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Invariant measures for substitutions on countable alphabets

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Abstract. In this work, we study ergodic and dynamical properties of symbolic dynamical system associated to substitutions on an infinite countable alphabet. Specifically, we consider shift dynamical systems associated to irreducible substitutions which have well-established properties in the case of finite alphabets. Based on dynamical properties of a countable integer matrix related to the substitution, we obtain results on existence and uniqueness of shift invariant measures.

Key words: substitutions on infinite alphabet, unique ergodicity, countable non-negative matrices, shift dynamical systems
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1. *Introduction*

Let A be a countable set (called an *alphabet*), A^* be the set of finite words on A, and $A^{\mathbb{Z}_+}$ be the set of infinite words on A, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. A *substitution* is a map $\sigma : A \to A^*$. We assume that for every letter $a \in A$, $\sigma(a)$ is not empty. We extend σ to A^* and $A^{\mathbb{Z}_+}$ by concatenation and, to simplify the notation, we also denote these extensions by σ. Hence, $\sigma(u_0 \ldots u_n) = \sigma(u_0) \ldots \sigma(u_n)$ for all $u_0 \ldots u_n \in A^*$ and

$\sigma(u_0u_1\ldots) = \sigma(u_0)\sigma(u_1)\ldots$ for all $u_0u_1\ldots \in A^{\mathbb{Z}_+}$. We assume that there exists a letter a in A such that the length of the finite word $\sigma^n(a)$ converges to infinity as n goes to infinity.

To any substitution σ, we can associate a *shift dynamical system* (Ω_σ, S), where

$$\Omega_\sigma = \{u \in A^{\mathbb{Z}_+} : \text{ any finite factor of } u \text{ occurs in } \sigma^n(a) \text{ for some } n \in \mathbb{N} \text{ and } a \in A\},$$

$\mathbb{N} = \{1, 2, \ldots\}$, and S is the shift map given by

$$S(u_0u_1\ldots) = u_1u_2\ldots \quad \text{for all } u = u_0u_1\ldots \in A^{\mathbb{Z}_+}.$$

Shift dynamical systems associated to substitutions provide many important examples in ergodic theory and they have been well studied in the literature when the alphabet is finite (see for instance [18, 19]). It is classical that if σ is a *primitive substitution* on $A = \{0, \ldots, d-1\}$, $d \geq 2$, i.e., there exists $k \in \mathbb{N}$ such that for all $a, b \in A$, the letter b occurs in the word $\sigma^k(a)$, then the dynamical system is minimal, uniquely ergodic with topological entropy 0 (see [16] and [19, Ch. 5]). Moreover, Ω_σ is the closure of the orbit of any periodic point of σ.

The unique shift invariant probability measure μ is given on cylinders $[w]$, where $w = w_0\ldots w_n$, $w_i \in A$ for $i = 0, \ldots, n$, is a finite word that occurs in u and $[w] = \{u_0u_1\ldots \in \Omega_\sigma,\ u_i = w_i,\ i = 0, \ldots, n\}$, by $\mu[w]$ which is the frequency of occurrences of w in the periodic point u. Moreover, the vector $(\mu[0], \ldots, \mu[d-1])$ is the normalized left Perron eigenvector associated to the dominant Perron–Frobenius eigenvalue of the matrix $M_\sigma = (M_{ij})_{0\leq i,j\leq d-1}$ associated to σ, where $M_{ij} := |\sigma(i)|_j$ is the number of occurrences of the letter j in the word $\sigma(i)$. However, it is known (see [2]) that if σ is of Pisot type, then the dynamical system (Ω_σ, S) has good geometrical properties, in particular, it is semi-conjugated to a translation on the torus $\mathbb{T}^{d-1}$.

When the alphabet A is a topological compact set, many results are given in [4, 13, 19].

When A is countably infinite, the situation is more complicated and there are already some work on the subject, see for instance [1, 4, 6, 13]. One of the difficulties in studying ergodic properties of the dynamical system (Ω_σ, S) in such cases lies in the fact that the countably infinite matrix M_σ may present a larger number of possible behaviors. Specifically, consider an irreducible countably infinite matrix $M = (M_{ij})_{i,j\in\mathbb{Z}_+}$, which means for all $i, j \in \mathbb{Z}_+$, there exists an integer $n \geq 1$ such that for all $k \geq n$, $M^k_{ij} > 0$, where for the sake of simplicity, we write $(M^n)_{ij} = M^n_{ij}$. It is known that for all $i, j \in \mathbb{Z}_+$, $\lim_{n\to\infty}(M^n_{ij})^{1/n} = \lambda$ exists. We say that M is *transient* if and only if $\sum_{n=0}^{+\infty}(M^n_{ij}/\lambda^n) < +\infty$, otherwise M is said to be *recurrent*. It is known that if M is recurrent, there are left and right eigenvectors l and r associated to λ, and when the scalar product $l \cdot r$ is finite, we say that M is *positive recurrent*, otherwise M is said to be *null recurrent*. Thus, for instance, if the countably infinite matrix M_σ is irreducible, then it could be either transient or null recurrent or positive recurrent and each of these cases may be associated to a distinct behavior of (Ω_σ, S).

For substitutions on countably infinite alphabets, an important study was initiated by Ferenczi in [6]. In that paper, several results were proved, in particular, it considered the squared drunken substitution defined on $A = 2\mathbb{Z}$ by $\sigma(n) = (n-2)nn(n+2)$, $n \in A$

and proved that the dynamical system (Ω_σ, S) is not minimal and has non-finite invariant measure. However, it is also shown that (Ω_σ, S) has an infinite invariant measure μ which is shift ergodic and has Krengel entropy equal to 0.

Let us recall that σ is called left determined or determined to order 1 if there exists a non-negative integer N such that every w of length at least N which occurs on some element of Ω_σ has a unique decomposition $w = w_1 \ldots w_s$, where each $w_i = \sigma(a_i)$ for some $a_i \in A$, except that w_1 may be only a suffix of $\sigma(a_1)$ and w_s may be only a prefix of $\sigma(a_s)$, and the a_i, $1 \le i \le s-1$ are unique.

The definition of determined to order 1 was introduced in [14] (see also [17, Definition 1]). In [6], the author used the same definition and called it left determined. It is known that this condition is stronger than recognizability, see [17].

In [6], it is also proved that if σ is of constant length, left determined, and has an irreducible aperiodic positive recurrent matrix M_σ, then the associated shift dynamical system admits an ergodic probability invariant measure.

In [1], the authors constructed stationary and non-stationary generalized Bratteli–Vershik models for left determined, irreducible, aperiodic, and recurrent substitutions on an infinite countable alphabet. As a consequence, they proved that for a left determined substitution $\sigma : \mathbb{Z} \to \mathbb{Z}$ with M_σ irreducible, aperiodic, and recurrent which is also of *bounded size* (the letters of all $\sigma(n)$ belong to the set $\{n-t, n-t+1, \ldots, n+t\}$, where $t \in \mathbb{Z}$ is independent of n), there exists a shift invariant measure μ on Ω_σ.

It is also worth mentioning that an arithmetic study of substitutions on countably infinite alphabets was done in [15].

In this paper, unless explicitly indicated, we consider $A = \mathbb{Z}_+$ and $\sigma : A \to A^*$ a bounded length substitution ($\sup\{|\sigma(a)|, a \in A\}$ is finite) such that σ has a periodic point u and $M = M_\sigma$ is irreducible and aperiodic. We prove that if M_σ satisfies

$$\lim_{n\to+\infty} \sup_{i\in A} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \quad \text{for all } j \in A, \tag{1.1}$$

then the dynamical system (Ω_σ, S) has no finite invariant measure. In particular, the last result holds for a subclass of substitutions σ such that M_σ is transient and σ has constant length, or M_σ is recurrent and has a left Perron eigenvector $l = (l_i)_{i\ge 0} \notin l^1$.

We also prove that if M_σ is positive recurrent, then the dynamical system (Ω_σ, S) has a shift invariant measure μ which is finite if and only if M_σ has a left Perron eigenvector $l \in l^1$. Moreover, if σ has constant length and M_σ has a power that is scrambling, then (Ω_σ, S) has a unique shift invariant probability measure μ. Let us recall that a non-negative matrix $M = (M_{ij})_{i,j\ge 0}$ is said to be *scrambling* if there exists $a > 0$ such that

$$\sum_{j=0}^{+\infty} \min(M_{ij}, M_{kj}) \ge a \quad \text{for all } i \ne k \in \mathbb{Z}_+.$$

Scrambling stochastic infinite countable matrices are very important, since a stochastic matrix $P = (P_{ij})_{i,j\ge 0}$ is strongly ergodic (see Definition 2.12) if and only if a power of P is scrambling.

We also consider the case where σ is not a constant length substitution. We introduce the notions of strongly ergodic and $\star$-strongly ergodic matrices M_σ related to the convergence of

$$\frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n}, \quad i,\ j \in A,$$

as $n \to \infty$. Then we show that if M_σ has a right Perron eigenvector in l^∞ and has a power that is scrambling (M_σ strongly ergodic), then (Ω_σ, S) is minimal and has a unique shift invariant probability measure μ.

A difference concerning substitutions on countable infinite alphabets that we should point out is that substitutions may not have a periodic point. In this paper, we consider M_σ irreducible and suppose the existence of a periodic point u, thus Ω_σ is the closure of the orbit of any periodic point of σ. However, our results will remain valid for σ that have no periodic point, since instead of using the left determined condition, we use the true fact that any finite word V occurring in some element of Ω_σ has a decomposition (not necessarily unique) as $V = v_0\sigma(Z)w_0$ where v_0, w_0, and Z finite words occurring in some elements of Ω_σ and $\max(|v_0|, |w_0|) \leq \sup\{|\sigma(a)|,\ a \in A\}$, where for all finite word $z \in A^*$, $|z|$ denotes the length of z.

The paper is organized as follows. In §2, we give notation, definitions, and preliminary results. Section 3 is devoted to the main results of the paper.

2. *Preliminaries and notation*

As in §1, let A be a countable set (called an alphabet), A^* be the set of finite words on A, and $A^{\mathbb{Z}_+}$ the set of infinite words on A. We denote a finite word on A by $u_0 \ldots u_{n-1}$ for some $n \geq 1$ and we call $n = |u_0 \cdots u_{n-1}|$ its length. An infinite word on A will be denoted by $u = u_0u_1 \ldots$. For $U = u_0 \ldots u_{n-1}$ and $V = v_0 \ldots v_{m-1}$ in A^*, where $n \geq m$ are positive integers, we denote

$$|U|_V = \{0 \leq k \leq n-m, u_k \ldots u_{k+m-1} = v_0 \ldots v_{m-1}\},$$

which is the number of occurrences of V in U. Let $u = u_0u_1 \ldots \in A^{\mathbb{Z}_+}$ and $V \in A^*$. We say that V occurs in u or V is a factor of u if $V = u_k \ldots u_l$ for some integer $0 \leq k \leq l$. We denote by F_u the set of all factors of u.

On $A^{\mathbb{Z}_+}$, we consider the discrete product topology, which is metrizable and generated by the metric d defined on $A^{\mathbb{Z}_+}$ by

$$d(u_0u_1 \ldots, v_0v_1 \ldots) = 0 \quad \text{if } u_0u_1 \ldots = v_0v_1 \ldots$$

and

$$d(u_0u_1 \ldots, v_0v_1 \ldots) = \frac{1}{2^{k_0}} \quad \text{where } k_0 = \min\{i \geq 0,\ u_i \neq v_i\} \text{ otherwise.}$$

A base for the discrete product topology is given by the cylinders

$$[w] = \{u_0u_1 \ldots \in A^{\mathbb{Z}_+},\ u_i = w_i \text{ for all } 0 \leq i \leq k\},$$

for $w = w_0 \dots w_k \in A^*$. The cylinders are clopen sets. When the alphabet A is finite, the set $A^{\mathbb{Z}_+}$ is compact and is homeomorphic to a Cantor set. If A is infinite, $A^{\mathbb{Z}_+}$ is closed but not compact.

Let $\sigma : A \to A^*$ be a substitution. We will assume without loss of generality that $A = \mathbb{Z}_+$ (and occasionally $A = \mathbb{Z}$ in some examples). We define the infinite matrix $M_\sigma = (M_{ij})_{i,j \in \mathbb{Z}_+}$ by $M_{ij} = |\sigma(i)|_j$. Observe that M_σ is the transpose of the substitution matrix given in [19]. It is easy to prove by induction that for all $i, j \in A$ and for all integers $n \in \mathbb{N}$,

$$|\sigma^n(i)|_j = M^n_{ij}, \quad |\sigma^n(i)| = \sum_{j=1}^{\infty} M^n_{ij}.$$

For example, if $\sigma(n) = 0(n+1)$ for all $n \in \mathbb{Z}_+$, then

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.1}$$

We say that a substitution $\sigma : A \to A^*$ is of *constant length* (respectively *bounded length*) if there exists an integer $L \geq 1$ such that $|\sigma(a)| = L$ (respectively $|\sigma(a)| \leq L$) for all $a \in A$.

