Mordell–Lang conjecture for function fields in characteristic zero, revisited

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ABSTRACT

We give an elementary proof of the function field (or geometric) case of the Mordell–Lang conjecture in characteristic zero.

1. Introduction

In this paper a short proof of the following is given.

THEOREM 1.1. *Let $k < K$ be algebraically closed fields of characteristic zero. Let A be a semi-abelian variety over K, X an irreducible subvariety of A also over K, and Γ a subgroup of $A(K)$ of finite rational rank (that is $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite-dimensional as a $\mathbb{Q}$-vector space). Suppose that $\Gamma \cap X(K)$ is Zariski-dense in X and that X has trivial stabilizer in A. Then, after possibly replacing X by a translate, there are a semi-abelian subvariety A_1 of A containing X, and an isomorphism f of A_1 with a semi-abelian variety A_2 defined over k, such that $f(X)$ is also defined over k.*

The proof uses the differential-algebraic set-up as in [Bui92b] and [Hru96] but replaces Buium's analytic arguments and Hrushovski's appeal to the deep results on Zariski geometries [HZ96] and some model theory of groups of finite Morley rank, by considerations from [PZ01] involving linear differential equations on k-jets. Buium [Bui92b] essentially proves Theorem 1.1 in the case where A is an abelian variety. The full statement of Theorem 1.1 is proved in [Hru96]. As stated in [PZ01] the main result (Theorem 2.5) of [PZ01] (which is stated there in model-theoretic language) can be plugged into the final part of Hrushovski's proof of Theorem 1.1 to give a substantial simplification. Nevertheless, it is considered worthwhile to give a direct account of the simplified proof, avoiding model-theoretic language.

It is worth remarking that a much stronger theorem than Theorem 1.1 is true (the absolute Mordell–Lang conjecture), proved in full generality by McQuillan [McQ95], using heavily work and ideas of Faltings [Fal91], Vojta [Voj96], Raynaud [Ray83], and Hindry [Hin88]. Namely under the hypotheses of Theorem 1.1, the conclusion is that X is trivial (a point). As far as it is understood the most difficult case is when A is defined over a number field and when Γ is finitely generated. For A an abelian variety this is treated in [Fal94] and in general (A semi-abelian) in [Voj96]. Specialization and other arguments are used to reduce to this case. A discussion of some of these specialization arguments appears in [Maz00]. Of course, Theorem 1.1 also provides a rather direct reduction to the number field case.

Let us also remark that included in Theorem 1.1 is Mordell's conjecture over function fields, which was proved in the 1960s by Manin [Man63], with proofs by Grauert [Gra65] and Parshin

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[Par68] following shortly thereafter. An account of the methods in these latter papers appears in Chapter VI of [Lan97].

The author considers it a major challenge to find an elementary proof of Theorem 1.1 in the *positive characteristic case* (and where prime to p division points are used rather than all division points). The only known proof is Hrushovski's [Hru96] which makes use of the Zariski geometries results applied to separably closed fields. As pointed out in [PZ01] the methods of the present paper work in the positive characteristic case if A is an ordinary abelian variety. But this case is already treated in [AV92].

The referee pointed out to us that the idea of performing descent to the constants using the theory of linear differential equations together with a 'Gauss map' from the variety in question to a certain Grassmanian, already appears in [Bui94], where Buium gives an algebraic proof of Mordell–Lang for function fields of characteristic zero in the 'non-hyperelliptic case'. In [Bui94], the Grassmanian in question was associated to a tangent space. In the present paper, we use a Grassmanian of subspaces of a suitable *jet space*, and the hypothesis of 'non-hyperellipticity' in [Bui94] can be removed.

2. Algebraic D-groups and the proof of Theorem 1.1

The main new aspect of our treatment is a rather direct proof of a (new, although implicit in [PZ01]) 'isotriviality' result for 'algebraic D-groups'. The category of algebraic D-groups was introduced by Buium in [Bui92a]. We give a quick description of the category and then state and prove our main lemma.

