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RETRACTS OF PARTIALLY ORDERED SETS

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Abstract

Let P be a finite, connected partially ordered set containing no crowns and let Q be a subset of P. Then the following conditions are equivalent:
(1) Q is a retract of P;
(2) Q is the set of fixed points of an order-preserving mapping of P to P;
(3) Q is obtained from P by dismantling by irreducibles.

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1. Introduction

Let P and Q be partially ordered sets. We call Q a *retract* of P if there are order-preserving mappings g of Q to P and f of P to Q such that $f \circ g$ is the identity mapping of Q. In particular, a subset Q of P is a retract of P provided that there is an order-preserving mapping f of P onto Q such that f is the identity mapping on Q; in this case, we call f a *retraction mapping* of P onto Q.

Retracts promise to play a significant role in combinatorial investigations of finite partially ordered sets. To give an example, let us consider the fixed point problem. A partially ordered set P has the *fixed point property* if every order-preserving map of P to P fixes an element of P: *P has the fixed point property if and only if every retract of P has the fixed point property*; in fact, *a finite partially ordered set P is fixed point free if and only if P has a retract with a fixed point free automorphism* (Duffus et al. (1977)).

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Here we examine a condition related to the fixed point property.

Let P be a partially ordered set. For elements $a > b$ in P, we say *a covers b* or a is an *upper cover* of b (denoted $a \succ b$) if, for all $c \in P$, $a \geqslant c > b$ implies $a = c$. An element a of P is *irreducible* in P if a has precisely one upper cover (denoted by a^*) or precisely one lower cover (a_*) in P. We let $I(P)$ denote the set of irreducible elements of P. P is called *connected* if for all $a, b \in P$ there is a sequence $a = a_0, a_1, \ldots, a_n = b$ of elements of P such that a_i is comparable with a_{i+1} $(i = 0, 1, \ldots, n-1)$; otherwise, P is *disconnected.*

Let P be finite. A nonempty subset Q of P is *obtained from P by dismantling* (*by irreducibles*) if $P - Q = \{a_1, a_2, \ldots, a_n\}$ and

$$a_i \in I(P - \{a_1, a_2, \ldots, a_{i-1}\}) \quad (i = 1, 2, \ldots, n).$$

We call *P dismantlable* (*by irreducibles*) if a singleton subset of P is obtained from P by dismantling by irreducibles. (Note that a dismantlable partially ordered set is connected.) Every dismantlable partially ordered set has the fixed point property (Rival (1976)). Moreover, *every retract of a dismantlable partially ordered set is dismantlable* (Duffus et al. (1977)).

Which subsets of a given partially ordered set are retracts? For instance, any subset of a partially ordered set P that, under the induced ordering, is a complete lattice, is a retract of P (Birkhoff (1937)). Beyond this fact very little is known about this question. The purpose of this paper is to answer the question in the case that P contains no crowns.

For an integer $n \geqslant 3$, a *2n-crown*, or simply a *crown*, is a $2n$-element partially ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ so that $x_i < y_i, x_{i+1} < y_i$ $(i = 1, 2, \ldots, n-1)$ and $x_1, x_n < y_n$ are the *only* comparabilities (see Fig. 1). A subset C of a partially ordered set P is a $2n$-crown in P provided that, with the partial ordering inherited from P, C is a $2n$-crown. A *four-crown* in P is a four-element subset $\{x_1, y_1, x_2, y_2\}$ of P so that $x_i < y_j$ $(i, j = 1, 2)$ are the *only* comparabilities *and* there is no $z \in P$ such that $x_i \leqslant z \leqslant y_j$ $(i, j = 1, 2)$.

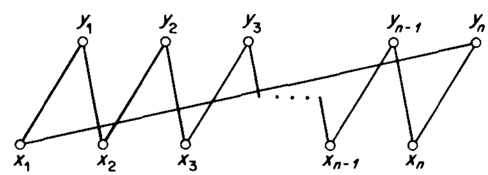

FIG. 1. A $2n$-crown.

Our main result provides a charàcterization of retracts of partially ordered sets without crowns; in fact, we provide a 'canonical' procedure for obtaining retracts of such partially ordered sets.