Observe that if σ has constant length L (respectively bounded length by L), then the sum of the coefficients of each line of the matrix M^n_σ, $n \in \mathbb{N}$ equals L^n (respectively $\leq L^n$).

In this paper, we will assume that σ is a bounded length substitution and there exists $a \in A$ such that $|\sigma^n(a)|$ tends to infinity as n converges to infinity.

We define *the language* of a substitution σ on A as the set F_σ of finite factors of $\sigma^n(a)$ for some integer $n \geq 0$ and $a \in A$.

We will need some classical definitions from the theory of countable non-negative matrices, see [9, 20].

Definition 2.1. Let $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ be an infinite non-negative matrix (not necessarily a substitution matrix). We say that M is *irreducible* if for all $i, j \in \mathbb{Z}_+$, there exists an integer $k = k(i, j) \geq 1$ such that $M^k_{ij} > 0$. Let $i \in \mathbb{Z}_+$. The number

$$p_i = \gcd\{n \in \mathbb{N},\ M^n_{ii} > 0\}$$

is called the *period* of the state i. If M is irreducible, then there exists $p \geq 1$ such that $p_i = p$ for every $i \in \mathbb{Z}_+$ and we say that M has *period* $p \geq 1$. We say that an irreducible matrix M is *aperiodic* if $p = 1$ and *periodic* otherwise.

Observe that M_σ in equation (2.1) is irreducible and aperiodic, and σ is a constant length substitution which has a fixed point $u = \lim_{n\to\infty} \sigma^n(0)$ since $\sigma(0) = 01$ begins with 0.

Remark 2.2. (See [9]) If a matrix $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ is irreducible and aperiodic, then for all $i, j \in \mathbb{Z}_+$, there exists an integer $n = n(i, j) \geq 1$ such that for all $k \geq n$, $M^k_{ij} > 0$.

Remark 2.3. Let $\sigma : A \to A^*$ be a substitution which has a fixed point and M_σ is irreducible. Since there exists $i \in \mathbb{Z}_+$ such that $M_{ii} > 0$, we deduce that M_σ is aperiodic.

Assume that $M = (M_{ij})_{i,j\in\mathbb{Z}_+}$ is an irreducible and aperiodic non-negative matrix until the end of this section. It is known (see [**19**, **21**]) that there exists $\lambda_M \in [0, \infty]$, called the Perron value of M, such that for all $i, j \in \mathbb{Z}_+$,

$$\lim_{n\to\infty} (M_{ij}^n)^{1/n} = \lambda_M. \tag{2.2}$$

For all $i, j \in \mathbb{Z}_+$, put as usual $M_{ij}^0 = \delta_{ij}$, then consider the series

$$\overline{M}_{ij}(z) = \sum_{n=0}^{+\infty} M_{ij}^n z^n, \quad z \in \mathbb{C}.$$

Observe that the convergence radius of the series $\overline{M}_{ij}(z)$ is equal to λ_M^{-1}. When there is no possibility of confusion, we will omit the subscript in λ_M and write simply λ.

Remark 2.4. Directly from the definition, if $\hat{M} = CM$ for some $C > 0$, then $\lambda_{\hat{M}} = C\lambda_M$. If σ is a substitution with constant length L, then $P = M/L$ is a stochastic matrix and $\lambda_M = L\lambda_P$. Moreover, for the stochastic matrix P, clearly $\lambda_P \le 1$ and if $\overline{P}_{ij}(1) = +\infty$, then $\lambda_P = 1$. Thus, $\lambda_M \le L$. Indeed it is enough to have σ with bounded length L, see Lemma 2.10.

We either have $\overline{M}_{ij}(1/\lambda) < \infty$ for every $i, j \in \mathbb{Z}_+$, in this case, we say that M is *transient*, or $\overline{M}_{ij}(1/\lambda) = \infty$ for every $i, j \in \mathbb{Z}_+$, and we say that M is *recurrent*. The class of irreducible, aperiodic recurrent matrices can be divided into two classes: positive recurrent matrices and null recurrent ones. To present the definitions, we need to introduce the series

$$L_{ij}(M, z) = L_{ij}(z) = \sum_{n=0}^{+\infty} l_{ij}(M, n) z^n,$$

where $l_{ij}(M, n) = l_{i,j}(n)$ is defined by: $l_{ij}(0) = 0,\ l_{ij}(1) = M_{ij}$ and

$$l_{ij}(n+1) = \sum_{s\neq i}^{+\infty} l_{is}(n) M_{sj} \quad \text{for all } n \ge 1.$$

The matrix M is said to be *positive recurrent* if

$$\sum_{n=0}^{+\infty} \frac{n l_{ii}(n)}{\lambda^n} < +\infty,$$

otherwise we say that M is null recurrent.

An interesting result is that if M is an irreducible, aperiodic, and recurrent matrix with finite Perron value $\lambda > 0$, then λ has strictly positive left and right eigenvectors l and r, unique up to multiples by a constant. Moreover, the scalar product $l \cdot r$ is finite if and only if M is positive recurrent.

Remark 2.5. In §3.2, we will give examples of null recurrent non-negative matrices with constant length L having Perron value strictly smaller than L. These cases are associated to stochastic matrices with Perron value strictly smaller than 1, so they are transient in probabilistic sense (see [5]), but they might be null recurrent according to the above definition. This is not a novelty, see [9]. What is important here is also that we provide substitution matrices in our examples.

To state the next result, we still need to introduce another important series

$$R_{ij}(M, z) = R_{ij}(z) = \sum_{n=0}^{+\infty} r_{ij}(M, n)z^n,$$

where $r_{ij}(M, n) = r_{ij}(n)$ is defined by $r_{ij}(0) = 0,\ r_{ij}(1) = M_{i,j}$ and

$$r_{ij}(n+1) = \sum_{s\neq j}^{+\infty} M_{is}r_{sj}(n) \quad \text{for all } n \geq 1.$$

LEMMA 2.6. (See [22] and [9, p. 211]) *Let M be a non-negative, irreducible, and aperiodic matrix, with finite Perron value $\lambda > 0$. Let $i, j \in \mathbb{Z}_+$.*

(1) *If M is positive recurrent, then*

$$\lim_{n\to\infty} \frac{M_{ij}^n}{\lambda^n} = \frac{L_{ij}(1/\lambda)}{\mu(i)} = \frac{R_{ij}(1/\lambda)}{\mu(j)} > 0,$$

where $\mu(i) = \sum_{n=1}^{+\infty} nl_{ii}(n)/\lambda^n$.

(2) *If M is transient or null recurrent, then $\lim_{n\to\infty} M_{ij}^n/\lambda^n = 0$.*

For all $i, j \in \mathbb{Z}_+$, let

$$l^{(i)} = (L_{ik}(1/\lambda))_{k\geq 0} \quad \text{and} \quad r^{(j)} = (R_{sj}(1/\lambda))_{s\geq 0}.$$

LEMMA 2.7. (See [9, p. 203]) *Let M be a non-negative, irreducible, aperiodic matrix, with finite Perron value $\lambda > 0$.*

(1) *If M is recurrent, then for all $i, j \in \mathbb{Z}_+$,*

$$l^{(i)}M = \lambda l^{(i)} \quad \textit{and} \quad Mr^{(j)} = \lambda r^{(j)}.$$

(2) *If M is transient, then for all $i, j \in \mathbb{Z}_+$,*

$$l^{(i)}M \leq \lambda l^{(i)} \quad \textit{and} \quad Mr^{(j)} \leq \lambda r^{(j)}.$$

Remark 2.8. Let $M = (M_{ij})_{i,j\geq 0}$ be a non-negative, irreducible, aperiodic positive recurrent matrix, with finite Perron value $\lambda > 0$. By item (1) of Lemma 2.6 and item (1) of Lemma 2.7, the vector

$$T_i = (t_{ij})_{j\geq 0} \quad \text{where } t_{ij} = \lim_{n\to\infty} M_{ij}^n/\lambda^n \tag{2.3}$$

is a left eigenvector for λ associated to M. Moreover, we have

$$\lim_{n\to\infty} \frac{M_{i,j}^n}{\lambda^n} = \frac{l_j r_i}{\sum_{k=0}^{+\infty} l_k r_k} \tag{2.4}$$

where $l = (l_k)_{k\geq 0}$ and $r = (r_k)_{k\geq 0}$ are respectively a left and a right Perron eigenvector of M (see [20]).

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LEMMA 2.9. (See [9, Proposition 7.1.11, p. 204]) *Let $M = (M_{ij})_{i,j\geq 0}$ be a non-negative, irreducible, aperiodic, and recurrent matrix with finite Perron value λ. Let $Z = (z_i)_{i\geq 0}$ be a sub invariant non-negative and non-zero eigenvector of M_σ associated to λ, that is, $(ZM)_i \leq \lambda z_i$ for all $i \geq 0$ and $Z \neq 0$, then Z is a left Perron eigenvector associated to M.*

LEMMA 2.10. *Let $M = (M_{ij})_{i,j\in\mathbb{Z}_+}$ be a non-negative, irreducible, and aperiodic matrix with finite Perron value λ. The following results hold.*

(1) *If M has line sums uniformly bounded by $L > 0$, then $\lambda \leq L$.*

(2) *If M is positive recurrent and has constant line sums equal to L, then $\lambda = L$. Moreover, L is the unique eigenvalue of M having non-negative probability left eigenvector.*

Proof. (1) Suppose that M has line sums bounded by L, then for all integers $j \geq 0$ and $n \geq 1$, we have

$$M^n_{jj} \leq \sum_{k=0}^{+\infty} M^n_{jk} \leq L^n.$$

We deduce by equation (2.2) that $\lambda \leq L$.

(2) If M is positive recurrent and has constant line sums equal to L, then there exists $l = (l_i)_{i\geq 0} \in l^1$ such that $\sum_{i=0}^{\infty} l_i = 1$ and $lM = \lambda l$, then $\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} l_i M_{ij} = \lambda$, then $L = \lambda$. Using the same idea, we obtain that L is the unique eigenvalue of M having non-negative probability left eigenvector. □

Definition 2.11. Let $M = (M_{ij})_{i,j\geq 0}$ be a non-negative matrix. We say that M is scrambling if there exists $a > 0$ such that

$$\sum_{j=0}^{+\infty} \min(M_{ij}, M_{kj}) \geq a \quad \text{for all } i \neq k \in \mathbb{Z}_+.$$

Note that a substitution matrix M_σ has a power that is scrambling if and only if for some $n \geq 1$,

$$\text{for all } i, k \in A, \quad \text{there exists } j \in A \text{ which occurs in } \sigma^n(i) \text{ and } \sigma^n(k). \tag{2.5}$$

Definition 2.12. Let $P = (P_{ij})_{i,j\geq 0}$ be a non-negative stochastic matrix. We say that P is:

- *ergodic* if $\lim_{n\to\infty} P^n_{ij} = \pi_j > 0$ for all $i, j \in \mathbb{N}$, where $(\pi_j)_{j\geq 0}$ is a probability vector;
- *strongly ergodic* if P is ergodic and if there exists a probability vector $(\pi_i)_{i\geq 0}$ of non-negative real numbers such that $\lim_{n\to\infty} \|P^n - Q\|_s = 0$, where Q is the infinite stochastic matrix with rows equal to $(\pi_j)_{j\geq 0}$ and $\|N\|_s = sup_{i\geq 0} \sum_{j=0}^{+\infty} |N_{ij}|$ for any infinite complex matrix $N = (N_{ij})_{i,j\geq 0}$. In other words,

$$\lim_{n\to\infty} \sup_{i\geq 0} \sum_{j=0}^{\infty} |P^n_{ij} - \pi_j| = 0.$$

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Remark 2.13. It was proved in [7] that if P is strongly ergodic, then P is uniformly geometrically ergodic, that is, there exist $\beta \in (0, 1)$ and a constant $C > 0$ such that

$$|P_{ij}^n - \pi_j| \leq C\beta^n \quad \text{for all } i, j, n \in \mathbb{Z}_+.$$

The converse is proved in [12]. In particular, it is shown that P is strongly ergodic if and only if for some $j \geq 0$ with $\pi_j \geq 0$, we have

$$\lim_{n\to\infty} \sup_{i\geq 0} |P_{ij}^n - \pi_j| = 0. \tag{2.6}$$

There is a nice characterization of the strong ergodicity (see [8]). It is defined as follows. If $P = (P_{ij})_{i,j\in\mathbb{N}}$ is a stochastic non-negative countable matrix, then P is strongly ergodic if and only if there exists an integer $n \geq 1$ such that $\delta(P^n) < 1$, where the δ coefficient of any non-negative countable stochastic matrix $N = (N_{ij})_{i,j\in\mathbb{N}}$ is

$$\delta(N) = \frac{1}{2} \sup_{i,k\in\mathbb{N}} \sum_{j=0}^{+\infty} |N_{ij} - N_{kj}|. \tag{2.7}$$

The number $\delta(N)$ is called Dobrushin coefficient of N or coefficient of ergodicity of N (see for instance [3, 7, 11, 12]). It is not difficult to show that

$$\delta(N) = 1 - \inf_{i\neq k} \sum_{j=0}^{+\infty} \min(N_{ij}, N_{kj}). \tag{2.8}$$

Observe that $\delta(N) < 1$ if and only if N is scrambling. Hence, P is strongly ergodic if and only if there exists an integer $n \geq 1$, such that P^n is scrambling.

