-finite) group, then the group ring AG has a right artinian right quotient ring. We give here a different proof (and a generalization) of this result using methods developed by Jategaonkar [3, 4]. Explicitly we prove

THEOREM. *Let A be a ring which has a right artinian right quotient ring, and let G be a group which has a (transfinite) ascending normal series with each factor either finite or cyclic, but only a finite number of finite factors. Then AG has a right artinian right quotient ring.*

We refer to Kuroš [5, p. 173] for the definition of an ascending normal series. We note that in the case of Smith's Theorem the group ring AG is always right noetherian (if A is) whereas this is not necessarily the case in our theorem.

REMARK. The restriction that there be only a finite number of finite factors in the ascending normal series in G is necessary, as we can see from the following example.

Let G be a group which is the union of a strictly ascending sequence of finite subgroups $\{G_i, i = 1, 2, \cdots\}$ where G_i is normal in G_{i+1} for each i. Let K be a field. For each i let $\omega(G_i)$ be the right ideal of KG generated by $\{g - 1; g \in G_i\}$. It is readily seen that $\omega(G_i)$ is the right annihilator of $r_i = \sum g \ (g \in G_i)$ for each i. It is easy to see that $\{\omega(G_i)\}$ is strictly ascending. We thus have a strictly ascending infinite sequence of right annihilators in KG.

Now suppose KG has an artinian Quotient ring Q. Certainly Q has the ascending chain condition on right annihilators and so any subring, in particular KG, has this property. This contradicts what we have shown in the previous paragraph.

NOTATION: Let R be a ring. We denote by $P(R)$ the prime radical of R and by $R^{\#}$ the ring $R/P(R)$. We denote by $Q(R)$ the right quotient ring of R when it exists. If ρ is an automorphism of R we denote by $\rho^{\#}$ the automorphism of $R^{\#}$ induced by ρ. If R is semi-simple artinian, we denote by $m(R)$ the number of minimal two-sided ideals of R.

Let G be a group and R be a ring with identity. Suppose (η, ω) is a factor set for G over R (cf, for example, Bovdi [2]). We denote by $R[G; \eta, \omega)$ the group ring (or crossed product) of G over R with factor set (η, ω).

For m a positive integer we denote by I_m the set $\{1, 2, \cdots, m\}$.

In the remainder of this paper quotient ring will mean right quotient ring and artinian will mean right artinian.

2. Group rings of infinite cyclic groups

Let G be an infinite cyclic group with generator x, and A a ring with identity. Suppose ρ is an automorphism of A. We denote by $A[G, \rho]$ the set of finite sums $\Sigma x^i a_i$ where i is an integer and a_i is in A. We define addition in the usual way, and multiplication by assuming the distributive law and the rule $ax = x\rho(a)$ for all a in A. It is straightforward to check that $A[G, \rho]$ is an associative ring with identity. We remark that $A[G, \rho]$ is very similar to a skew polynomial ring as defined in [3]; however in our case negative powers of x occur. But our situation is somewhat more special in that ρ is an automorphism of A and not merely a monomorphism of A into itself.

Let Q be a semi-simple artinian ring, and suppose $\{f_i;\ i \in I_m\}$ is a complete set of primitive central idempotents in Q. Let ρ be an automorphism of Q. Then, clearly, $\rho(f_i) = f_{\pi(i)}$ where π is a permutation of I_m. Let $\pi = \pi_1 \pi_2 \cdots \pi_t$ be the (unique) decomposition of π into disjoint cycles (we write 1-cycles also). For each j in I_t let $g_j = \Sigma f_i (i \in \pi_j)$ where $i \in \pi_j$ denotes that i appears in the cycle notation for π_j. We denote π_j by $(j1\, j2 \cdots jm(j))$. Thus π_j is a cycle of length $m(j)$ and $m = \Sigma m(j)\ (j \in I_t)$. If $t = 1$ we say that Q is ρ-transitive.