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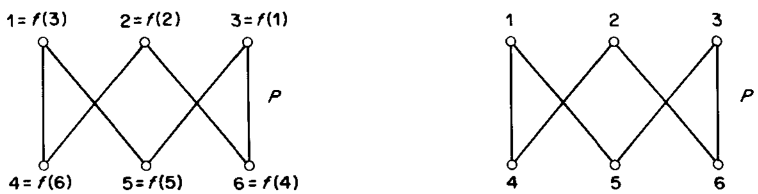
THEOREM. *Let P be a finite connected partially ordered set containing no crowns and let Q be a subset of P. Then the following conditions are equivalent:*

(1) *Q is a retract of P;*

(2) *Q is the set of fixed points of an order-preserving mapping of P to P;*

(3) *Q is obtained from P by dismantling by irreducibles.*

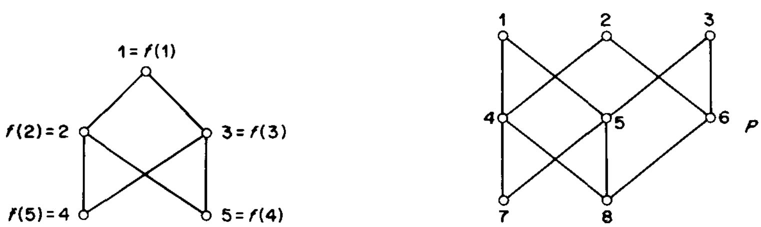
Actually, (1) follows from (3), and (2) follows from (1) for any finite partially ordered set P. However neither converse, (2) implies (1), nor (1) implies (3), holds for arbitrary finite connected partially ordered sets (see Figs. 2 and 3). A finite connected partially ordered set containing no crowns is dismantlable (Duffus and Rival (1976)); however, the examples of Fig. 3 demonstrate that the theorem cannot hold for an arbitrary dismantlable partially ordered set. The connectivity hypothesis of the theorem ensures against degenerate cases: indeed, every subset of an unordered set is a retract.



(a) $Q = \{x \in P \mid f(x) = x\}$ is not a retract of P.

(b) $Q = \{1, 2, 4, 5\}$ is a retract of P, $I(P) = \emptyset$.

FIG. 2.



(a) $Q = \{x \in P \mid f(x) = x\}$ is not a retract of P.

(b) $Q = \{3, 5, 6, 7, 8\}$ is a retract of P, $I(P) \subseteq Q$.

FIG. 3.

2. Preliminaries

Let P be a partially ordered set. Let P^P denote the set of all order-preserving mappings of P to P. For $f \in P^P$, we let $P(f) = \{x \in P \mid f(x) = x\}$. Also, let f^0 be the identity mapping on P, let $f^1 = f$ and $f^i = f \circ (f^{i-1})$ $(i = 1, 2, \ldots)$. For $a \in P$, we set $f_a = (f^i(a) \mid i = 0, 1, \ldots\}$.

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LEMMA 1. *Let P be a finite partially ordered set and let $f \in P^P$. Then there exists a positive integer n such that $f' = f | f^n(P)$ is an automorphism of $f^n(P)$ and $f^n(P)$ is a retract of P.*

PROOF. Since P is finite there is a positive integer n such that

$$P \supseteq f(P) \supseteq f^2(P) \supseteq \ldots \supseteq f^n(P) = f^{n+1}(P);$$

that is, $f' = f | f^n(P)$ is an automorphism of $f^n(P)$. We choose a positive integer k such that $(f')^k$ is the identity mapping of $f^n(P)$. Then f^{nk} is a retraction mapping of P onto $f^n(P)$. This completes the proof.

A *fence* is a partially ordered set obtained from a crown by deleting either a single element or a comparable pair of elements from the crown (see Fig. 4). The elements of a fence comparable with only one other element of the fence are *endpoints.*

For two elements x and y of a partially ordered set, we write $x || y$ provided that x is noncomparable with y.

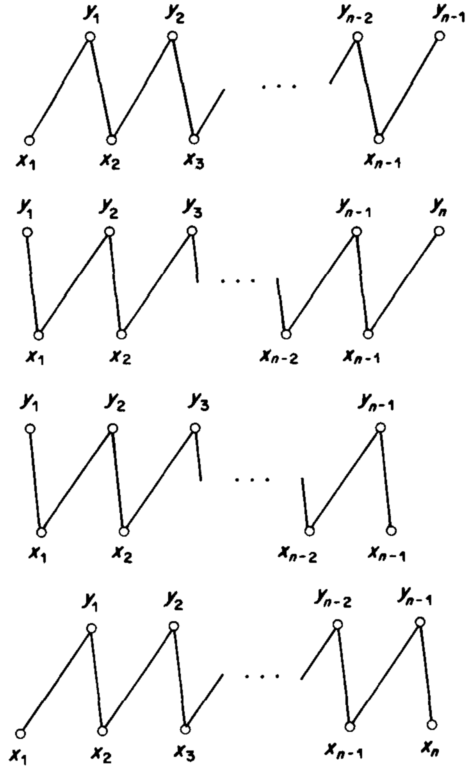

FIG. 4. Fences.