3. *Irreducible aperiodic substitutions*

3.1. *Non-existence of finite invariant measure.*

In [6], the author proved that if $A = \mathbb{Z}$ and $\sigma(n) = (n-1)nn(n+1)$, $n \in A$, then the dynamical system (Ω_σ, S) has no finite invariant measure. We will extend this result in the next theorem.

THEOREM 3.1. *Let $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+^*$ be a bounded length substitution such that σ has a periodic point u and $M = M_\sigma$ is irreducible and aperiodic. If M satisfies*

$$\lim_{n\to+\infty} \sup_{i\in\mathbb{Z}_+} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \quad \textit{for all } j \in \mathbb{Z}_+, \tag{3.1}$$

then the dynamical system (Ω_σ, S) has no finite invariant measure.

Remark 3.2. One natural question is if the condition in equation (3.1) can be replaced by the weaker condition

$$\lim_{n\to+\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \quad \text{for all } i, \ j \in \mathbb{Z}_+. \tag{3.2}$$

This last condition is more natural and holds for a large class of substitutions σ such that M_σ is transient or null recurrent and σ has constant length, or M_σ is positive recurrent with left Perron eigenvector $l = (l_k)_{k\geq 0} \notin l^1$, see Lemma 3.4 at the end of this section and also Remark 3.3 just below.

Proof of Theorem 3.1. Assume without loss of generality that $u = u_0u_1 \ldots$ is a fixed point of σ. By equation (3.1), we have that for all $j \in \mathbb{Z}_+$,

$$\lim_{n\to+\infty} \sup_{a\in A} \frac{|\sigma^n(a)|_j}{|\sigma^n(a)|} = 0. \tag{3.3}$$

Now, assume that (Ω_σ, S) has a finite invariant measure, then there exists a finite ergodic invariant measure μ. By Birkhoff's ergodic theorem, we deduce that for μ almost all $x \in \Omega_u$,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{0 \le k \le N-1 : \ S^k(x) \in [j]\} = \mu[j] \quad \text{for all } \ j \in \mathbb{Z}_+. \tag{3.4}$$

Now, let $x \in \Omega_\sigma$ satisfying equation (3.4) and $N \in \mathbb{N}$. Let $V = u_m \ldots u_{m+N-1}$, $m \in \mathbb{N}$ be a prefix of x. The word V can be written as

$$V = v_0\sigma(v_1)\ldots\sigma^{n-1}(v_{n-1})\sigma^n(v_n)\sigma^{n-1}(w_{n-1})\ldots\sigma(w_1)w_0, \tag{3.5}$$

where $n \ge 1$ is an integer and v_i, $i \in \{0, \ldots, n\}$, w_j, $j \in \{0, \ldots, n-1\}$ are elements of F_u possibly empty words of lengths smaller or equal to $K = \max\{|\sigma(b)|, \ b \in A\}$ and v_n is not empty. Equation (3.5) comes from the fact that since $u = \sigma(u)$, there exists $a \in A$ and $n \in \mathbb{N}$ such that V is a factor of $\sigma^{n+1}(a)$ and V is not a factor of $\sigma^n(a)$. Hence, there exist v_0, w_0, V_1 in F_u such that

$$V = v_0\sigma(V_1)w_0$$

and $|v_0|, |w_0| \le K$. We proceed analogously with V_1, continuing by induction until the process stops and we obtain equation (3.5).

With our choice of x and its prefix V, from equations (3.4) and (3.5), we have that

$$\begin{aligned}
&\frac{1}{N}\operatorname{card}\{0 \le k \le N-1, S^k(x) \in [j]\} \\
&\quad = \frac{|V|_j}{|V|} = \frac{|\sigma^n(v_n)|_j + \sum_{k=0}^{n-1}(|\sigma^k(v_k)|_j + |\sigma^k(w_k)|_j)}{|\sigma^n(v_n)| + \sum_{k=0}^{n-1}(|\sigma^k(v_k)| + |\sigma^k(w_k)|)}.
\end{aligned}$$

By equation (3.3), we deduce that

$$\lim_{k\to\infty} \sup \left\{ \frac{|\sigma^k(v)|_j}{|\sigma^k(v)|}, \ v \in F_u, \ |v| \le K \right\} = 0. \tag{3.6}$$

Using equation (3.6) and the Stolz–Cesaro theorem, we deduce that

$$\lim_{n\to\infty} \frac{|\sigma^n(v_n)|_j + \sum_{k=0}^{n-1}(|\sigma^k(v_k)|_j + |\sigma^k(w_k)|_j)}{|\sigma^n(v_n)| + \sum_{k=0}^{n-1}(|\sigma^k(v_k)| + |\sigma^k(w_k)|)} = 0.$$

Therefore,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{0 \le k \le N-1, S^k(x) \in [j]\} = \mu[j] = 0.$$

Since j is arbitrary, $\mu(\Omega) = 0$, which yields a contradiction. □

Remark 3.3. It is important to notice that the condition in equation (3.1) may or may not hold on both the transient and the null recurrent cases. To see this, we consider examples where M is a multiple of an irreducible stochastic matrix P. In this situation, equation (3.1) is equivalent to

$$\lim_{n\to+\infty} \sup_{i\in A} P_{ij}^n = 0 \quad \text{for all } j \in A. \tag{3.7}$$

It is simple to find examples of stochastic matrices for which equation (3.7) does not hold. So we start with a first example that can be adapted to both transient and recurrent cases. Consider $A = \mathbb{Z}$ and set $P_{-n,-n-1} = q_n = 1 - P_{-n,-n+1}$ for $n \geq 1$, where $q_n \in (0, 1)$ and $\sum_{n=1}^{+\infty} q_n < \infty$. Also put $P_{0,-1} = P_{0,1} = 1/2$ and $P_{m,-n} = 0$ for $m, n \geq 1$. No matter how we complete the definition of P to obtain a irreducible and aperiodic matrix which may be recurrent or transient, we have that

$$\sup_{a\in A} P_{a,0}^n \geq P_{-n,0}^n \geq \prod_{k=1}^{+\infty}(1 - q_k) > 0 \quad \text{for every } n \geq 1.$$

Thus, equation (3.7) does not hold. However, since $\lim_{n\to\infty} q_n = 0$, there is no multiple of P which is a matrix M associated to a substitution. Thus, we will provide another example.

Again we consider $A = \mathbb{Z}$ and set $P_{-2^n,0} = 1/2 = P_{-2^n,-2^n-1}$ and $P_{-2^n-j,-2^n-j-1} = 1$ for $j = 1, \ldots, 2^n - 1$ and $n \geq 1$. We can check that $P^{2^n}_{-2^{n-1}-1,0} = 1/2$. Again, no matter how we complete the definition of P, which may be recurrent or transient, we have that

$$\limsup_{n\to\infty} \sup_{a\in A} P_{a,0}^n \geq 1/2 > 0,$$

thus equation (3.7) does not hold. In this case, we could define $M_{-2^n,0} = 1 = M_{-2^n,-2^n-1}$ and $M_{-2^n-j,-2^n-j-1} = 2$ and complete the definition for the other entries for M to have an irreducible and aperiodic matrix associated to a substitution of constant length equal to two. We have that $P = M/2$, thus equation (3.1) does not hold.

As a third example, we consider P as the transition matrix of a simple random walk on $\mathbb{Z}$, that is, we fix $p \in (0, 1)$ and set $P_{n,n+1} = p = 1 - P_{n,n-1}$ for every $n \in \mathbb{Z}$ (for basic properties of random walks, the reader can check [5]). Notice that this Markov chain is irreducible with period two which is null recurrent if $p = 1/2$ and transient otherwise. The stochastic matrix P is irreducible and we can use P^2 instead of P for an example with an aperiodic chain. A standard computation using the binomial distribution and Stirling formula shows that

$$\sup_{w\in\mathbb{Z}} P_{w,\tilde{w}}^n = \sup_{w\in\mathbb{Z}} P_{0,\tilde{w}-w}^n \leq \max_{0\leq k\leq n} \binom{n}{k} p^k(1-p)^{n-k} = O(n^{-1/2}).$$

Thus, equation (3.7) holds. Here, we also have $P = M/2$, where M is a substitution matrix of constant length equal to 2 defined as

$$M_{n,n+1} = M_{n,n-1} = 1 \quad \text{for all } n \in \mathbb{Z}.$$

Thus, M satisfies equation (3.1).

Question 3.1. Under the hypothesis of Theorem 3.1, is the dynamical system (Ω_σ, S) not minimal?

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Question 3.2. Is the result of Theorem 3.1 still true if M_σ is transient, or recurrent with a left Perron eigenvector $l = (l_i)_{i \geq 0} \notin l^1$ and without the condition in equation (3.1)? Even in a little less general setting, is the result of Theorem 3.1 still true if M_σ satisfies the weaker condition in equation (3.2)?

We finish this section proving a result with conditions that imply the condition in equation (3.2).

LEMMA 3.4. *Let $M = (M_{ij})_{i,j \in \mathbb{Z}_+}$ be a non-negative, irreducible, and aperiodic matrix with finite Perron value λ. If M is transient with constant line sums, or M is positive recurrent with a left Perron eigenvector $l = (l_k)_{k \geq 0} \notin l^1$, then*

$$\lim_{n \to +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \quad \textit{for all } i, j \in \mathbb{Z}_+.$$

Proof. Assume that M is transient with constant line sums equal to L. Let $i, j \in \mathbb{Z}_+$. Since $\lambda \leq L$ and $\lim_{n \to +\infty} M_{ij}^n / \lambda^n = 0$, then

$$\frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = \frac{M_{ij}^n}{L^n} \leq \frac{M_{ij}^n}{\lambda^n} \to 0 \quad \text{as } n \to \infty.$$

Now, let us suppose that M is positive recurrent and $l = (l_k)_{k \geq 0} \notin l^1$ is a left Perron eigenvector. Let $i, j \in \mathbb{Z}_+$. Since M is positive recurrent, we have by Remark 2.8 that

$$\lim_{n \to \infty} \frac{M_{ik}^n}{\lambda^n} = c l_k \quad \text{for all } k \in \mathbb{Z},$$

where $c > 0$. Using the Fatou lemma for series and the fact $l = (l_k)_{k \geq 0} \notin l^1$, we deduce that

$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{+\infty} M_{ik}^n}{M_{ij}^n} \geq \frac{\sum_{k=0}^{+\infty} l_k}{l_j} = +\infty$$

and we are done. □

Question 3.3. (1) If M is transient with non-constant line sums, is

$$\lim_{n \to +\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = 0 \quad \text{for all } i, j \in \mathbb{Z}_+?$$

Note that the answer is affirmative if

$$\liminf_{n \to +\infty} \frac{M^n \mathbf{1}}{\lambda^n} > 0,$$

which is simple to verify in the finite dimensional case from linear algebra arguments. It is also true to check in the infinite dimensional case when M is transient and has a right Perron eigenvector $r = (r_i)_{i \geq 0} \in l^\infty$ such that $\inf\{r_j, j \geq 0\} > 0$, since for all $j \geq 0$,

$$\frac{\inf_j r_j}{\sup_j r_j} \leq \frac{1}{\lambda^n} \sum_{k=0}^{+\infty} M_{ik}^n \leq \frac{\sup_j r_j}{\inf_j r_j}. \tag{3.8}$$

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(2) Assume that M is recurrent with a left Perron eigenvector $l = (l_k)_{k\geq 0} \in l^1$. Does there exist $i, j \in A$ such that

$$\lim_{n\to+\infty} \frac{M^n_{ij}}{\sum_{k=0}^{+\infty} M^n_{ik}} > 0?$$

Again, from item (2) in Lemma 2.6, item (2) in Lemma 2.10, and equation (3.3), it is simple to check that this holds when M is positive recurrent and has a right Perron eigenvector $r = (r_i)_{i\geq 0} \in l^\infty$ such that $\inf\{r_j, j \geq 0\} > 0$. In particular, in the case of lines with constant sums.

