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Rearranging transfinite series of ordinals

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A new, simplified proof is given of a theorem of J.L. Hickman (to be published; see also *J. London Math. Soc.* (2) 9 (1974), 239-244), giving an upper bound for the sums of a well-ordered series of ordinals under permutation.

In this paper we consider the following problem: let $s = \langle s_\beta : \beta < \alpha \rangle$ be a sequence of type α of ordinals where α is an ordinal. How many distinct sums can we obtain by permuting the members of the series? To put it in a more precise form, let $\Pi(\alpha)$ be the set of all permutations of α, $\Sigma(\pi) = \Sigma\langle s_{\pi(\beta)} : \beta < \alpha \rangle$ for $\pi \in \Pi(\alpha)$, $S(s) = \{\Sigma(\pi) : \pi \in \Pi(\alpha)\}$, and $\sigma(s) = |S(s)|$. Given s we want to get upper estimates for $\sigma(s)$. An old theorem of Sierpiński [4] states that $\sigma(s) < \omega$ provided $\mathrm{dom}(s) = \alpha = \omega$. A more recent result of Hickman [2, 3] gives the following upper estimate

$$\sigma(s) \leq |\alpha|$$

for an arbitrary sequence of ordinals s with $\mathrm{dom}(s) = \alpha \geq \omega$.

Our aim is to give a new and considerably simpler proof of Hickman's Theorem.

Our notation is standard. Lower case Latin and Greek letters denote ordinals. $\alpha + \beta$, $\alpha.\beta$, α^β denote ordinal operations; thus $\alpha^{\beta+1} = \alpha^\beta.\alpha$, $\alpha^{\sup A} = \sup\{\alpha^\gamma : \gamma \in A\}$.

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First we formulate some preliminary statements.

1. *If* $\sum_{i<\delta} \omega^{\alpha_i} = \omega^\gamma$, *then* $\sup_{i<\delta} \alpha_i \leq \gamma \leq (\sup_{i<\delta} \alpha_i) + \delta$.

Proof. Immediate.

2. *Assume* $\kappa \geq \delta$ *a cardinal,* F *is a set of ordinals,* $|F| \leq \kappa$, $\alpha < \kappa^+$. *Let* $F_0 = F \cup \{x : x \text{ is a limit point of } F\}$ *and*

$$F_1 = \{x+\delta : x \in F_0, \delta \leq \alpha\} .$$

Then $|F_1| \leq \kappa$, *and, moreover, if* $\sum_{i<\delta} \omega^{\alpha_i} = \omega^c$ *where* $\alpha_i \in F$, $\delta \leq \alpha$, *then* $c \in F_1$.

Proof. Straightforward.

3. *If* $\sum_{i<\delta} \omega^{\alpha_i} = \omega^{c_1} + \ldots + \omega^{c_n}$ *where* $c_1 \geq \ldots \geq c_n$ $(n < \omega)$ *then there exists a* $\delta_1 \leq \delta$ *such that*

$$\sum_{i<\delta_1} \omega^{\alpha_i} = \omega^{c_1} .$$

Proof. Put $\delta_1 = \min\left\{\gamma : \sum_{i<\gamma} \omega^{\alpha_i} \geq \omega^{c_1}\right\}$. If $\sum_{i<\delta_1} \omega^{\alpha_i} \geq \omega^{c_1} + 1$, the ω^{c_1}-th member of the sum is in some $\omega^{\alpha_{i_0}}$. So $\sum_{i\leq i_0} \omega^{\alpha_i} \geq \omega^{c_1} + 1$, and $\sum_{i<i_0} \omega^{\alpha_i} = \beta < \omega^{c_1}$ and $\delta_1 = i_0 + 1$. But

$$\omega^{\alpha_{i_0}} \leq \sum_{i<\delta} \omega^{\alpha_i} \leq n.\omega^{c_1} < \omega^{c_1+1} ,$$

so $\alpha_{i_0} \leq c_1$. As ω^{c_1} is additively indecomposable, $\alpha_{i_0} = c_1$, and $\sum_{i\leq i_0} \omega^{\alpha_i} = \beta + \omega^{c_1} = \omega^{c_1}$. //

4. *If* $\sum_{i<\delta} \omega^{\alpha_i} = \omega^{c_1} + \ldots + \omega^{c_n}$ $(c_1 \geq \ldots \geq c_n)$ *then there are* $0 = \delta_0 < \delta_1 < \ldots < \delta_n = \delta$, *such that*

$$\sum_{\delta_t \leq i < \delta_{t+1}} \omega^{\alpha_i} = \omega^{c_{t+1}} \quad (t = 0, 1, \ldots, n-1) .$$

Proof. By iterating 3.

THEOREM (Hickman [3]). *Assume* X *is a set of ordinals,* α *is an ordinal,* $|X| \leq \kappa$ *and* $\alpha < \kappa^+$. *Then those* α*-series formed from members of* X *give at most* κ *distinct sums. (Multiple occurrence is admitted.)*

Proof. We may assume that every element of X is a power of ω . As a well-known matter of fact, every ordinal is a finite sum of powers of ω . Hence if we prove the theorem for sums of lengths less than or equal to $\alpha.\omega$ and X having ω^x-type members only, then the general statement follows for sums of length α . So let $X = \{\omega^x : x \in F\}$. Then by 2 and 4 the cardinality of $\left\{\sum_{i<\alpha} \omega^{\alpha_i} : \alpha_i \in F\right\}$ is at most as large as the set of finite subsets of F_1 , which is less than or equal to κ .

References

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