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LEMMA 2 (Kelly and Rival (1974)). *Let P be a finite partially ordered set containing no crowns and let* $F = \{x_1, x_2, \ldots, x_m\}$ *be a fence contained in P with* $m \geq 3$ *and the comparabilities* $x_1 > x_2$, $x_2 < x_3$, $x_3 > x_4$, ,... $x_{m-1} > x_m$. *If* x_{m+1} *belongs to P and satisfies* $x_{m+1} > x_m$, $x_{m+1} \| x_{m-1}$ *and* $x_{m+1} \| x_{m-2}$ *then* $F \cup \{x_{m+1}\}$ *is a fence.*

PROOF. We need to show that $x_{m+1} \| x_j$ for $1 \leq j \leq m-3$. If $x_{m+1} < x_j$ then $x_m < x_j$ and, since F is a fence, $x_j = x_{m-1}$. Therefore, x_{m+1} is not less than any element of F. Suppose x_{m+1} is greater than some element of $F - \{x_m\}$. Let k be the greatest integer such that $x_k < x_{m+1}$. Since $x_{m-2} \nless x_{m+1}$, $1 \leq k < m-3$. Also, by our choice of k, x_k is a minimal element of F. It follows that $\{x_k, x_{k+1}, \ldots, x_{m-1}, x_m, x_{m+1}\}$ is a $2n$-crown ($n \geq 3$).

Finally, we shall repeatedly apply the following fact in the proof of the theorem: if a partially ordered set P contains no crowns then every retract of P contains no crowns.

3. Proof of the theorem

From the definition of a retract of a partially ordered set it follows that (1) implies (2) in any partially ordered set.

To see that (3) implies (1) let $P - Q = \{a_1, a_2, \ldots, a_n\}$, $a_i \in I(P - \{a_1, a_2, \ldots, a_{i-1}\})$, and let f_i be the mapping of $P - \{a_1, a_2, \ldots, a_{i-1}\}$ to $P - \{a_1, a_2, \ldots, a_i\}$ that is the identity on $P - \{a_1, a_2, \ldots, a_i\}$ and maps a_i to a'_i where a'_i is either the unique upper cover or unique lower cover of a in $P - \{a_1, a_2, \ldots, a_{i-1}\}$ $(i = 1, 2, \ldots, n)$. Then $f = f_n \circ f_{n-1} \circ \ldots \circ f_1$ is a retraction mapping of P onto the subset Q.

The rest of the proof is devoted to showing that (2) implies (3). Let $f \in P^P$ and let $Q = P(f)$. We proceed by induction on $|P - Q|$. Let $P - Q = \{a\}$. As P is connected, we may assume that a has two upper covers b and c. If $f(a) \| a$ then $\{a, b, f(a), c\}$ is a four-crown in P. Clearly, $f(a) \ngtr a$; therefore, $f(a) < a$ and $f(a)$ must be the unique lower cover of a in P.

Let n be the positive integer guaranteed by Lemma 1, let $R = f^n(P)$, and choose k so that f^{nk} is a retraction mapping of P onto R. Then $R = P(f^{nk})$, $Q \subseteq R$ where $Q = R(f')$ and $f' = f|R$, and R contains no crowns. If $|P - R| < |P - Q|$ and $|R - Q| < |P - Q|$ then, by our induction hypothesis, Q is obtained from R by dismantling by irreducibles and R is obtained from P by dismantling by irreducibles. Consequently, Q is obtained from P by dismantling. Therefore, either $P = R$ or $R = Q$.

Let us suppose that $P = R$; that is, f is an automorphism of P. In this case we shall show that there exists $a \in I(P) - Q$. Let us first verify that this is sufficient to complete the proof when $P = R$. Since f is an automorphism of P and $a \in I(P) - Q$,

$f_a = \{a = f^k(a), f(a), f^2(a), \ldots, f^{k-1}(a)\}$, $k \geqslant 2$, and

$$f^i(a) \in I(P - \{a, f(a), \ldots, f^{i-1}(a)\}) \quad (i = 1, 2, \ldots, k-1).$$

Since $P' = P - f_a$ is obtained from P by dismantling by irreducibles, P' contains no crowns. We have $Q = P'(f|P')$ and, applying the induction hypothesis to $P' - Q$, we conclude that Q is obtained from P by dismantling by irreducibles.

Suppose that $I(P) \subseteq Q$ and choose $a \in P - Q$. Since P is connected a has two upper covers x_1 and y_1. Assuming $x_1 \notin Q$, we choose x_2 such that $x_2 \prec x_1$ and $x_2 \| a$. Suppose we have obtained the fence $F = \{a, x_1, x_2, \ldots, x_m\} \subseteq P - Q$ with comparabilities $a \prec x_1$, $x_1 \succ x_2$, $x_2 \prec x_3$, $\ldots$, $x_{m-1} \succ x_m$. Since $x_m \notin I(P)$, we can choose x_{m+1} such that $x_m \prec x_{m+1}$; by the covering relations in F, $x_{m+1} \| x_{m-1}$ and $x_{m+1} \| x_{m-2}$. By Lemma 2, $F \cup \{x_{m+1}\}$ is a fence. Since P is finite and contains no crowns there is an integer $s \geqslant 1$ such that $F_1 = \{a, x_1, x_2, \ldots, x_s\}$ is a fence in P satisfying $F_1 - \{x_s\} \subseteq P - Q$, $x_s \in I(P)$, and with comparability relations