3.2. *A class of examples.* Let $\sigma := \sigma_{a,b,c}$ be defined by

$$\sigma(0) = 0^{a+b}1^c \quad \text{and} \quad \sigma(n) = (n-1)^a n^b (n+1)^c \quad \text{for all } n \geq 1,$$

where a, b, c are non-negative integers such that $a > 0$, $c > 0$, and $i^k = ii \ldots i$ (k times). The matrix M_σ is irreducible and aperiodic. We have

$$M_\sigma = \begin{bmatrix} a+b & c & 0 & 0 & 0 & 0 & 0 & \cdots \\ a & b & c & 0 & 0 & 0 & 0 & \cdots \\ 0 & a & b & c & 0 & 0 & 0 & \cdots \\ 0 & 0 & a & b & c & 0 & 0 & \cdots \\ 0 & 0 & 0 & a & b & c & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that σ is a substitution of constant length $L = a + b + c$. The stochastic matrix $P = M_\sigma/L$ is the transition matrix of a homogeneous nearest-neighbor random walk in $\{0, 1, 2, \ldots\}$ partially reflected at the boundary, see also the last example in Remark 3.3. It is well known, see [**5**], that the random walk is (in the probabilistic sense) positive recurrent if $c < a$, null recurrent if $c = a$, and transient if $c > a$. The difference for the matrix theoretical definition is that we also have null recurrence in the case $c > a$, see also [**9**, Example 7.1.28] for the case $b = 0$ and $a = c$.

PROPOSITION 3.5. *The following properties hold:*

- *if $c < a$, then M_σ is positive recurrent;*
- *if $c \geq a$, then M_σ is null recurrent and (Ω_σ, S) has no finite invariant measure.*

Proof. For the cases $c < a$ and $c = a$, we have $\lambda_P = 1$, thus $\lambda_{M_\sigma} = L$. From the probabilistic results on transience/recurrence of random walks, we have that M_σ is positive recurrent for $c < a$ and null recurrent for $c = a$.

Before we deal with the case $c > a$, let us point out that we can prove the result in the cases $c < a$ and $c = a$ by directly computing the Perron eigenvectors.

Let λ be the Perron value of M_σ, then by Lemma 2.10, we have $\lambda \leq L = a + b + c$. Let $l = (l_i)_{i\geq 0}$ be a left eigenvector of M_σ associated to L. A simple computation implies that $l_1 = c/al_0$ and

$$cl_n + al_{n+2} = (a + c)l_{n+1} \quad \text{for all } n \geq 0.$$

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Hence,

$$l_n = \left(\frac{c}{a}\right)^n l_0 \quad \text{for all } n \geq 1.$$

Assume that M_σ is positive recurrent, then by Lemma 2.10, we deduce that $\lambda = L$. Thus, $l \in l^1$ (since a right Perron eigenvector of M_σ has constant entries) and we deduce that $c < a$.

Now assume that $c \leq a$. If $\lambda = L$, then l is a left Perron eigenvector, and hence M_σ is positive recurrent if $c < a$ and null recurrent if $c = a$. Now suppose that $\lambda < L$ and let $u = (u_i)_{i\geq 0}$ be a non-zero non-negative left Perron sub-invariant eigenvector of M_σ associated to λ. Thus, $uM < LM$. Hence,

$$u_{n+1} \leq \frac{c}{a} u_n \quad \text{for all } n \geq 0,$$

and there exists a real number $s > 0$ and an integer $k \geq 1$ such that $u_k = (c/a)u_{k-1} - s$. Since $cu_{k-1} + au_{k+1} \leq (a + c)u_k$, we deduce that $u_{k+1} \leq (c/a)u_k - s$. Thus,

$$u_{n+1} \leq \frac{c}{a} u_n - s \quad \text{for all } n \geq k. \tag{3.9}$$

Therefore,

$$u_n \leq \left(\frac{c}{a}\right)^{n-k} u_k - s \quad \text{for all } n \geq k + 1.$$

If $c < a$, we deduce that there exists a positive integer N such that $u_n < 0$ for all integers $n \geq N$. This is absurd, then $u = u_0 l$. Therefore, $\lambda = L$ and hence M_σ is positive recurrent.

If $c = a$, we deduce by equation (3.9) that

$$u_n \leq u_k - (n - k)s \quad \text{for all } n \geq k + 1.$$

Then $\lambda = L$ and M_σ is null recurrent.

Now consider the case $c > a$. We will consider a probabilistic approach to show that $\lambda_P < 1$ and that $\overline{P}_{00}(1/\lambda_P) = \infty$, this implies null recurrence. Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P and $\mathbb{P}^x$ the distribution of $(X_n)_{n\geq 0}$, when $X_0 = x$ for $x \in \mathbb{Z}_+$. Set $p = c/(a + c)$, which is the conditional probability that the random walk jumps to the right when it necessarily leaves its current position and this is not 0, that is,

$$p = \mathbb{P}^x\big(X_{n+1} = X_n + 1 \big| X_{n+1} \neq X_n\big) \quad \text{for all} \;\; x \neq 0.$$

We want to estimate P^n_{00}, that is, the probability that the random walk is visiting state 0 at time n given that it has also started at 0. For this last event to happen, necessarily, we must have a number of jumps to the right equal to the number of jumps to the left. Here we need two important observations.

(i) Note that $\#\{1 \leq j \leq n : \;X_{n+1} \neq X_n\}$ counts the total number of jumps to the right or to the left. There exist strictly positive constants $c, \delta \in (0, 1)$ such that

$$\mathbb{P}^0\big(\#\{1 \leq j \leq n : \;X_{n+1} \neq X_n\} \geq cn\big) > 1 - (\delta q)^n.$$

(ii) In $2k$ transitions to the left or to the right, the number of transitions to the right is distributed as a binomial random variable with parameters $2k$ and p. Thus, the probability of having an equal number of jumps to the left or to the right is

$$\mathbb{P}\big(Bin(2k,p)=k\big)=\binom{2k}{k}p^k(1-p)^k\approx\frac{C}{\sqrt{k}}\big(4p(1-p)\big)^k$$

(the approximation could be appropriately described using Stirling's formula). Note that $q=4p(1-p)<1$.

Using observations (i) and (ii), we are able to show that P_{00}^n is of order $O(q^n/\sqrt{n})$. This implies that $\lambda_P=q$ and $\overline{P}_{00}(1/\lambda_P)=\overline{P}_{00}(1/q)=\infty$. Therefore, P and M_σ are null recurrent matrices.

It is worth mentioning that M_σ satisfies equation (3.1) for every $a\leq c$ and b. This follows as in the last example in Remark 3.3, in the case $a=c$, and from computation as in the proof of Proposition 3.5. Indeed, one can prove that $\sup_{i\in\mathbb{Z}_+}P_{i,j}^n$ is of order $O(1/\sqrt{n})$, which implies that M_σ satisfies equation (3.1). This can also be proved using the local central limit theorem for simple random walks [10, Theorem 1.2.1]. □

Remark 3.6. It is worth mentioning that apparently small modifications on the matrix can completely change its behavior. For instance, consider the case $b=0$ and $a=c$ which implies that M_σ is null recurrent. Instead of $\sigma(0)=0^a1^c$, put $\sigma(0)=1^c$, then, from [9, (i) in Example 7.1.29], we have that M_σ is transient. For the case $b>0$, $a\leq c$, and $\sigma(0)=1^c$, we also have transience as a consequence of our Proposition 3.5 and [9, Lemma 7.1.23].

Remark 3.7. We consider the substitution σ of [6] defined on $A=\mathbb{Z}$ by $\sigma(n)=(n-1)nn(n+1)$. The associated matrix is null recurrent and satisfies the condition in equation (3.1). Hence, by using Theorem 3.1, we deduce that the dynamical system (Ω_σ,S) associated to σ has non-finite invariant measure.

Let $\sigma:=\sigma_{a_n,b_n,c_n}$ be defined by

$$\sigma(0)=0^{a_0+b_0}1^{c_0}\quad\text{and}\quad\sigma(n)=(n-1)^{a_n}n^{b_n}(n+1)^{c_n}\quad\text{for all } n\geq 1,$$

where a_n,b_n,c_n are non-negative integers such that $a_n>0$, $c_n>0$ for every $n\geq1$, and $L=\sup\{a_n+b_n+c_n : n\geq1\}<\infty$. The matrix M_σ is irreducible and aperiodic with bounded length L and can be represented as

$$M_\sigma=\begin{bmatrix} a_0+b_0 & c_0 & 0 & 0 & 0 & 0 & 0 & \cdots\\ a_1 & b_1 & c_1 & 0 & 0 & 0 & 0 & \cdots\\ 0 & a_2 & b_2 & c_2 & 0 & 0 & 0 & \cdots\\ 0 & 0 & a_3 & b_3 & c_3 & 0 & 0 & \cdots\\ 0 & 0 & 0 & a_4 & b_4 & c_4 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We will see in Proposition 3.8 below that (Ω_σ,S) is not minimal for these substitutions. We do not discuss the transience/recurrence in this general case, but we discuss an example. Consider $(a_n,b_n,c_n)=(2,1,1)$ for n even and $(a_n,b_n,c_n)=(1,1,1)$ otherwise. Our first step is to compute the Perron value λ. For this, we will estimate $(M_\sigma^n)_{0,0}$. Consider a matrix $\hat{M}_\sigma^n$ which has the form above but with $(a_n,b_n,c_n)=(2,1,1)$ for every n. The Perron eigenvalue of $\hat{M}_\sigma^n$ is 4, since it has constant row sums equal to 4. Now for each path of length n leaving and returning to n, we will have the number of jumps to the left equal to the number of jumps to the right. So for a total of $2m \leq n$ jumps with m jumps to the left and m jumps to right (neglecting jumps from 0 to 1 and jumps from an state to itself), $m/2$ jumps to the left are made from an odd position and these jumps contribute with a factor of $(\sqrt[4]{2})^{2m} = 2^{m/2}$ to the product of weights $(\hat{M}_\sigma^n)_{0,0}$. This shows that

$$\left(\frac{\sqrt[4]{2}}{4}\right)^n (M_\sigma^n)_{0,0} \geq \frac{(\hat{M}_\sigma^n)_{0,0}}{4^n},$$

which implies $\lambda \geq 4/\sqrt[4]{2}$. However, $(M_\sigma^n)_{0,0} \leq (\hat{M}_\sigma^n)_{0,0}$ and $\lambda \leq 4$. Thus, $\lambda \in [4/\sqrt[4]{2}, 4]$. With this bound on λ, we can show that M_σ is positive recurrence. For this, we follow the computation in [9, Example 7.1.29(iii)] to obtain that

$$\lim_{n\to\infty} \sqrt[2n]{\ell_{00}(2n)} = 2^{5/4} < \lambda,$$

then apply [9, Lemma 7.1.25] to conclude.

PROPOSITION 3.8. *Let (Ω_σ, S) be the shift dynamical system associated to σ_{a_n,b_n,c_n}, then it is not minimal.*

Proof. For all $n \geq 2$, we have

$$\sigma^{k-1}(k) = (\sigma^{k-2}(k-1))^{a_n}(\sigma^{k-2}(k))^{b_n}(\sigma^{k-2}(k+1))^{c_n}.$$

Hence, the infinite word w beginning with $\sigma^{k-1}(k)$ for all $k \geq 2$ is well defined and belongs to Ω_σ. Moreover, the letter 0 does not occur in w. Thus, the orbit of w does not visit the cylinder [0], and hence (Ω_σ, S) is not minimal. □

Remark 3.9. The last proposition gives examples of positive or null recurrent, aperiodic irreducible substitutions such that its shift dynamical systems are not minimal. The first example was given by Ferenczi in [6] by considering $\sigma(n) = (n-1)nn(n+1),\ n \in \mathbb{Z}$.

We will see in Theorem 3.33 that given a substitution σ on $A = \mathbb{Z}_+$, not necessarily with constant length such that σ has a periodic point u and M_σ is irreducible, aperiodic, and has a scrambling positive power, then (Ω_σ, S) is minimal. Observe that the matrices associated to substitutions σ_{a_n,b_n,c_n} do not have a scrambling power, since for any positive integer k, there is no letter occurring both in the words $\sigma^k_{a_n,b_n,c_n}(k)$ and $\sigma^k_{a_n,b_n,c_n}(4k)$.

To end this section, we describe an interesting substitution whose matrix is transient. The construction of the matrix is based on multidimensional random walks in dimension greater or equal to 3. Thus, we set $A = \mathbb{Z}^d$, let $\{e_j : 1 \leq j \leq d\}$, and define the substitution

$$\sigma(x) = (x+e_1)(x-e_1)(x+e_2)(x-e_2)\ldots(x+e_d)(x-e_d).$$

We have that M_σ is a matrix of length $2d$. The stochastic matrix $P = M_\sigma/2d$ is transient with $\lambda_P = 1$. Indeed, from classical results in probability theory, one has that $P_{00}^n \sim O(n^{-d/2})$ and $\overline{P}_{00}(1) < \infty$. Therefore, M_σ is a transient matrix with $\lambda_{M\sigma} = 2d$. Using again the local central limit theorem [10, Theorem 1.2.1], we have that M_σ also satisfies equation (3.1).