We work in characteristic zero. Let F be a field equipped with a derivation ∂. Let k be the field of constants of F (set of elements killed by ∂). We will take (F, ∂) to be *differentially closed*, that is, any finite system of differential polynomial equations and inequations over F with a common solution in a differential field extension of (F, ∂) already has a solution in F. (However, all the definitions make sense for arbitrary (F, ∂).) Buium defines an algebraic D-variety to be an irreducible variety X over F equipped with a derivation ∂_X on the structure sheaf of X which extends ∂. Algebraic D-varieties form a category in the natural way. If G is a connected algebraic group over F, and ∂_G respects multiplication and inversion (in the obvious sense), he speaks of an algebraic D-group. ∂_X as above is a kind of twisted vector field on X and it will be useful to make this formalism more explicit. We describe briefly a functor τ defined on varieties over F, taking account of the derivation ∂. If the variety X is locally defined by a system of polynomial equations $P_1(X_1, \ldots, X_n) = 0, \ldots, P_r(X_1, \ldots, X_n) = 0$ over F, then $\tau(X)$ will be locally defined by these equations together with

$$\sum_{i=1,\ldots,n} \left(\frac{\partial P_j}{\partial X_i}\right)(X_1, \ldots, X_n)v_i + P_j^{\partial}(X_1, \ldots, X_n) = 0,$$

for $j = 1, \ldots, r$ where P^{∂} is obtained from P by hitting the coefficients of P with the derivation ∂. $\tau(X)$ is a variety over F, irreducible if X is smooth, and comes with a canonical surjection to X. If X happens to be defined over k, $\tau(X)$ is precisely $T(X)$, the Zariski tangent bundle of X. τ is a functor, so if G is an algebraic group over F, then $\tau(G)$ is equipped with an algebraic group structure such that $\tau(G) \to G$ is a homomorphism. (So for G defined over k, this is the usual algebraic group structure on $T(G)$.) See [Mar00] for more details on the functor τ applied to algebraic groups.

We define an algebraic D-group to be a pair (G, s) where G is a connected algebraic group over F and $s : G \to \tau(G)$ is a section which is also a homomorphism of algebraic groups (over F). If X is a subvariety of G we say that X is a D-subvariety of (G, s) if $s|X : X \to \tau(X) \subseteq \tau(G)$. If $f : G_1 \to G_2$ is a homomorphism of algebraic groups then the functoriality of τ yields a homomorphism

65

$\tau(f) : \tau(G_1) \to \tau(G_2)$ commuting with the canonical surjections to G_1, G_2. So a morphism between algebraic D-groups (G_1, s_1) and (G_2, s_2) is a rational homomorphism $f : G_1 \to G_2$ such that $\tau(f) \cdot s_1 = s_2 \cdot f$. This agrees with Buium's definitions.

We will call the algebraic D-group (G, s) *split*, or *strongly isotrivial*, if there is an algebraic group G_0 defined over k such that (G, s) is isomorphic to (G_0, s_0) where $s_0 : G_0 \to T(G_0)$ is the 0-section.

We now give some final notation. Given an algebraic D-variety (X, s), $(X, s)^\sharp$ (or just $X^\sharp$ if s is understood) denotes $\{x \in X(F) : \partial(x) = s(x)\}$. This is a Zariski-dense subset of $X(F)$. If, moreover, (G, s) is an algebraic D-group, then $G^\sharp$ is a subgroup of $G(F)$. ($G^\sharp$ is a finite-dimensional differential algebraic group in the sense of Kolchin, and every such group arises this way.)

Our main result is as follows.

THEOREM 2.1. *Suppose that (G, s) is an algebraic D-group, X is a D-subvariety of (G, s), and $\mathrm{Stab}(X) = \{g \in G : g \cdot X = X\}$ is trivial, namely equals $\{e\}$ where e is the identity element of G. Assume also that $e \in X$ and X generates G in the sense that multiplication takes some $X^{\pm 1} \times X^{\pm 1} \times \cdots \times X^{\pm 1}$ onto G. Then (G, s) is strongly isotrivial.*

Before giving the proof let us remark that it is related to (and even motivated by) Campana's proof [Cam80] of a result of Ueno: if A is a complex torus and X an analytic subvariety of A with trivial stabilizer, then X is an algebraic variety.