$$a \prec x_1, x_1 \succ x_2, x_2 \prec x_3, \ldots, x_{s-1} \succ x_s.$$

(The case $x_s \succ x_{s-1}$ is similar.) Also, there is an integer $t \geqslant 1$ such that $F_2 = \{a, y_1, y_2, \ldots, y_t\}$ is a fence in P satisfying $F_2 - \{y_t\} \subseteq P - Q$, $y_t \in I(P)$ and

$$a \prec y_1, y_1 \succ y_2, y_2 \prec y_3, \ldots, y_{t-1} \prec y_t.$$

(Again, the case $y_{t-1} \succ y_t$ in F_2 is similar.) Since P contains no crowns, $F_1 \cap F_2 = \{a\}$ and $F_1 \cup F_2$ is a fence in P (Lemma 2). Set

$$F' = \left(\bigcup_{i=1}^{s} f_{x_i}\right) \cup \left(\bigcup_{j=1}^{t} f_{y_j}\right) \cup f_a.$$

Let us consider $f^i(x_j)\,(1 \leqslant j \leqslant s-1)$ and suppose $x_j \succ x_{j-1}$, $x_j \succ x_{j+1}$. Then $f^i(x_j) \succ f^i(x_{j-1})$, $f^i(x_j) \succ f^i(x_{j+1})$ and $x_j \neq x_{j+1}$ implies $f^i(x_j) \neq f^i(x_{j+1})$. Also, $x_{s-1} \neq f(x_{s-1})$ $(y_{t-1} \neq f(y_{t-1}))$; therefore, x_s is covered by at least two distinct elements of F' (y_t covers two distinct elements of F'). Since F' is a connected partially ordered set containing no three-element chains and each element of F' has two upper covers or two lower covers in F', F' contains a crown.

We turn to the case that $R = Q$. Then f is a retraction mapping of P onto Q; in fact, by the induction hypothesis, Q is a maximal proper retract of P and $I(P) \subseteq Q$. We shall complete the proof by constructing a crown in $P - Q$. Since the construction is long and detailed, we divide it into several steps.

First, we record some straightforward observations. If $x \in \max(Q)$, the set of maximal elements of Q, and $x \prec y$ in P then $f(y) = x$; hence, the mapping f' of P to P defined by

$$f'(z) = \begin{cases} y, & \text{if } z \geqslant y, \\ f(z) & \text{otherwise} \end{cases}$$

is a retraction mapping of P onto $Q \cup \{y\}$. This contradicts the maximality of Q. Therefore, $\max(Q) \subseteq \max(P)$ and $\min(Q) \subseteq \min(P)$. Also, if $y \in \max(P) - Q$ and $f(y) < y$ then f', as defined above, is again a retraction mapping. Let $N = \{x \in P \mid x \| f(x)\}$. We have

$$(\max(P) \cup \min(P)) - Q \subseteq N.$$

Moreover, we may take $f(y) \in \max(Q)$ $(\min(Q))$ for any $y \in \max(P)$ $(\min(P))$.

Since P contains no four-crowns, a subset of P that is bounded above (below) has a supremum (infimum). Now, if $z \in P$ and $z = \sup(X) = \inf(Y)$ for $X, Y \subseteq Q$ then $z \in Q$. Of course, each $z \in P$ is the supremum (infimum) of $\{y \in P \mid y \leqslant z\}$ $(\{y \in P \mid y \geqslant z\})$. In fact, since every chain in P is finite, z is the supremum (infimum) of those elements $y \leqslant z$ $(y \geqslant z)$ that have at most one lower (upper) cover. Since $I(P) \subseteq Q$, it follows that $(\max(P) \cup \min(P)) \subseteq Q$.

A. *$P - Q$ contains a maximal chain C of P such that* $\sup(C) \in N$ *and* $\inf(C) \in N$.

Let $a \in \max(P) - Q$; $a \notin I(P)$ so $a = \sup(y \mid a \succ y)$. If each lower cover of a belongs to Q then $a \in Q$. Choose $a_1 \in P - Q$ such that $a_1 \prec a$. If $a_1 \notin \min(P)$, we choose $a_2 \prec a_1$ such that $a_2 \in P - Q$. Continuing in this manner we obtain a maximal chain of P contained in $P - Q$.

Of course, $\sup(C) \in N$ and $\inf(C) \in N$.