3.3. *Shift invariant measures and unique ergodicity.* In this subsection, we prove the following results.

THEOREM 3.10. *Let σ be a bounded length substitution on $A = \mathbb{Z}_+$ such that M_σ is irreducible, aperiodic, positive recurrent, then the dynamical system (Ω_σ, S) has a shift invariant measure μ which is finite if and only if any left Perron eigenvector l belongs to l^1.*

Remark 3.11. Theorem 3.10 improves [1, Theorem 7.6], where it is assumed the additional hypothesis where σ is a bounded size left determined substitution.

THEOREM 3.12. *Let σ be a constant length substitution on $A = \mathbb{Z}_+$ such that σ has a periodic point u and M_σ is irreducible and aperiodic. If there exists a positive integer n such that M_σ^n is scrambling, then there exists a unique probability shift invariant measure of (Ω_σ, S).*

Remark 3.13. The same proof of Theorem 3.12 will show that if σ is a constant length substitution on $A = \mathbb{Z}_+$ without periodic point such that M_σ is irreducible, aperiodic, and M_σ^n is scrambling positive integer n, then there exists a unique probability shift invariant measure of (Ω_σ, S).

Before proving Theorems 3.10 and 3.12, we need to introduce some notation and state some preliminary results.

Let $\sigma : A \to A^*$ be a bounded length substitution, not necessarily with constant length. Let $t \geq 2$ be an integer and A_t be the set of finite words of length t that occur in u. Now, consider a substitution σ_t on the alphabet A_t defined in the following way: if $w = w_0 \ldots w_{t-1} \in A_t$ and $\sigma(w) = y_0 \ldots y_{|\sigma(w_0)|-1} y_{|\sigma(w_0)|} \ldots y_{|\sigma(w)|-1}$, then

$$\sigma_t(w) = (y_0 \ldots y_{t-1})(y_1 \ldots y_t) \ldots (y_{|\sigma(w_0)|-1} \ldots y_{|\sigma(w_0)|+t-2}). \tag{3.10}$$

Considering that $|\sigma_t(w)|$ counts letters in A_t (not in A), note that

$$|\sigma_t(w_0 \ldots w_{t-1})| = |\sigma(w_0)|, \tag{3.11}$$

and for all $i_1 \ldots i_t \in A_t$, we have

$$|\sigma(w_0)|_{i_1 \ldots i_t} \leq |\sigma_t(w_0 \ldots w_{t-1})|_{i_1 \ldots i_t} \leq |\sigma(w_0)|_{i_1 \ldots i_t} + t. \tag{3.12}$$

We extend σ_t by concatenation to A_t^* and to $A_t^{\mathbb{Z}_+}$. The substitution σ_t was defined in [19] (in the case of substitutions on finite alphabets).

For example, for $A = \{0, 1\}$ and $\sigma(0) = 01$, $\sigma(1) = 0$. We have $A_2 = \{00, 01, 10\}$ and

$$\sigma_2(00) = (01)(10), \ \sigma_2(01) = (01)(10), \ \sigma_2(10) = (00).$$

If $A = \mathbb{Z}_+$ and $\tau(n) = 0(n+1)$ for all $n \in A$, then $A_2 = \{0n, \ n0, \ n \geq 1\}$ and

$$\tau_2(0n) = (01)(10), \ \tau_2(n0) = (0(n+1))((n+1)0) \quad \text{for all } n \geq 1.$$

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LEMMA 3.14. *The following results hold:*

(1) *for all integers $n \geq 1$ and $t \geq 2$, we have $(\sigma^n)_t = (\sigma_t)^n$;*

(2) *let $u = u_0 u_1 \ldots$ be a periodic point of σ, then for all integers $t \geq 2$, the infinite word $(u_0 \ldots u_{t-1}).(u_1 \ldots u_t) \ldots (u_i \ldots u_{t+i-1}) \ldots$ is a periodic point (with the same period) of σ_t;*

(3) *if M_σ is irreducible and aperiodic, then so is M_{σ_t} for all integers $t \geq 2$.*

Proof. The proof is analogous to that for the case of a finite alphabet, which is given in [19, pp. 138–139]. □

LEMMA 3.15. *Let $A = \mathbb{Z}_+$ and $\sigma : A \to A^*$ be a bounded length substitution such that M_σ irreducible and aperiodic with Perron value λ, then for all integers $t \geq 2$, the matrix $M_t = M_{\sigma_t}$ associated to σ_t also has Perron value λ. Moreover, if M_σ is positive recurrent (respectively null recurrent, transient), then M_{σ_t} is also positive recurrent (respectively null recurrent, transient).*

Proof. Let $t \geq 2$ be an integer and denote by λ_t the Perron value of M_t. First observe that by item (3) in Lemma 3.14, M_t is irreducible and aperiodic. For $i_1 \ldots i_t,\ j_1 \ldots j_t \in A_t$, we have

$$|\sigma_t^n(i_1 \ldots i_t)|_{j_1 \ldots j_t} \leq |\sigma^n(i_1)|_{j_1 \ldots j_t} + t \leq |\sigma^n(i_1)|_{j_1} + t.$$

Hence,

$$(M_t^n)_{i_1 \ldots i_t, j_1 \ldots j_t} \leq (M^n)_{i_1, j_1} + t \quad \text{for all } n \in \mathbb{N}. \tag{3.13}$$

We deduce by equation (2.2) that $1 \leq \lambda_t \leq \lambda$.

However, let $k, m \in \mathbb{N}$ such that $j_1 \ldots j_t$ is a factor of $\sigma^m(k)$. Hence,

$$|\sigma^{n+m}(i_1)|_{j_1 \ldots j_t} \geq |\sigma^n(i_1)|_k.$$

Thus, for all $n \in \mathbb{N}$, we have $|\sigma_t^{n+m}(i_1 \ldots i_t)|_{j_1 \ldots j_t} \geq |\sigma^n(i_1)|_k$. Therefore,

$$(M_t^{n+m})_{i_1 \ldots i_t, j_1 \ldots j_t} \geq (M^n)_{i_1, k} \quad \text{for all } n \in \mathbb{N}. \tag{3.14}$$

Thus, $\lambda_t \geq \lambda$ and hence $\lambda_t = \lambda$.

Assume that M_σ is positive recurrent. By equation (3.14), we have

$$\liminf_{n \to +\infty} \frac{(M_t^{n+m})_{i_1 \ldots i_t, j_1 \ldots j_t}}{\lambda^{n+m}} \geq \lambda^{-m} \lim_{n \to +\infty} \frac{(M^n)_{i_1, k}}{\lambda^n}. \tag{3.15}$$

Hence, by equation (3.15) and Lemma 2.6, we deduce that $\lim_{n \to +\infty}((M_t^n)_{i_1 \ldots i_t, j_1 \ldots j_t}/\lambda_t^n) > 0$. Thus, M_{σ_t} is positive recurrent.

Now suppose that M_σ is null recurrent, then we have by equation (3.14) that

$$\sum_{n=0}^{+\infty} \frac{(M_t^n)_{i_1 \ldots i_t, j_1 \ldots j_t}}{\lambda^n} = +\infty. \tag{3.16}$$

Hence, by equation (3.13), we deduce that

$$\lim_{n \to \infty} \frac{(M_t^n)_{i_1 \ldots i_t, j_1 \ldots j_t}}{\lambda^n} = 0. \tag{3.17}$$

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By equations (3.16) and (3.17), we deduce that M_{σ_t} is null recurrent.

Finally, if M_σ is transient, we deduce by equation (3.13) that

$$\sum_{n=0}^{+\infty} \frac{(M_t^n)_{i_1\ldots i_t, j_1\ldots j_t}}{\lambda^n} < +\infty.$$

Hence, M_{σ_t} is transient. □

Before proving Theorem 3.10, we need the following lemma.

LEMMA 3.16. *Let σ be a bounded length substitution on $A = \mathbb{Z}_+$ such that M_σ is irreducible, aperiodic, recurrent, and has finite Perron value λ. Let $r = (r_i)_{i\geq 0}$ be a right Perron eigenvector of M_σ. For all integers $t \geq 2$, let $r^{(t)} = (r_I)_{I\in A_t}$ be an infinite vector defined by*

$$r_I = r_{i_0} \quad \textit{for all } I = i_0 \ldots i_{t-1} \in A_t,$$

then $r^{(t)}$ is a right Perron eigenvector of $M_t = M_{\sigma_t}$ associated to λ.

Proof. Let $I = i_0 \ldots i_{t-1} \in A_t$. We have

$$(M_t r^{(t)})_I = \sum_{J=j_0\ldots j_{t-1}\in A_t} |\sigma_t(I)|_J \, r_{j_0} = \sum_{j_0\in A} r_{j_0} \sum_{J^*=j_1\ldots j_{t-1}, j_0 J^*\in A_t} |\sigma_t(I)|_{j_0 J^*}.$$

However, for all $j_0 \in A$, we have

$$\sum_{J^*=j_1\ldots j_{t-1}, j_0 J^*\in A_t} |\sigma_t(I)|_{j_0 J^*} \leq |\sigma(i_0)|_{j_0} = M_{i_0 j_0}.$$

Thus,

$$(M_t r^{(t)})_I \leq \sum_{j_0\in A} M_{i_0 j_0} r_{j_0} = \lambda r_{i_0} = \lambda (r^{(t)})_I.$$

Since M_t is an aperiodic, irreducible, and recurrent matrix, Lemma 2.9 implies that $r^{(t)}$ is a right eigenvector of M_t associated to λ. □

Proof of Theorem 3.10. Let $u = u_0 u_1 \ldots = \sigma(u)$ be an element of Ω_σ. For $j \in A$, set

$$\mu[j] := \lim_{n\to\infty} \frac{|\sigma^n(u_0)|_j}{\lambda^n} = \lim_{n\to\infty} \frac{M^n_{u_0,j}}{\lambda^n}.$$

The last limit exists since M_σ is positive recurrent with Perron eigenvalue λ. For integers $t \geq 2$ and $I_t = i_1 \ldots i_t \in A_t$, set

$$\mu[i_1 \ldots i_t] = \lim_{n\to\infty} \frac{|\sigma^n(u_0)|_{i_1\ldots i_t}}{\lambda^n}. \tag{3.18}$$

Applying equation (3.12) for σ^n in place of σ and the fact that $\lambda > 1$, we deduce that

$$\mu[i_1 \ldots i_t] = \lim_{n\to\infty} \frac{|\sigma_t^n(u_0 \ldots u_{t-1})|_{i_1\ldots i_t}}{\lambda^n} = \lim_{n\to\infty} \frac{(M_t^n)_{U_t, I_t}}{\lambda^n},$$

where $U_t = u_0 \ldots u_{t-1}$ and $I_t = i_1 \ldots i_t$. Observe that $\lim_{n\to\infty} \frac{(M_t^n)_{U_t,I_t}}{\lambda^n}$ exists since M_t is positive recurrent with Perron value $\lambda_t = \lambda$.

By the Kolmogorov consistency theorem, there exists a unique measure μ with cylinder specification in equation (3.18) if for every integer $t \geq 1$ and $I = i_1 \ldots i_t \in A_t$, we have

$$\mu[I] = \sum_{b \in A, Ib \in A_{t+1}} \mu[Ib] \tag{3.19}$$

and

$$\mu[I] = \sum_{a \in A, aI} \mu[aI]. \tag{3.20}$$

For the proof of equation (3.19), let $l = (l_i)_{i \geq 0}$ and $r = (r_i)_{i \geq 0}$ be respectively left and right Perron eigenvectors of M such that the scalar product $l \cdot r = 1$. For all $t \geq 2$, let $l^{(t)} = (l_I)_{I \in A_t}$ and $r^{(t)} = (r_I)_{I \in A_t}$ be left and right Perron eigenvectors of M_t such that

$$r_{i_1 \ldots i_t} = r_{i_1} \quad \text{for all } i_1 \ldots i_t \in A_t \text{ and } l^{(t)} \cdot r^{(t)} = 1.$$

We could choose $r_{i_1 \ldots i_t} = r_{i_1}$ because of Lemma 3.16.