Now $Q = \oplus\, Qf_i\ (i \in I_m)$; and clearly for each j in I_t there is a division ring D_j and a positive integer $n(j)$ such that for each $i (i \in \pi_j)$ there is an isomorphism

$$\theta_i \colon Qf_i \to M_{n(j)}(D_j).$$

Also, if we denote $\theta_{j1} \rho^{m(j)} \theta_{j1}^{-1} \,|\, D_j$ by ϕ_j then it is easy to see that ϕ_j is an automorphism of D_j. Also, for each j in I_t, we have $\rho(g_j) = g_j$ and if we denote $\rho \,|\, Qg_j$ by ρ_j then ρ_j is an automorphism of Qg_j such that Qg_j is ρ_j-transitive We have

PROPOSITION 1. *Let G be an infinite cyclic group, Q a semi-simple artinian ring and ρ an automorphism of Q. Then there is an isomorphism*

$$\psi \colon Q[G, \rho] \to \oplus\, M_{m(j)}(M_{n(j)}(D_j[G, \phi_j])) \qquad (j \in I_t)$$

such that for q in Q, the jth component of $\psi(q)$ *is given by a diagonal* $m(j) \times m(j)$ *matrix, namely*

$$\psi(q)_j = \operatorname{diag}(\theta_{j1}(qf_{j1}), \theta_{j2}(qf_{j2}), \cdots, \theta_{jm(j)}(qf_{jm(j)}))$$

for each j in I_t, *where the notation is as described in the previous two paragraphs.*

We omit the proof since we use essentially the same arguments that are used n the proofs of Theorem 2.1(a), Lemma 3.1(a) and Lemma 3.2 of [3].

We can now prove

PROPOSITION 2. *Let A be a ring which has an artinian quotient ring Q and suppose* ρ *is an automorphism of A. Let G be an infinite cyclic group. Then*

(a) $A[G,\rho]$ *has an artinian quotient ring R.*

(b) (i) $P(R) = P(Q)R$

(ii) $P(Q)^k = 0 \rightarrow P(R)^k = 0$ (*k a positive integer.*)

(c) $R^{\#}$ *is the quotient ring of* $Q^{\#}[G,\rho^{\#}]$

(d) $m(R^{\#}) \leqq m(Q^{\#})$

(e) *If* $m(Q^{\#}) = m(R^{\#}) = m$ (say) *then for each j in* I_m *there is a positive integer* $n(j)$ *and two division rings* D_j *and* E_j *where* D_j *is contained in* E_j *such that there is a commutative diagram*

$$\begin{array}{ccccc} Q & \longrightarrow & Q^{\#} & \longleftrightarrow & \oplus_j M_{n(j)}(D_j) \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & R^{\#} & \longleftrightarrow & \oplus_j M_{n(j)}(E_j) \end{array}$$

where the two double-headed arrows denote isomorphisms and for the rest, each homomorphism in the diagram is the obvious one.

PROOF. We apply Proposition 1 to $Q^{\#}$ and use the same notation as we did in that proposition. By the usual argument using the division algorithm, for each j we see that $D_j[G,\phi_j]$ is a principal right ideal domain and so has a quotient division ring which we denote by E_j. It follows from Proposition 1, that $Q^{\#}[G,\rho^{\#}]$ has a semi-simple artinian quotient ring which is isomorphic to $\oplus_j M_{m(j)n(j)}(E_j)$ $(j \in I_t)$.

The proofs of (a), (b) (i) and (b) (ii) are now essentially the same as the proof of Theorem 3.1 of [4]. We therefore omit their proofs. Now (c) follows from (b) (i) using a routine argument; (d) follows from (c) since $m(R^{\#}) = t \leqq m = m(Q^{\#})$.

We now prove (e). Since we are supposing $m(Q^{\#}) = m(R^{\#})$, we see that $m(j) = 1$ for each j. We may clearly assume that π_j is the 1-cycle (j).