B. *There is a fence $F = \{u, x_1, x_2, \ldots, x_m, v\}$ $(m \geqslant 2)$ in P with endpoints $u, v \in Q$. For each i, $2 \leqslant i \leqslant m-1$, $x_i \in N$ and one of* (1), (1′), (2) *or* (2′) *obtains*:

(1) $u < x_1$, $x_1 \succ x_2, \ldots, x_{m-1} \prec x_m$, $x_m > v$, *and* $f(x_1) \ngtr x_1$, $f(x_m) \ngtr x_m$;

(1′) $u > x_1$, $x_1 \prec x_2, \ldots, x_{m-1} \succ x_m$, $x_m < v$, *and* $f(x_1) \nless x_1$, $f(x_m) \nless x_m$;

(2) $u < x_1$, $x_1 \succ x_2, \ldots, x_{m-1} \succ x_m$, $x_m < v$, *and* $f(x_1) \ngtr x_1$, $f(x_m) \nless x_m$;

(2′) $u > x_1$, $x_1 \prec x_2, \ldots, x_{m-1} \prec x_m$, $x_m > v$, *and* $f(x_1) \nless x_1$, $f(x_m) \ngtr x_m$.

For the proof of **B** we shall require some further notation. Let $x \succ y$ in P. We set

$$S(x,y) = \{z \in P \mid z < x \text{ and } z \nleqslant y\}$$

and

$$T(x,y) = \{z \in P \mid z \ngeqslant x \text{ and } z > y\}.$$

If $x \in P - Q$ then $S(x,y) \neq \varnothing$ and if $y \in P - Q$ then $T(x,y) \neq \varnothing$.

Let E be the set of ordered pairs (x,y) satisfying the following properties:

(i) $x > y$ in P;

(ii) $x, y \in N$;

(iii) $\{z \in P \mid y \leqslant z \leqslant x\}$ contains a maximal chain C such that $C \subseteq P - Q$.

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Now, let D be that subset of E consisting of ordered pairs (u,v) such that, if $(x,y)\in E$ and $u\geqslant x>y\geqslant v$, then $u=x$ and $v=y$. Note that an ordered pair (u,v) in E belongs to D precisely if $[v,u]$ is minimal (with respect to set inclusion) in $\{[y,x]\,|\,(x,y)\in E\}$.

By **A**, there is a maximal chain C of P such that $C\subseteq P-Q$ and $\inf(C), \sup(C)\in N$. Therefore, D is nonempty.

The proof of **B** is divided into two cases. In the first case we assume there exists $(x,y)\in D$ such that $x \not\succ y$ in P. (The reader may find it helpful to refer to the schematic diagram in Fig. 5 which illustrates the construction.)

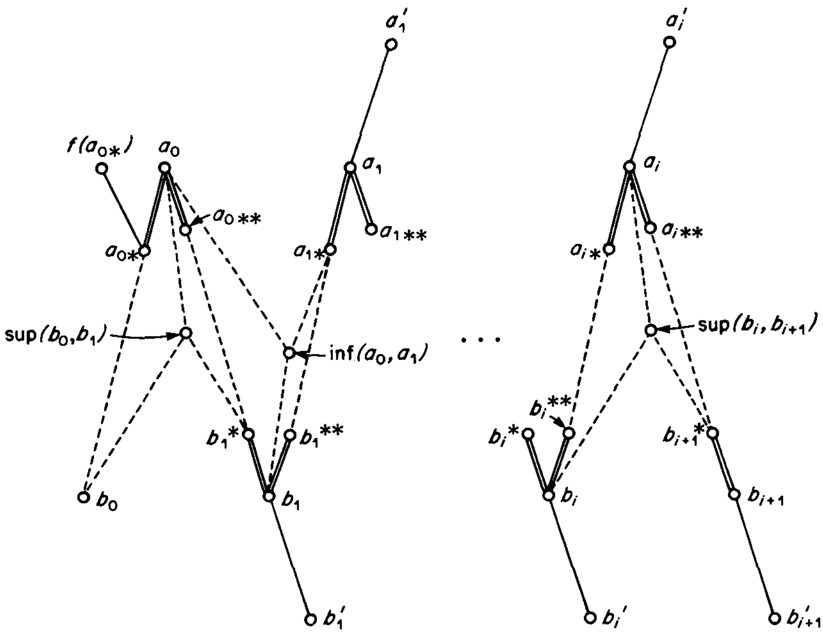

FIG. 5. Construction scheme.