For all $I = i_1 \ldots i_t,\ t \geq 2$, we have by equation (2.4) that

$$\mu[I] = r_{U_t} l_I = r_{u_0} l_I.$$

Hence, equation (3.19) is equivalent to

$$l_I = \sum_{b \in A, Ib \in A_{t+1}} l_{Ib}. \tag{3.21}$$

However, for all $I = i_1 \ldots i_t \in A_t$, we have by Fatou's lemma that

$$\sum_{b \in A, Ib \in A_{t+1}} \mu[Ib] \leq \lim_{n \to \infty} \frac{1}{\lambda^n} \sum_{b \in A, Ib \in A_{t+1}} |\sigma^n_{t+1}(u_0 \ldots u_t)|_{Ib} = \lim_{n \to \infty} \frac{1}{\lambda^n} |\sigma^n_t(u_0 \ldots u_{t-1})|_I.$$

Hence,

$$\sum_{b \in A, Ib \in A_{t+1}} \mu[Ib] \leq \mu[I],$$

that is,

$$\sum_{b \in A, Ib \in A_{t+1}} l_{Ib} \leq l_I. \tag{3.22}$$

Since $\sum_{I \in A_t} r_I l_I = \sum_{J \in A_{t+1}} r_J l_J = 1$ and $r(Ib) = r(I)$ for all $I \in A_t$ and $Ib \in A_{t+1}$, we deduce that

$$\sum_{I \in A_t} r_I l_I = \sum_{I \in A_t} r_I \sum_{b \in A, Ib \in A_{t+1}} l_{Ib} = 1.$$

Using this last equality and equation (3.22), we obtain equation (3.21) and hence we get equation (3.19).

Analogously, equation (3.20) is equivalent to

$$l_I = \sum_{a \in A, aI \in A_{t+1}} l_{aI}. \tag{3.23}$$

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Using Fatou's lemma, we have for all $I \in A_t$,

$$\sum_{a\in A, aI\in A_{t+1}} \mu[aI] \le \lim_{n\to\infty} \frac{1}{\lambda^n} \sum_{a\in A, aI\in A_{t+1}} |\sigma_{t+1}^n(u_0 \ldots u_t)|_{aI}.$$

Note that

$$\beta_I := \sum_{a\in A, aI\in A_{t+1}} |\sigma_{t+1}^n(u_0 \ldots u_t)|_{aI} - |\sigma_t^n(u_0 \ldots u_{t-1})|_I \in \{-1, 0, 1\},$$

indeed $\beta_I = -1$ if the first letter of $\sigma_{t+1}^n(u_0 \ldots u_t)$ begins with I and the last letter of $\sigma_{t+1}^n(u_0 \ldots u_t)$ does not end with I. The number $\beta_I = 1$ if the first letter of $\sigma_{t+1}^n(u_0 \ldots u_t)$ does not begin with I and the last letter of $\sigma_{t+1}^n(u_0 \ldots u_t)$ ends with I. In the complementary case, we have $\beta_I = 0$.

Since $\lambda > 1$, we deduce that

$$\lim_{n\to\infty} \frac{1}{\lambda^n} \sum_{a\in A, aI\in A_{t+1}} |\sigma_{t+1}^n(u_0 \ldots u_t)|_{aI} = \lim_{n\to\infty} \frac{1}{\lambda^n} |\sigma_t^n(u_0 \ldots u_{t-1})|_I.$$

Hence,

$$\sum_{a\in A, Ia\in A_{t+1}} \mu[aI] \le \mu(I),$$

that is,

$$\sum_{a\in A, Ia\in A_{t+1}} l_{aI} \le l_I. \tag{3.24}$$

However, by equation (3.21), we have

$$\sum_{J\in A_{t+1}} l_J = \sum_{I\in A_t} \bigg(\sum_{b\in A, Ib\in A_{t+1}} l_{Ib} \bigg) = \sum_{I\in A_t} l_I.$$

Thus,

$$\sum_{I\in A_t} \bigg(\sum_{a\in A, aI\in A_{t+1}} l_{aI} \bigg) = \sum_{I\in A_t} l_I.$$

By using equation (3.24), we obtain equation (3.20). Hence, μ is an invariant measure for (Ω_u, S). □

3.3.1. *Constant length substitution and unique ergodicity.* Let σ be a substitution on $A = \mathbb{Z}_+$ with constant length $L > 0$. By equation (2.8), the stochastic matrix M_σ/L is strongly ergodic if and only if there exists a positive power of M_σ which is scrambling.

As an example, the dyadic substitution σ defined by $\sigma(n) = 0(n+1)$ has a strongly ergodic matrix $M_\sigma/2$ since the matrix M_σ is scrambling.

Another way to see that $M_\sigma/2$ is strongly ergodic comes from the fact that for all i, $j \in \mathbb{Z}_+$,

$$\lim_{n\to\infty} \sup_{i\in\mathbb{Z}_+} \sum_{j=0}^{+\infty} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - \frac{1}{2^{j+1}} \right| = 0. \tag{3.25}$$

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Indeed, for all integers $n \in \mathbb{N}$, $i, j \in \mathbb{Z}_+$, we have

$$|\sigma^n(i)|_j = |\sigma^{n-1}(i)|_{j-1} = 2^{n-j-1} \quad \text{for all } 0 \leq j < n$$

and

$$|\sigma^n(i)|_j = |\sigma(i)|_{j+1-n} \quad \text{for all } j \geq n.$$

Thus, for all $j \geq n$,

$$|\sigma^n(i)|_j = 1 \quad \text{if } j = i + n \text{ and } 0 \text{ otherwise.}$$

Hence,

$$\sum_{j=0}^{+\infty} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - \frac{1}{2^{j+1}} \right| = \sum_{j=n}^{+\infty} \frac{1}{2^{j+1}} + \left(\frac{1}{2^n} - \frac{1}{2^{i+n}} \right) \quad \text{for all } i \geq 0,$$

which implies that

$$\sup_{i \in \mathbb{Z}_+} \sum_{j=0}^{+\infty} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - \frac{1}{2^{j+1}} \right| = \frac{1}{2^{n-1}},$$

and we obtain equation (3.25).

Another example are the substitutions $\sigma_{a,b,c}$, $a, b, c \in \mathbb{N}$ and $a > c$. We have seen in the proposition that for all positive integers a, b, c with $a > c$, the matrix $M_{\sigma_{a,b,c}}$ is positive recurrent. Furthermore, the stochastic matrix $M_{\sigma_{a,b,c}}/(a + b + c)$ is not strongly ergodic since $M_{\sigma_{a,b,c}}$ does not have a scrambling power (see Remark 3.9).

Remark 3.17. Let σ be a substitution on $A = \mathbb{Z}_+$ with constant length $L > 0$ and a periodic point u such that the stochastic matrix M_σ/L is strongly ergodic. Then M_σ is positive recurrent and, by Theorem 3.10, Ω_σ has a finite invariant measure.

LEMMA 3.18. *Assume that σ is a constant length substitution on $A = \mathbb{Z}_+$ and M_σ is irreducible and aperiodic. If M_σ is strongly ergodic, then for all integers $t \geq 2$, M_{σ_t} is also strongly ergodic.*

Proof. Let $L > 0$ be the length of σ. Fix and integer $t \geq 2$ and $i_1 \ldots i_t, k_1 \ldots k_t \in A_t$. Since M_σ is strongly ergodic, then equation (2.5) implies that there exists an integer $n > 0$ and $j_1 \in \mathbb{N}$ such that

$$j_1 \quad \text{occurs in } \sigma^n(i_1) \text{ and } \sigma^n(k_1).$$

Let $m > 0$ be an integer such that

$$m = \left[\frac{\ln(2t)}{\ln L} \right] + 1,$$

then

$$\sigma^m(j_1) = a_1 \ldots a_s \quad \text{where } s \geq 2t.$$

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Hence, $a_1 \dots a_{2t}$ occurs in $\sigma^{n+m}(i_1)$ and $\sigma^{n+m}(k_1)$. Thus,

$$a_1 \dots a_t \quad \text{occurs in } \sigma_t^{n+m}(i_1 \dots i_t) \text{ and } \sigma_t^{n+m}(k_1 \dots k_t),$$

and we are done again by equation (2.5). □

Proof of Theorem 3.12. Assume without loss of generality that σ has a fixed point $u = \sigma(u) = u_0 u_1 \dots = \lim_{n\to\infty} \sigma(u_0)$ and let $L > 0$ be the length of σ. Recall that for all $i, j \in \mathbb{Z}_+$ and $n \ge 0$,

$$\frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} = \frac{M_{ij}^n}{L^n}.$$

Since M_σ is irreducible, aperiodic, and strongly ergodic, we have that $\lambda = L$ and

$$\lim_{n\to+\infty} \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} = v_j > 0$$

independently of i. Moreover, strong ergodicity implies that there exist $c > 0$ and $0 < \beta < 1$ such that

$$\sup_{i\ge 0} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - v_j \right| \le c\beta^n \quad \text{for all } n \ge 0.$$

To compute $\lim_{n\to+\infty}(|\sigma^n(i)|_w/|\sigma^n(i)|)$, where w is a word of length $t \ge 2$, we will consider a substitution σ_t on the alphabet A_t. From Lemmas 3.14 and 3.18, we deduce that M_{σ_t} is irreducible, aperiodic, and strongly ergodic. Thus, if $w = w_0 \dots w_{t-1}$ and $B = b_0 \dots b_{t-1} \in A_t$, then there exists $d_B > 0$ such that $\lim_{n\to+\infty}(|\sigma_t^n(w)|_B/|\sigma_t^n(w)|) = d_B$ independently of w. Now, since

$$|\sigma_t^n(w)| = |\sigma^n(w_0)| = L^n \quad \text{and} \quad |\sigma^n(w_0)|_B \le |\sigma_t^n(w)|_B \le |\sigma^n(w_0)|_B + t,$$

we obtain

$$\lim_{n\to+\infty} \frac{|\sigma^n(w_0)|_z}{|\sigma^n(w_0)|} = d_B. \tag{3.26}$$

Moreover, there exists $c_t > 0$ and $0 < \beta_t < 1$ such that

$$\sup_{w_0\ge 0} \left| \frac{|\sigma^n(w_0)|_B}{|\sigma^n(w_0)|} - d_B \right| \le c_t\beta_t^n \quad \text{for all } n \ge 0. \tag{3.27}$$

To finish the proof, we have to show the following claim.

Claim. Let $t \ge 2$ and $B = b_1 \dots b_t \in F_u$. Then $\lim_{N\to\infty}(1/N)|u_k \dots u_{k+N-1}|_B = d_B$ uniformly on k.

The proof is the same as that for a finite alphabet and σ primitive given in [19, Theorem 4.6, pp. 141–142]. Indeed, let $V_k = u_k \dots u_{k+N-1} \in F_u$, $k \in \mathbb{Z}_+$, $N \in \mathbb{N}$. As cited in the proof of Theorem 3.1, the word V_k can be written as

$$V_k = v_0\sigma(v_1) \dots \sigma^{n-1}(v_{n-1})\sigma^n(v_n)\sigma^{n-1}(w_{n-1}) \dots \sigma(w_1)w_0, \tag{3.28}$$

where $n \ge 0$ is an integer and v_i, $i \in \{0, \dots, n\}$, w_j, $j \in \{0, \dots, n-1\}$ are elements of F_u possibly empty words of lengths $\le L$ and v_n is not empty.

Since $L \geq 2$, there exist $C > 0$ and $1 < \tau < L$ such that $C\tau^n \geq c_t(2n-1)((\max(\beta_t L, 1))^n$ for every $n \geq 1$. Now for $V_k = u_k \ldots u_{k+N-1}$, $k \in \mathbb{Z}_+$, for $N \in \mathbb{N}$, and $B \in F_u$ such that $|B| < N$, we use equations (3.28) and (3.27) to obtain that

$$\begin{aligned} ||V_k|_B - d_B N| &\leq \sum_{j=0}^{n} \big||\sigma^j(v_j)|_B - d_B|\sigma^j(v_j)|\big| + \sum_{j=0}^{n-1} \big||\sigma^j(w_j)|_B - d_B|\sigma^j(w_j)|\big| \\ &\leq \sum_{j=0}^{n} c_t(\beta_t L)^j + \sum_{j=0}^{n-1} c_t(\beta_t L)^j \leq c_t(2n-1)((\max(\beta_t L, 1))^n \leq C\tau^n. \end{aligned} \tag{3.29}$$

Since $N \geq |\sigma^n(v_n)| = L^n$, we obtain that

$$\sup_{k \geq 0} \left| \frac{|V_k|_B}{N} - d_B \right| \leq C\gamma^n \tag{3.30}$$

for some $\gamma < 1$ and we obtain the claim. □

3.3.2. *Strong ergodicity for non-constant bounded length substitution*

Definition 3.19. Let $M = (M_{ij})_{i,j \geq 0}$ be a non-negative matrix such that M is irreducible, aperiodic, and positive recurrent with finite Perron value $\lambda > 0$. Let $P = (P_{ij})_{i,j \geq 0}$ be the stochastic matrix defined by

$$P_{ij} = \frac{M_{ij} r_j}{\lambda r_i} \quad \text{for all } i, j \geq 0,$$

where $r = (r_k)_{k \geq 0}$ is a right Perron eigenvector of M. We say that M is strongly ergodic if $P = (P_{ij})_{i,j \geq 0}$ is too.