We let $Q^{\#} \leftrightarrow \oplus_j M_{n(j)}(D_j)$ $(j \in I_m)$ be the isomorphism defined by the θ_j and let $R^{\#} \leftrightarrow \oplus_j M_{n(j)}(E_j)$ $(j \in I_m)$ be the isomorphism defined by ψ is Proposition 1.

We have thus obtained the required diagram. The commutativity of the left square follows from (b) (i) and that of the right square from the formula for $\psi(q)$ (q in $Q^{\#}$) given in Proposition 1. The proof is complete.

3. Proof of theorem

We assume the hypotheses of the theorem. A straight-forward argument shows that we may replace A by its quotient ring. We may thus assume that A is an artinian ring with identity.

Let the ascending normal series in G be $\{G_\alpha, \alpha \text{ an ordinal}, G_\Gamma = G\}$. For each $\alpha < \Gamma$ we denote $G_{\alpha+1}/G_\alpha$ by X_α. By assumption X_α is either finite or infinite cyclic. In the case where X_α is finite, we fix a set of coset representatives for G_α in $G_{\alpha+1}$ containing the identity of G_α. It is easy to see that a factor set $(\eta_\alpha, \omega_\alpha)$ for X_α over AG_α is then determined and that the obvious map from $AG_{\alpha+1}$ to $AG_\alpha[X_\alpha; \eta_\alpha, \omega_\alpha]$ is an isomorphism.

In the case where X_α is infinite cyclic, we choose as set of coset representatives of G_α in $G_{\alpha+1}$ the powers of an appropriate element x in $G_{\alpha+1}$. The map $\rho_\alpha\colon G_\alpha \to G_\alpha$ given by $\rho_\alpha(g) = x^{-1}gx$ for each g in G_α extends uniquely to an automorphism of AG_α. It is easy to see that the obvious map from $AG_{\alpha+1}$ to $AG_\alpha[X_\alpha, \rho_\alpha]$ is an isomorphism.

For each $\alpha \leqq \Gamma$ we prove by transfinite induction on α that

(a) AG_α has an artinian quotient ring, which we denote by Q_α.

(b) For $\gamma < \beta \leqq \alpha$ such that there are no finite jumps between γ and β

(i) $m(Q_\gamma^{\#}) \geqq m(Q_\beta^{\#})$

(ii) $P(Q_\beta) = P(Q_\gamma)Q_\beta$

(iii) $P(Q_\gamma)^k = 0 \Rightarrow P(Q_\beta)^k = 0$ (k a positive integer).

If in addition, $m(Q_\gamma^{\#}) = m(Q_\beta^{\#}) = m$ (say) then

(iv) For each j in I_m there is a positive integer $n(j)$ and division rings $D_{\gamma,j}$ and $D_{\beta,j}$ such that $D_{\gamma,j} \subseteq D_{\beta,j}$ and such that there is a commutative diagram

$$\begin{array}{ccccc} Q_\gamma & \longrightarrow & Q_\gamma^{\#} & \longleftrightarrow & \oplus_j M_{n(j)}(D_{\gamma,j}) \qquad (j \in I_m) \\ \downarrow & & \downarrow & & \downarrow \\ Q_\beta & \longrightarrow & Q_\beta^{\#} & \longleftrightarrow & \oplus_j M_{n(j)}(D_{\beta,j}) \end{array}$$

where the two double-headed arrows denote isomorphisms and for the rest each homomorphism in the diagram is the obvious one.

https://doi.org/10.1017/S1446788700015202 Published online by Cambridge University Press

We now prove these assertions. Suppose firstly that α is not a limit ordinal. If $X_{\alpha-1}$ is finite then as indicated above AG_α is isomorphic to

$$AG_{\alpha-1}[X_{\alpha-1}; \eta_{\alpha-1}, \omega_{\alpha-1}].$$

It is clear that the factor set $(\eta_{\alpha-1}, \omega_{\alpha-1})$ can be extended to $Q_{\alpha-1}$. It is then straightforward to prove that $Q_{\alpha-1}[X_{\alpha-1}; \eta_{\alpha-1}, \omega_{\alpha-1}]$ is the quotient ring of AG_α and that it is artinian. This proves (a) in this case. It is obvious that (b) is true also.