Proof of Theorem 2.1. By assumption $e \in X$. For $p \geqslant 1$ let $j_p(G)_e$, the p-jet of G at e, be the dual space to m/m^{p+1} where m is the maximal ideal of the local ring of G at e. $j_p(G)_e$ is a finite-dimensional F-vector space. Similarly, if Y is a subvariety of G passing through e, we obtain $j_p(Y)_e$ as a F-subspace of $j_p(G)_e$. If Y is a member of an algebraic family of subvarieties, all passing through e, then Y is determined (in this family) by $j_p(Y)_e \subseteq j_p(G)_e$ for sufficiently large p. As $\mathrm{Stab}(X)$ is trivial, for $t_1, t_2 \in G$, $Xt_1^{-1} = Xt_2^{-1}$ if and only if $t_1 = t_2$. For $t \in X$, Xt^{-1} contains e. Thus, for sufficiently large p, the map taking $t \in X$ to $j_p(Xt^{-1})_e$ gives a birational embedding h of X into $Gr_r(j_p(G)_e)$, the variety of r-dimensional subspaces of $j_p(G)_e$ for suitable values of r. Let us fix such p. By virtue of s, the local ring of G at e is equipped with a derivation extending ∂, making it into a differential ring. As $e \in G^\sharp$, all powers of the maximal ideal m of this local ring are differential ideals. It follows that $V = m/m^{p+1}$ is equipped with the structure of a ∂-module (V, D_V) over F. The dual space $j_p(G)_e = V^*$ of V is equipped with the dual connection D_{V^*}: $(D_{V^*}(\lambda))(v) = \partial(\lambda(v)) - \lambda(D_V(v))$. So (V^*, D_{V^*}) is a ∂-module. As F is differentially closed, the theory of linear differential equations gives us a fundamental system of solutions of the equation $D_{V^*} = 0$ on V^*, that is a tuple $d = (d_1, \ldots, d_n)$ of elements of V^* which is simultaneously an F-basis of V^* and a k-basis of the solution space $(V^*)^\partial$ of $D_{V^*} = 0$. Note that $(V^*)^\partial$ is precisely the set of $\lambda \in V^*$ such that $\lambda(D_V(v)) = \partial(\lambda(v))$ for all $v \in V$. In any case, the basis d yields an identification of V^* with F^n and $(V^*)^\partial$ with k^n.

Now suppose that $t \in X^\sharp$. Then Xt^{-1} is a D-subvariety of G and $e \in (Xt^{-1})^\sharp$. $(Xt^{-1}, s|(Xt^{-1}))$ imposes on the local ring of Xt^{-1} at e a differential ring structure which is clearly induced by that on the local ring of G at e. As $e \in (Xt^{-1})^\sharp$ the maximal ideal m_t, say, of the local ring of (Xt^{-1}) at e is again a differential ideal. Let $W_t = j_p(Xt^{-1})_e$. Then W_t is a ∂-submodule of (V^*, D_{V^*}). As above, the solution space W_t^∂ of $D_{V^*} = 0$ on W_t is Zariski-dense in W_t. Identifying W_t with an r-dimensional subspace of F^n, via the basis d, we see that the set of k-points of W_t is Zariski-dense, whence W_t is defined over k. Thus $W_t = j_p(Xt^{-1})_e$ is a k-rational point of $Gr_r(F^n)$. We have obtained a birational isomorphism h of X with a subvariety Y of $Gr_r(F^n)$ such that, for generic $t \in X^\sharp$, $h(t)$ is rational over k. Note that, as $X^\sharp$ is Zariski-dense in X, Y is defined over k. It is now rather routine, using Weil's theorem, to find an algebraic group G_0 defined over k satisfying the conclusion of Theorem 2.1. We will give some details.