Remark 3.20. (1) It is easy to see that the stochastic matrix P defined in the last definition satisfies $P_{ij} = M_{ji} l_j / \lambda l_i$ for all $i, j \geq 0$, where $l = (l_k)_{k \geq 0}$ is a left Perron eigenvector of M. Furthermore, we have that $P^n_{i,j} = M^n_{ij} r_j / \lambda^n r_i$ for all integers $n \geq 1$.

(2) Definition 3.19 appeared in [20] in the case where M is a finite irreducible, aperiodic matrix. It is also an extension of the definition in the case where M has constant row sums L. This comes from the fact that $r_i = 1$ for all $i \geq 0$ and $\lambda = L$.

Remark 3.21. Let $M = (M_{ij})_{i,j \geq 0}$ be a non-negative, irreducible, aperiodic, and positive recurrent matrix with finite Perron value $\lambda > 0$. Then M is strongly ergodic if and only if there exists positive integer n and a vector of probability $(\pi_j)_{j \geq 0}$ such that

$$\lim_{n \to \infty} \sup_{i \geq 0} \sum_{j=0}^{+\infty} \left| \frac{M^n_{ij} r_j}{\lambda^n r_i} - \pi_j \right| = 0. \tag{3.31}$$

Furthermore, $\pi_j = l_j r_j$, where l and r are respectively Perron left and right eigenvectors such that $l \cdot r = 1$. By using Remark 2.13, we deduce that M is strongly ergodic if and only if there exists a positive integer n and a positive constant a such that for all integers $i \neq k$,

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we have

$$\sum_{j=0}^{+\infty} \min\left(\frac{M_{ij}^n r_j}{\lambda^n r_i}, \frac{M_{kj}^n r_j}{\lambda^n r_k}\right) \geq a. \tag{3.32}$$

Remark 3.22. Let $M = (M_{ij})_{i,j\geq 0}$ be a non-negative, irreducible, aperiodic, and positive recurrent matrix with finite Perron value $\lambda > 0$. Assume that M has a right Perron eigenvector $r = (r_i)_{i\geq 0} \in l^\infty$ which satisfies $\inf\{r_i : i \geq 0\} > 0$, then by Remark 3.21, we deduce that M is strongly ergodic if and only if there exists a positive integer n such that M^n is scrambling.

THEOREM 3.23. *Let σ be a non-constant bounded length substitution on $A = \mathbb{Z}_+$ with a periodic point u and such that $M = M_\sigma$ is irreducible, aperiodic, positive recurrent and has a finite Perron value. Assume that M_σ has a right Perron eigenvector $r = (r_i)_{i\geq 0} \in l^\infty$ and there exists a positive integer such that M_σ^n is scrambling. Then the dynamical system (Ω_σ, S) has a unique invariant probability measure.*

For the proof, we need the following results.

LEMMA 3.24. *Let $M = (M_{ij})_{i,j\in\mathbb{Z}_+}$ be an irreducible, aperiodic, positive recurrent non-negative matrix such that $\|M\| = \sup\{\sum_{j=0}^{+\infty} M_{ij},\ i \in \mathbb{Z}_+\} < \infty$ and $\inf\{M_{ij} : i, j \in \mathbb{Z}_+,\ M_{ij} > 0\} > 0$. Assume that there exists a positive integer such that M^n is scrambling. Then M has a right Perron eigenvector $r = (r_i)_{i\geq 0}$ which satisfies* $\inf\{r_i : i \geq 0\} > 0$.

Proof. Assume without loss of generality that M is scrambling. Let $r = (r_i)_{i\geq 0}$ be a non-negative right Perron eigenvector of M. Since M is irreducible, $r_i > 0$ for every $i \geq 0$. Moreover, $\|M\| < \infty$ and $\inf\{M_{ij} : i, j \in \mathbb{Z}_+,\ M_{ij} > 0\} > 0$ imply that there exists $L > 0$ such that $M_{0k} = 0$ for all $k > L$. Since M is scrambling, then for all $i \in \mathbb{Z}_+$, there exists $k_i \in \{0, \ldots, L\}$ such that $M_{i,k_i} > 0$. Since $\sum_{k=0}^{+\infty} M_{ik} r_k = \lambda r_i$, we deduce that

$$r_i \geq \frac{M_{i,k_i} r_{k_i}}{\lambda} \geq \frac{C \inf\{r_k :\ 0 \leq k \leq L\}}{\lambda} > 0 \quad \text{for all } i \in \mathbb{Z}_+,$$

where $C = \inf\{M_{ij} :\ i, j \in \mathbb{Z}_+,\ M_{ij} > 0\}$. □

LEMMA 3.25. *Let σ be a non-constant bounded length substitution on $A = \mathbb{Z}_+$ with a periodic point u such that $M = M_\sigma$ is irreducible, aperiodic, and has a finite Perron value. Assume that M_σ is strongly ergodic and has a right Perron eigenvector $r = (r_i)_{i\geq 0} \in l^\infty$ which satisfies* $\inf\{r_i : i \geq 0\} > 0$. *Then for all integers $t \geq 2$, M_{σ_t} is strongly ergodic and has a right Perron eigenvector $r^{(t)} = (r_I)_{I\in A_t} \in l^\infty$ such that* $\inf\{r_I : I \in A_t\} > 0$.

Proof. Using the same proof given in Lemma 3.18, we can show that if M_σ is strongly ergodic, then M_{σ_t} is also strongly ergodic. Moreover, since $r = (r_i)_{i\geq 0} \in l^\infty$ and $\inf\{r_i : i \geq 0\} > 0$, Lemma 3.16 implies that $r^{(t)} = (r_I)_{I\in A_t} \in l^\infty$ and $\inf\{r_I : I \in A_t\} > 0$. □

LEMMA 3.26. *Let $M = (M_{ij})_{i,j\geq 0}$ be a non-negative strongly ergodic matrix with finite Perron value $\lambda > 0$. Assume that M has a right Perron eigenvector $r = (r_i)_{i\geq 0} \in l^\infty$ which*

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satisfies $\inf\{r_i : i \geq 0\} > 0$. *Then*

$$\lim_{n\to+\infty} \sup_{i\geq 0} \sum_{j=0}^{\infty} \left| \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} - z_j \right| = 0,$$

where $z_j = l_j / \sum_{k=0}^{+\infty} l_k$ *and* $l = (l_i)_{i\geq 0} \in l^1$ *is a left Perron eigenvector of* M.

Proof. Since M is strongly ergodic, we deduce by equation (3.31) that for all $i, j \in \mathbb{Z}_+$,

$$\lim_{n\to\infty} \frac{M_{ij}^n}{\lambda^n} = \pi_j \frac{r_i}{r_j}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{+\infty} \frac{M_{ik}^n}{\lambda^n} = \sum_{k=0}^{+\infty} \pi_k \frac{r_i}{r_k},$$

where the last two limits are finite and uniform on i. Hence,

$$\lim_{n\to+\infty} \sup_{i\geq 0} \sum_{j=0}^{\infty} \left| \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} - z_j \right| = 0,$$

where

$$z_j = \frac{\pi_j / r_j}{\sum_{k=0}^{+\infty} \pi_k / r_k} = \frac{l_j}{\sum_{k=0}^{+\infty} l_k} \quad \text{for all } j \geq 0.$$

□

COROLLARY 3.27. *Let* $M = (M_{ij})_{i,j\geq 0}$ *be a non-negative strongly ergodic matrix with finite Perron value* $\lambda > 0$. *Assume that* M *has a right Perron eigenvector* $r = (r_i)_{i\geq 0} \in l^\infty$ *which satisfies* $\inf\{r_i : i \geq 0\} > 0$. *Then*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{+\infty} M_{ij}^n}{\lambda^n} = c r_i$$

for some $c > 0$.

Proof of Theorem 3.23. Without loss of generality, assume that σ has a fixed point $u = \sigma(u) = u_0 u_1 \ldots$. For all $i, j \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, we have

$$\frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} = \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n}.$$

Hence, by Lemma 3.26, we have

$$\lim_{n\to+\infty} \sup_{i\geq 0} \sum_{j=0}^{\infty} \left| \frac{|\sigma^n(i)|_j}{|\sigma^n(i)|} - \frac{l_j}{\sum_{k=0}^{+\infty} l_k} \right| = 0.$$

Let $j \in \mathbb{Z}_+$ and put

$$\mu[j] = \lim_{n\to\infty} \frac{|\sigma^n(u_0)|_j}{|\sigma^n(u_0)|} = \frac{l_j}{\sum_{k=0}^{+\infty} l_k}.$$

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Let $t \geq 2$ be an integer and $I_t = i_1 \ldots i_t \in A_t$, and put

$$\mu[i_1 \ldots i_t] = \lim_{n\to\infty} \frac{|\sigma^n(u_0)|_{i_0\ldots i_{t-1}}}{|\sigma^n(u_0)|}.$$

By equation (3.12) and the fact that $\lambda > 1$, we deduce that

$$\mu[i_1 \ldots i_t] = \lim_{n\to\infty} \frac{|\sigma_t^n(u_0 \ldots u_{t-1})|_{i_1\ldots i_t}}{|\sigma_t^n(u_0 \ldots u_{t-1})|} = \lim_{n\to\infty} \frac{(M_t^n)_{U_t,I_t}}{\sum_{J\in A_t} M^n_{U_t,J}},$$

where $U_t = u_0 \ldots u_{t-1}$, $I_t = i_1 \ldots i_t$. By Lemmas 3.25 and 3.26, we have

$$\mu[i_1 \ldots i_t] = \frac{l^{(t)}_{I_t}}{\sum_{J\in A_t} l^{(t)}_J},$$

where $(l^{(t)}_I)_{I\in A_t}$ is a left Perron eigenvector of M_{σ_t} associated to its Perron value λ.

The measure μ is the same as that given in the proof of Theorem 3.10. Hence, μ is a shift invariant measure. The uniqueness is a direct consequence of the following claim.

Claim. Let E be a measurable subset of Ω_σ such that $\mu(E) > 0$. For all $x \in E$, we have

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{0 \leq k \leq N-1,\ S^k(x) \in E\} = \mu(E). \tag{3.33}$$

It remains to prove the claim. First, assume that $E = [i_0]$. Suppose $x = u = \sigma(u) = u_0u_1 \ldots$ and $N = |\sigma^n(u_0)|$. Then

$$\lim_{n\to\infty} \frac{1}{N} \operatorname{card}\{0 \leq k \leq N-1,\ S^k(x) \in E\} = \lim_{n\to\infty} \frac{|\sigma^n(u_0)|_{i_0}}{|\sigma^n(u_0)|} = \mu(E).$$

Now, let $x \in \Omega_\sigma$ and $N \in \mathbb{N}$. Let $V = u_k \ldots u_{k+N-1} \in F_u$, $k \in \mathbb{Z}_+$ be a prefix of x. As seen before, the word V can be written as a concatenation of at most $2n+1$ words $v_0, \sigma(v_1), \ldots, \sigma^n(v_n), \sigma^{n-1}(w_{n-1}) \ldots w_0$ that is

$$V = v_0\sigma(v_1) \ldots \sigma^{n-1}(v_{n-1})\sigma^n(v_n)\sigma^{n-1}(w_{n-1}) \ldots \sigma(w_1)w_0,$$

where $n \geq 1$ is an integer and v_i, $i \in \{0, \ldots, n\}$, w_j, $j \in \{0, \ldots, n-1\}$ are elements of F_u possibly empty words of lengths $\leq K = \max\{|\sigma(b)|,\ b \in A\}$ and v_n is not empty. Thus,

$$\begin{aligned}\frac{1}{N} \operatorname{card}\{0 \leq k \leq N-1,\ S^k(x) \in E\} &= \frac{|V|_{i_0}}{|V|}\\ &= \frac{|\sigma^n(v_n)|_{i_0} + \sum_{i=0}^{n-1}(|\sigma^i(v_i)|_{i_0} + |\sigma^i(w_i)|_{i_0})}{|\sigma^n(v_n)| + \sum_{i=0}^{n-1}(|\sigma^i(v_i)| + |\sigma^i(w_i)|)}.\end{aligned} \tag{3.34}$$

Since M_σ is strongly ergodic, we have

$$\lim_{k\to\infty} \sup \left\{\frac{|\sigma^k(j)|_{i_0}}{|\sigma^k(j)|},\ j \in A\right\} = \lim_{k\to\infty} \sup \left\{\frac{M^k_{j,i_0}}{\sum_{i=0}^{+\infty} M^k_{j,i}},\ j \in A\right\} = \mu[i_0].$$

We deduce that

$$\lim_{k\to\infty} \sup \left\{\frac{|\sigma^k(v)|_{i_0}}{|\sigma^k(v)|},\ v \in F_u,\ |v| \leq K\right\} = \mu[i_0]. \tag{3.35}$$

Using equations (3.34), (3.35), and the Stolz–Cesaro theorem, we obtain that

$$\lim_{N\to\infty}\frac{1}{N}\operatorname{card}\{0\le k\le N-1,\ S^k(x)\in E\}=\mu[i_0]=\mu(E).$$

Hence, we obtain the claim for $E=[i_0]$.