Let $(a_0,b_0)\in D$ with the associated chain C and choose a_{0*}, b_0^* in C such that $a_0\succ a_{0*}\geqslant b_0^*\succ b_0$. Then $f(a_{0*})$ is comparable with a_{0*} and $f(b_0^*)$ is comparable with b_0^*. Since C is finite and any $z\in C$ with $a_{0*}\leqslant z\leqslant b_0^*$ is comparable with $f(z)$, either $f(a_{0*})>a_{0*}$ or $f(b_0^*)<b_0^*$. Let us suppose the former. If there exists $c\in S(a_0,a_{0*})\cap Q$ then $\{f(a_0),c,a_0,a_{0*}\}$ is a four-crown. Therefore,

$$S(a_0,a_{0*})\subseteq P-Q.$$

Take b_1' to be a minimal element of $S(a_0,a_{0*})$. Since $b_1'\notin I(P)$, b_1' has no lower covers or at least two lower covers in P. If $b_1'\succ x,y$ then $\{b_1',x,y,a_{0*}\}$ is a four-crown in P. Hence, $b_1'\in\min(P)$ and $b_1'\in N$. Now we choose b_1 to be a maximal

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element of $S(a_0, a_{0*})$ satisfying $b_1 \in N$. It follows that $(a_0, b_1) \in D$. Clearly $b_1 \nleqslant b_0$; moreover, $b_1 > b_0$ contradicts $(a_0, b_0) \in D$. We now choose b_1^* such that $\sup(b_0, b_1) \geqslant b_1^* \succ b_1$ and, if $a_0 \not\succ b_1$, we choose a_{0**} such that $a_0 \succ a_{0**} \geqslant b_1^*$. Notice that if $b_1^* \neq a_0$ then $b_1^*, a_{0**} \in P - Q$ and $b_1^*, a_{0**} \notin N$.

We now *assume* that $T(b_1^*, b_1) \subseteq P - Q$ and choose a maximal element a_1' of $T(b_1^*, b_1)$. Then $a_1' \in \max(P)$ and, therefore, $a_1' \in N$. Let a_1 be minimal in $T(b_1^*, b_1)$ satisfying $a_1 \in N$. Again, $(a_1, b_1) \in D$. Let b_1^{**}, a_{1*} satisfy

$$a_1 \succ a_{1*} \geqslant \inf(a_0, a_1), \quad a_1 \geqslant b_1^{**} \succ b_1$$

and, provided that $a_1 \not\succ b_1$, $a_{1*} \geqslant b_1^{**}$. Now $a_1 \ngeqslant a_0$ and $a_1 < a_0$ contradicts $(a_0, b_1) \in D$. Therefore $a_1 \| a_0$. Of course $b_0 \ngeqslant a_1$; also, $a_1 \ngeqslant b_1^*$ implies $a_1 \ngeqslant b_0$.

Suppose we have obtained the fence $\{b_0, a_0, b_1, \ldots, b_i, a_i\}$ $(i \geqslant 2)$ as above. *Assuming* that $S(a_i, a_{i*}) \subseteq P - Q$, we let b_{i+1} be a maximal element of $S(a_i, a_{i*})$ satisfying $b_{i+1} \in N$. Then $(a_i, b_{i+1}) \in D$. Since $b_{i+1} \nleqslant a_{i*}$, $b_{i+1} \nleqslant b_i$. Since $b_{i+1} > b_i$ contradicts $(a_i, b_i) \in D$, $b_{i+1} \| b_i$. Since $a_i \ngeqslant a_{i-1}$, $b_{i+1} \ngeqslant a_{i-1}$. If $b_{i+1} < a_{i-1}$ then

$$b_{i+1} < \inf(a_i, a_{i-1}) \leqslant a_{i*}.$$

Therefore, by the dual of Lemma 2, $\{b_0, a_0, \ldots, b_i, a_i, b_{i+1}\}$ is a fence in P. We choose b_{i+1}^*, a_{i**} in P satisfying

$$\sup(b_i, b_{i+1}) \geqslant b_{i+1}^* \succ b_{i+1}, \quad a_i \succ a_{i**} \geqslant b_{i+1}$$

and, provided that $a_i \not\succ b_{i+1}$, $a_{i**} \geqslant b_{i+1}^*$. We proceed in a similar manner to adjoin a_{i+1} to $\{b_0, a_0, \ldots, b_i, a_i, b_{i+1}\}$ under the assumption that $T(b_{i+1}^*, b_{i+1}) \subseteq P - Q$.

Since P is a finite, connected partially ordered set and P contains no crowns, there is a least integer n such that

$$T(b_n^*, b_n) \nsubseteq P - Q$$

or

$$S(a_n, a_{n*}) \nsubseteq P - Q \quad (n \geqslant 1).$$

Let us take $c \in T(b_n^*, b_n) \cap Q$. If $f(b_n^*) < b_n^*$ then $\{b_n^*, f(b_n), b_n, c\}$ is a four-crown in P. Therefore, *either* $f(b_n^*) \| b_n^*$, implying $a_{n-1} = b_n^* \succ b_n$, *or* $f(b_n^*) > b_n^*$. In the latter case, $f(a_{n-1**}) > a_{n-1**}$; hence, *either* $f(a_{n-1*}) \| a_{n-1*}$, implying

$$a_{n-1} \succ a_{n-1*} = b_{n-1},$$

or $f(a_{n-1*}) < a_{n-1*}$. Continuing, we obtain either $a_i \succ b_i$ or $a_{i-1} \succ b_i$ $(i \geqslant 1)$ for otherwise we contradict $f(a_{0*}) > a_{0*}$. Let us take $a_i \succ b_i$ and $a_i \succ a_{i**} > b_{i+1}$. (The cases $a_i \succ b_{i+1}$, $b_{i+1}^{**} > a_{i+1}$ $(i<n-1)$ and $a_{n-1} \succ b_n$ are similar.) Now choose the integer j least such that