Let K be an algebraically closed subfield of F of finite transcendence degree over which all the data G, s, X, d are defined and such that K is algebraically independent from k over $k_0 = K \cap k$. There is no harm in assuming F to be a universal differential field (in the sense of Kolchin). We can find a generic point t of X over K such that $t \in X^\sharp$. From the previous paragraph we find some tuple c from k, such that t and c are birational over K, that is $K(t) = K(c)$. Write $c = f(t)$ where f is a K-rational generically defined generically invertible function on X. Let $t_1, \ldots, t_s$, say, be generic independent (over K) elements of X such that the product g of the t_i in G is a generic point of G over K, and such that all $t_i \in X^\sharp$. Note that $g \in G^\sharp$. Let $c_i = f(t_i)$. So $c_i \in k$ are independent over k_0, and we find $g' \in k_0(c_1, \ldots, c_s)$ birational with g over K. Write $g' = f'(g)$; again f' is K-rational and invertible. Let g_1, g_2 be chosen exactly like g, and independent over K. Let $g_i' = f'(g_i)$ for $i = 1, 2$. Let g_3 be the product of g_1 and g_2 in G. So $g_3 \in G^\sharp$ and $f'(g_3) = g_3'$ is rational over k too. g_1', g_2', g_3' are generic points of a variety W say defined over k_0, and each is k_0-rational over the other two. By virtue of the construction, W is equipped with a generic group law and so is birationally isomorphic over k_0 to an algebraic group G_0 defined over k_0 such that $g_3' = g_1' \cdot g_2'$. Moreover, f' induces a K-rational isomorphism of G with G_0. By construction f' takes $G^\sharp$ to $G_0(k)$. It follows that f' is also an isomorphism of algebraic D-groups between (G, s) and (G_0, s_0), where $s_0 : G_0 \to T(G_0)$ is the 0-section. $\qquad\square$

We now complete the proof of Theorem 1.1. The global structure of the proof remains that of Hrushovski [Hru96], and uses Buium's construction [Bui92b] of a 'finite-dimensional differential algebraic' subgroup of A containing Γ.

Proof of Theorem 1.1. We may assume Γ to be Zariski-dense in A. By possibly enlarging K we can choose a derivation ∂ on K whose field of constants is precisely k and such that (K, ∂) is differentially closed. We may assume (K, ∂) is universal in the sense of Kolchin. Buium ($\S$ 2 of [Bui92b]) produces a finite-dimensional differential algebraic subgroup of $A(F)$ containing Γ. Strictly speaking he does this assuming A to be an abelian variety. In [Hru96] it is stated that this also goes through in the semi-abelian case. Alternatively, it follows from statement C5 in Appendix C of [Bui92a]. In any case, in the algebraic D-group formalism, Buium's construction amounts to the following.

There is a connected commutative algebraic D-group (G, s) and a surjective homomorphism (of algebraic groups) $\pi : G \to A$ such that:

i) $L = \ker(\pi)$ is unipotent (and is thus the unipotent radical of G, as A is semi-abelian);

ii) $\pi | G^\sharp$ is injective;

iii) $\Gamma \subset \pi(G^\sharp)$.

Let X_1 be the Zariski closure in G of $\pi^{-1}(X) \cap G^\sharp$. Then X_1 is a (possibly reducible) D-subvariety of G. Replacing X_1 by a suitable irreducible component, we obtain a D-subvariety X_2 of G such that π maps X_2 into a Zariski-dense subset of X (using the Zariski-denseness of $\Gamma \cap X$ in X).

CLAIM. X_2 has trivial stabilizer in G.

Proof. $\mathrm{Stab}(X_2)$ is a D-subgroup of G, so if non-trivial it has a non-trivial point a in $G^\sharp$. But then $\pi(a)$ is a non-trivial point in $\mathrm{Stab}(X)$ in A, contradiction. $\qquad\square$

Replacing X_2 by a translate by a point of $X_2^\sharp$ we may assume that $e \in X_2$. Let G_0 be the connected algebraic subgroup of G generated by X. Then G_0 is a D-subgroup of (G, s) and X_2 a D-subvariety. By Theorem 2.1, we may assume that (G_0, s) is defined over k and s is the 0-section. But then $G_0^\sharp$ is precisely $G_0(k)$. As (by item i) $L = \ker(\pi)$ is the unipotent radical

of G, $L_1 = \ker(\pi|G_0)$ is the unipotent radical of G_0, so if non-trivial it has non-trivial points in $G_0(k)$. But this contradicts item ii. Thus $\pi|G_0$ is an isomorphism with an algebraic subgroup G_1 of G. Clearly $\pi(X_2)$ is a translate of X. Finally, as X_2 is a D-subvariety of G_0 and $s = 0$ we see that X_2 is defined over k. This completes the proof of Theorem 1.1. □

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