Now, suppose that $I=i_0\ldots i_{t-1}$ and $E=[I]$. Proceeding as in the case $E=[i_0]$, we have

$$\frac{\operatorname{card}\{0\le k\le N-1,\ S^k(x)\in E\}}{N}=\frac{|\sigma^n(v_n)|_I+\sum_{i=0}^{n-1}(|\sigma^i(v_i)|_I+|\sigma^i(w_i)|_I)+C_n}{|\sigma^n(v_n)|+\sum_{i=0}^{n-1}(|\sigma^i(v_i)|+|\sigma^i(w_i)|)},$$

where C_n is the cardinality of times such that $i_0\ldots i_{t-1}$ occurs in the concatenation of at least two consecutive words among the $2n+1$ words forming V. Observe that $0\le C_n\le 2n$.

Now for all $j\in A$, we have

$$\lim_{k\to\infty}\frac{|\sigma^k(j)|_{i_0\ldots i_{t-1}}}{|\sigma^k(j)|}=\lim_{k\to\infty}\frac{|\sigma_t^k(jz_1\ldots z_{t-1})|_{i_0\ldots i_{t-1}}}{|\sigma_t^k(jz_1\ldots z_{t-1})|}=\mu[i_0\ldots i_{t-1}],$$

where $jz_1\ldots z_{t-1}\in F_u$. We deduce by using the fact that σ_t is strongly ergodic that

$$\lim_{k\to\infty}\sup\left\{\frac{|\sigma^k(j)|_{i_0\ldots i_{t-1}}}{|\sigma^k(j)|},\ j\in\mathbb{Z}_+\right\}=\mu[i_0\ldots i_{t-1}].$$

Thus,

$$\lim_{k\to\infty}\sup\left\{\frac{|\sigma^k(v)|_{i_0\ldots i_{t-1}}}{|\sigma^k(v)|},\ v\in F_u,\ |v|\le K\right\}=\mu[i_0\ldots i_{t-1}]. \tag{3.36}$$

Using equation (3.36) and the Stolz–Cesaro theorem, we deduce that

$$\lim_{n\to\infty}\frac{|\sigma^n(v_n)|_I+\sum_{i=0}^{n-1}(|\sigma^i(v_i)|_I+|\sigma^i(w_i)|_I)}{|\sigma^n(v_n)|+\sum_{i=0}^{n-1}(|\sigma^i(v_i)|+|\sigma^i(w_i)|)}=\mu[i_0\ldots i_{l-1}].$$

Since $0\le C_n\le 2n$, $\lim_{n\to\infty}\frac{|\sigma^n(v_n)|_0}{\lambda^n}>0$ and $\lambda>1$, we deduce that

$$\lim_{N\to\infty}\frac{1}{N}\operatorname{card}\{0\le k\le N-1,\ S^k(x)\in E\}=\mu[i_0\ldots i_{t-1}],$$

and this finishes the proof. □

PROPOSITION 3.28. *Let σ be a bounded length substitution on $A=\mathbb{Z}_+$ such that σ has a periodic point u and M_σ is irreducible and aperiodic. Assume that there exists an integer n such that M_σ^n is scrambling. Then (Ω_σ, S) is minimal.*

Proof. Assume without loss of generality that $u=u_0u_1\ldots$ is a fixed point and M_σ is scrambling. Let $V=u_k\ldots u_{k+N}$, $k,N\in\mathbb{Z}_+$ be a factor of u. Let us prove that V occurs infinitely on u with bounded gaps. Indeed, let $n_0\in\mathbb{N}$ such that V occurs in $\sigma^k(u_0)$ for all $k\ge n_0$ and put

$$\sigma(u_0)=t_0\ldots t_s,\qquad s\in\mathbb{N}.$$

Let $m_0 \in \mathbb{N}$ such that u_0 occurs on $\sigma^k(t_i)$ for all $k \geq m_0$ and $i = 0, \ldots, s$. Hence, V occurs in $\sigma^k(t_i)$ for all $k \geq n_0 + m_0$ and $i = 0, \ldots, s$. However, since M_σ is scrambling, then for all $i \in \mathbb{N}$, there exists $j_i \in \{0, \ldots, s\}$ such that t_{j_i} occurs in $\sigma(u_i)$. Hence, V occurs in $\sigma^k(u_i)$ for all $k \geq n_0 + m_0$ and $i \in \mathbb{Z}_+$. Since $u = \sigma^k(u) = \sigma^k(u_0)\sigma^k(u_1) \ldots$, we are done. □

Examples.

(1) Let σ (infinite Fibonacci) be given by

$$\sigma(2n) = 0(2n+1),\ \ \sigma(2n+1) = 2n+2 \quad \text{for all } n \geq 0.$$

We can prove by induction that

$$|\sigma^n(0)| = F_n \quad \text{and} \quad |\sigma^n(0)|_0 = F_{n-1} \quad \text{for all } n \geq 1,$$

where $(F_n)_{n\geq 0}$ is the Fibonacci sequence defined by

$$F_0 = 1,\ F_1 = 2,\ \ F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2.$$

The substitution matrix is given by

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is irreducible, aperiodic, and its Perron eigenvector is the Golden number $\beta = (1+\sqrt{5})/2 = \lim_{n\to\infty}(F_{n+1})/F_n$. A right Perron and a left Perron eigenvector are respectively

$$l = (1, 1/\beta, \ldots, 1/\beta^n, \ldots) \text{ and } r = (1, 1/\beta, 1, 1/\beta, 1, 1/\beta, \ldots).$$

Hence, M_σ is positive recurrent. Furthermore, M_σ^2 is scrambling, $r \in l^\infty$, and σ has a fixed point $u = \lim_{n\to\infty} \sigma^n(0)$, thus Theorem 3.23 implies that the dynamical system (Ω_σ, S) has a unique probability invariant measure.

(2) Let τ be given by

$$\tau(n) = 0^{a_n}(n+1) \quad \text{for all } n \geq 0,$$

where $0 \leq a_i \leq C$ for all $i \geq 0$ for some fixed $C > 0$ and $a_0 > 0$, and $\lim sup\ a_n \geq 1$.

The substitution matrix is given by

$$M_\tau = \begin{pmatrix} a_0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ a_1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ a_2 & 0 & 0 & 1 & 0 & 0 & \ldots \\ a_3 & 0 & 0 & 0 & 1 & 0 & \ldots \\ a_4 & 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Perron eigenvalue of M_τ is the unique real number $\lambda > 1$ satisfying

$$1 = \sum_{i=0}^{\infty} a_i \lambda^{-i-1}.$$

A right Perron and a left Perron eigenvector are respectively

$$l = (1, 1/\lambda, \ldots, 1/\lambda^n, \ldots) \quad \text{and} \quad r = (1, \alpha_1, \ldots, \alpha_n, \ldots),$$

where

$$\alpha_n = \lambda^n - \sum_{i=0}^{n-1} a_i \lambda^{n-i-1} \quad \text{for all } n \geq 1.$$

Observe that

$$\alpha_n = \sum_{i=1}^{+\infty} a_{n+i-1} \lambda^{-i} \quad \text{for all } n \geq 1.$$

Since $l \cdot r$ is finite, M_σ is positive recurrent.

If there exists $k \geq 1$ such that $a_{kn} \geq 1$ for all $n \in \mathbb{Z}_+$, then $\inf\{\alpha_n,\ n \in \mathbb{Z}_+\} > 0$.. Moreover, M_τ^k is scrambling. Furthermore, τ has a fixed point $u = \lim_{n\to\infty} \tau^n(0)$, thus Theorem 3.23 implies that the dynamical system (Ω_u, S) has a unique probability invariant measure.

Question 3.4. It will be interesting to study dynamical properties of (Ω_u, S) associated to τ in the case where $\inf\{\alpha_n,\ n \in \mathbb{Z}_+\} = 0$.

3.3.3. $\star$ *strong ergodicity for non-constant bounded length substitution*

Definition 3.29. Let $M = (M_{ij})_{i,j\geq 0}$ be a non-negative matrix such that M is irreducible, aperiodic, positive recurrent and $\|M\| < +\infty$. We say that M is $\star$ *ergodic* if for all i, $j \in \mathbb{Z}_+$,

$$\lim_{n\to+\infty} \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} = z_j > 0, \tag{3.37}$$

where the vector $(z_j)_{j\geq 0}$ has 1 as coordinates sum and that M is $\star$ *strongly ergodic* if there exists a vector $(z_j)_{j\geq 0}$ of positive real numbers such that $\sum_{j=0}^{+\infty} z_j = 1$ and

$$\lim_{n\to\infty} sup_{i\geq 0} \sum_{j=0}^{\infty} \left| \frac{M_{ij}^n}{\sum_{k=0}^{+\infty} M_{ik}^n} - z_j \right| = 0.$$

Remark 3.30. If M is $\star$ strongly ergodic, then it is clear that M is $\star$ ergodic.

Question 3.5. Is $M = (M_{ij})_{i,j\geq 0}$ $\star$ ergodic equivalent to M positive recurrent with right Perron eigenvector in l^1? The last question has a positive answer when M is a multiple of a stochastic matrix.

An important result is the following.

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PROPOSITION 3.31. *Let $M = (M_{ij})_{i,j\geq 0}$ be an irreducible, aperiodic matrix with finite Perron value λ. Assume that M has a right Perron eigenvector $r = (r_i)_{i\geq 0} \in l^\infty$ satisfying* $\inf\{r_i,\ i \in \mathbb{Z}_+\} > 0$. *If M is strongly ergodic, then M is $\star$ strongly ergodic.*

Proof. It is just Lemma 3.26. □

Question 3.6. Does there exist a non-negative matrix $M = (M_{ij})_{i,j\geq 0}$ which is strongly ergodic (respectively $\star$ strongly ergodic), but not $\star$ strongly ergodic (respectively strongly ergodic)?

LEMMA 3.32. *Let $M = (M_{ij})_{i,j\geq 0}$ be a $\star$ ergodic matrix with finite Perron value λ and with right Perron eigenvector $r = (r_i)_{i\geq 0}$, then any left Perron eigenvector of M belongs to l^1. Moreover,*

$$\lim_{n\to\infty} \sum_{k=0}^{+\infty} \frac{M^n_{ik}}{\lambda^n} = \sum_{k=0}^{+\infty} \lim_{n\to\infty} \frac{M^n_{ik}}{\lambda^n} = c\, r_i \quad \textit{for all } i \in \mathbb{Z}_+, \tag{3.38}$$

for some $c > 0$, and

$$\lim_{n\to+\infty} \frac{M^n_{ij}}{\sum_{k=0}^{+\infty} M^n_{ik}} = \frac{l_j}{\sum_{k=0}^{+\infty} l_k} > 0 \quad \textit{for all } i, j \in \mathbb{Z}_+. \tag{3.39}$$

Proof. By $\star$ ergodicity,

$$\lim_{n\to+\infty} \frac{M^n_{ij}}{\sum_{k=0}^{+\infty} M^n_{ik}} = z_j.$$

Moreover, since M is positive recurrence, we have that

$$\lim_{n\to+\infty} \frac{M^n_{ij}}{\lambda^n} = l_j r_i, \tag{3.40}$$

where $l = (l_j)_{j\geq 1}$ is a left Perron eigenvector such that $l \cdot r = 1$. Thus,

$$\lim_{n\to+\infty} \sum_{k=0}^{+\infty} \frac{M^n_{ik}}{\lambda^n} = \lim_{n\to+\infty} \frac{\sum_{k=0}^{+\infty} M^n_{ik}}{M^n_{ij}} \frac{M^n_{ij}}{\lambda^n} = \frac{l_j}{z_j} r_i.$$

The left-hand side above does not depend on j, and thus l is a multiple of $(z_j)_{j\geq 1} \in l^1$. Thus, $l \in l^1$ and we also have equation (3.39), and equation (3.38) follows from the last equality and equation (3.40). □

THEOREM 3.33. *Let σ be a bounded length substitution on $A = \mathbb{Z}_+$ with non-constant length such that σ has a periodic point u and M_σ is irreducible, aperiodic. If M_σ and M_{σ_t}, $t \geq 2$ are $\star$ strongly ergodic, then the dynamical system (Ω_σ, S) has a unique probability shift invariant measure.*

Proof of Theorem 3.33. Similar to the proof of Theorem 3.23. □

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