If $X_{\alpha-1}$ is infinite cyclic, then by the above AG_α is isomorphic to

$$AG_{\alpha-1}[X_{\alpha-1}, \rho_{\alpha-1}]$$

and both (a) and (b) are true by Proposition 2.

We may thus assume that α is a limit ordinal. Since our ascending normal series in G has only a finite number of finite factors, there is an ordinal $\delta < \alpha$ such that G_δ contains every finite jump preceeding G_α. By applying (b) (i) of the inductive hypothesis it is clear, by enlarging δ if necessary, that we may assume that $m(Q_\gamma^\#) = m(Q_\delta^\#) = m(\text{say})$ for all γ satisfying $\delta \leqq \gamma < \alpha$.

We consider $Q_\alpha = \text{inj limit } Q_\gamma \, (\delta \leqq \gamma < \alpha)$. It is easy to see that it is a quotient ring of AG_α (we use the fact that if an element is regular in AG_γ then it is regular in AG_α for $\gamma < \alpha$).

For each j in I_m we let $D_{\alpha j} = \text{inj limit } D_{\gamma j} \; (\delta \leqq \gamma < \alpha)$. We also let $Q_\alpha^+ = \text{inj limit } Q_\gamma^\#(\delta \leqq \gamma < \alpha)$. Using (b)(iv) of the inductive hypothesis, it is straightforward to show that if in the diagram in (b) (iv) we replace β by α and $Q_\beta^\#$ by Q_α^+ then the diagram is commutative, where the maps are the obvious ones. It follows that Q_α^+ is semi-simple artinian.

If we can show that $Q_\alpha^+ = Q_\alpha^\#$ then (b) follows easily. To prove (a), that is that Q_α is artinian, it suffices to show (cf [1], p. 71) that

(1) $Q_\alpha^\#$ is a semi-simple artinian ring

(2) $P(Q_\alpha)$ is a nilpotent

(3) $P(Q_\alpha)$ is finitely generated as a right ideal of Q_α.

Using (b) (ii) of the inductive hypothesis it is easy to see that $I_\alpha = \ker(Q_\alpha \to Q_\alpha^+)$ is equal to $P(Q_\gamma)Q_\alpha$ for any γ satisfying $\delta \leqq \gamma < \alpha$. For each such γ, we have that Q_γ is an artinian ring with identity and so right noetherian. It follows that I_α is finitely generated as a right ideal of Q_α. By applying (b) (iii) of the inductive hypothesis it is easy to see I_α is a nilpotent ideal and since $Q_\alpha / I_\alpha = Q_\alpha^+$ is semi-simple artinian it follows that $I_\alpha = P(Q_\alpha)$. Thus $Q_\alpha^\# = Q_\alpha^+$ as required and (1), (2) and (3) are proved. Thus Q_α is artinian and the inductive proof is complete. The theorem follows by letting $\alpha = \Gamma$.

References

[1] N. Bourbaki, *Éléments de Mathématique*, Algèbre chapitre 8, Modules et anneaux semi-simples, (Hermann, Paris, 1958.)

[2] A. A. Bovdi, 'Crossed products of a semigroup and a ring', *Dokl. Akad. Nauk SSSR* 137 (1961), 1267–1268.

[3] A. V. Jategaonkar, 'Skew polynomial rings over semisimple rings', *J. Algebra* 19 (1971), 315–328.

[4] A. V. Jategaonkar, 'Skew polynomial rings over orders in artinian rings', *J. Algebra* 21 (1972), 51–59.

[5] A. Kurosh, *The Theory of Groups*, Vol. 2, (Chelsea, New York, 1956.)

[6] P. F. Smith, 'Quotient rings of group rings', *J. London Math. Soc.* (2) 3 (1971), 645–660.

Department of Mathematics
Queen's University
Kingston, Ontario, Canada