(a) $a_j \succ b_{j+1}, b_{j+1} \prec a_{j+1}, \ldots, b_i \prec a_i \quad (0 \leqslant j \leqslant i-1)$

or

(b) $b_j \prec a_j, a_j \succ b_j, \ldots, b_i \prec a_i \quad (1 \leqslant j \leqslant i)$.

If (a) holds and $f(a_{j*}) > a_{j*}$ then we consider the fence

$$F' = \{f(a_{j*}), a_{j*}, a_j, b_{j+1}, \ldots, b_i, a_i, a_{i**}, f(a_{i**})\}.$$

F' satisfies condition (1′) on F specified in **B** (this case is illustrated in Fig. 6). Similarly, if the comparabilities in (b) obtain and $f(b_j^*) < b_j^*$ we consider the fence

$$F'' = \{f(b_j^*), b_j^*, b_j, a_j, \ldots, b_i, a_i, a_{i**}, f(a_{i**})\}.$$

F'' satisfies condition (2) on F in B.

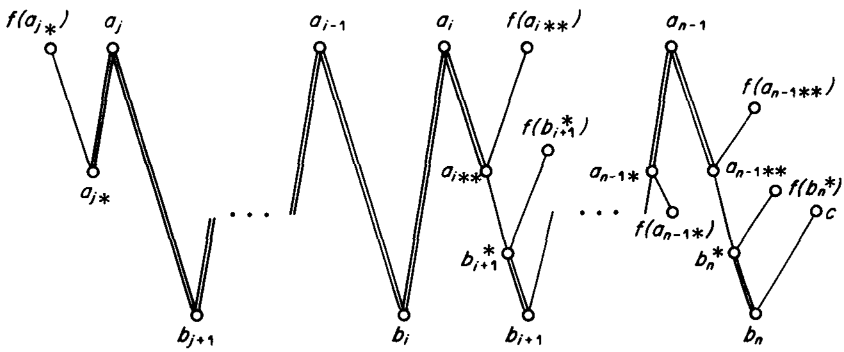

FIG. 6. Construction scheme.

Let us suppose (a) holds and $f(a_{j*}) < a_{j*}$. (Again, the case that (b) obtains and $f(b_j^*) > b_j^*$ is similar.) Then $f(b_j^{**}) < b_j^{**}$ and, therefore, *either* $a_{j-1} \succ b_j$ *or* $f(b_j^*) > b_j^*$. In any event we obtain $a_k \succ b_k$ or $a_k \succ b_{k-1}$ $(k<j)$ and repeat the argument above. Eventually we obtain a fence F with properties specified in **B**.

We now consider the case that $x \succ y$ whenever $(x,y) \in D$. Let $(a_0, b_0) \in D$. If there exist $c \in S(a_0,b_0) \cap Q$ and $d \in T(a_0,b_0) \cap Q$ then either $\{a_0, b_0, f(a_0), f(b_0), c, d\}$ is a six-crown or one of its four-element subsets is a four-crown in P. If $c \in S(a_0,b_0) \cap Q$ and $T(a_0,b_0) \subseteq P-Q$ then, as above, we obtain a fence F satisfying the conditions specified in **B** with $c=u$. If both $S(a_0,b_0) \subseteq P-Q$ and $T(a_0,b_0) \subseteq P-Q$, we construct two fences in P, as above, one beginning with $\{a_0, b_0, b_1'\}$, $b_1' \in S(a_0,b_0)$ and the other with $\{a_0, b_0, a_1''\}$, $a_1'' \in T(a_0,b_0)$. Then the union of the two fences is the required fence (this case is illustrated in Fig. 7). The proof of **B** is complete.

For the remainder of the proof we fix the fence $F=\{u,x_1,x_2,\ldots,x_m,v\}$ $(m\geqslant 2)$ with $x_i\in N$ $(2\leqslant i\leqslant m-1)$, $u,v\in Q$ and satisfying, say, (1) of **B** if $m\geqslant 3$ and satisfying (2) of **B** if $m=2$.

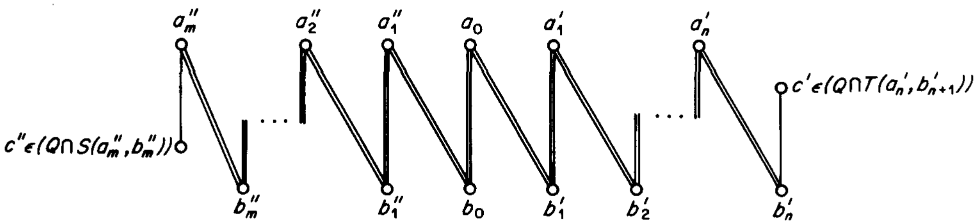

FIG. 7. Construction scheme.

C. *Each of the sets* $T(x_1,x_2)$, $S(x_3,x_2)$, $T(x_3,x_2)$, $S(x_3,x_4)$, $T(x_3,x_4),\ldots,$ $S(x_{m-2},x_{m-1})$, $T(x_{m-2},x_{m-1})$, $T(x_m,x_{m-1})$ *is contained in* $P-Q$.

To establish **C** it is enough to take F to be a fence of minimum cardinality in P satisfying the conditions listed in **B**.

Finally, we show

D. $F\cup f(F)=\{u,x_1,f(x_1),x_2,f(x_2),\ldots,x_m,f(x_m),v\}$ *contains a crown.*

Let us first suppose that $m=2$. Let $f(x_2)\|x_2$. If, in addition, $f(x_1)\|x_1$ then F' is a six-crown or a four-element subset forms a four-crown in P. If $f(x_1)<x_1$ then $\{v,x_1,x_2,f(x_2)\}$ is a four-crown in P. Therefore, $f(x_2)>x_2$. But now $f(x_1)\|x_1$, since otherwise $x_1>f(x_1)\geqslant f(x_2)>x_2$ contradicts $x_1\succ x_2$, and $\{f(x_1),u,x_1,x_2\}$ is a four-crown in P.

If $m=3$ an argument similar to that above demonstrates that P contains a crown.

Let us assume $m\geqslant 4$. If $f(x_1)<x_j$ for some j, $2\leqslant j\leqslant m-1$, then $f(x_1)<x_j$ for $j=2,3,\ldots,m-1$, by **C**. But then $u\leqslant f(x_1)<x_2$, which is nonsense. Similarly, $f(x_1)>x_j$ for some j, $2\leqslant j\leqslant m-1$, implies $f(x_1)>x_1$: this contradicts (1) in **B**. Hence, $f(x_1)\|x_j$ $(j=2,3,\ldots,m-1)$ and, similarly, $f(x_m)\|x_j$ $(j=2,3,\ldots,m-1)$. It is easy to see that each element of $\{f(x_i)\,|\,i=1,2,\ldots,m\}$ is noncomparable with each x_i $(i=2,3,\ldots,m-1)$. Also, if some $f(x_i)$ $(1\leqslant i\leqslant m)$ is comparable with $x_1(x_m)$ then $f(x_i)<x_1$ $(f(x_i)<x_m)$.

Suppose there exist integers j, k, $2\leqslant j$, $k\leqslant m-1$ such that $f(x_j)<x_1$ and $f(x_k)<x_m$. We can choose x_s and x_t in $\{x_i\,|\,i=2,3,\ldots,m-1\}$ so there is a fence $T\subseteq\{f(x_i)\,|\,i=2,3,\ldots,m-1\}$ containing $f(x_s)$, $f(x_t)$ such that $y\in T$,

$$y\notin\{f(x_s),f(x_t)\}$$

implies $y\|x_1$ and $y\|x_2$. Since each element of $T-\{f(x_s),f(x_t)\}$ is noncomparable with each element of $\{x_i\,|\,i=1,2,\ldots,m\}$, $T\cup\{x_1,x_2,\ldots,x_m\}$ contains a crown.

Let us suppose $x_1 \| f(x_i)$ $(i = 2, 3, \ldots, m-1)$ and $f(x_i) < x_m$ for some such i. Then $x_1 \| f(x_1)$ and we can choose s, $2 \leqslant s \leqslant m-1$, so that $f(x_s) < x_m$ and in order that there is a fence $T' \subseteq \{f(x_i) \mid i = 1, 2, \ldots, m-1\}$ with endpoints $f(x_1)$ and $f(x_s)$ and containing no other element comparable with an element of $\{u, x_1, x_2, \ldots, x_m\}$. Then $T \cup \{u, x_1, x_2, \ldots, x_m\}$ contains a crown.

Finally, if $x_1 \| f(x_i)$ and $x_m \| f(x_i)$ $(i = 2, 3, \ldots, m-1)$ then $x_1 \| f(x_1)$, $x_m \| f(x_m)$; that is, $\{f(x_1), f(x_m)\} \cup F$ is a fence. Clearly, $F \cup f(F)$ contains a crown.

The proof of the theorem is now complete.

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