Nagoya Math. J., **227** (2017), 86–159
DOI 10.1017/nmj.2016.50

SEMISTABLE SHEAVES WITH SYMMETRIC c_1 ON A QUADRIC SURFACE

TAKESHI ABE

Abstract. For moduli spaces of sheaves with symmetric c_1 on a quadric surface, we pursue analogy to some results known for moduli spaces of sheaves on a projective plane. We define an invariant *height*, introduced by Drezet in the projective plane case, for moduli spaces of sheaves with symmetric c_1 on a quadric surface and describe the structure of moduli spaces of height zero. Then we study rational maps of moduli spaces of positive height to moduli spaces of representation of quivers, effective cones of moduli spaces, and strange duality for height-zero moduli spaces.

§1. Introduction

It is a fundamental problem to decide whether there exists a semistable sheaf with a given numerical invariant or not. In the case of $\mathbb{P}^2$, Drezet and Le Potier [DL] solved this problem completely; they determined the set of pairs (μ, Δ) such that there exists a semistable sheaf having μ as its slope and Δ as its discriminant. In their result, exceptional bundles play an important role. In [D], Drezet refined the result by introducing a function $\delta : \mathbb{Q} \to \mathbb{Q}$, which is also defined using exceptional bundles. He proved that the inequality $\Delta \geqslant \delta(\mu)$ is a necessary and sufficient condition for $(\mu, \Delta) \in \mathbb{Q}^2$ to be the pair of the slope and discriminant of a nonexceptional stable sheaf. The function δ allowed him to define an invariant *height* of a positive-dimensional moduli space of semistable sheaves on $\mathbb{P}^2$, and he proved that a moduli space of height zero is isomorphic to a moduli space of representations of the Kronecker quiver [D, Theorem 2].

The Picard group of a positive-dimensional moduli space of semistable sheaves on $\mathbb{P}^2$ is a free abelian group of rank 1 (resp. 2) if its height is zero (resp. positive).

Received March 14, 2016. Revised July 31, 2016. Accepted August 14, 2016.
2010 Mathematics subject classification. Primary 14J60; Secondary 14D20.
This work was supported by JSPS KAKENHI Grant Numbers 15K04824, 24224001 and JSPS Grant-in-Aid (S), No. 25220701.

When the height of a moduli space M of semistable sheaves on $\mathbb{P}^2$ is positive, Coskun, Huizenga and Woolf [CHW] constructed a rational map of M to a moduli space of representations of a Kronecker quiver, and using it, determined completely the effective cone of the moduli space M. In [A15], using Coskun, Huizenga and Woolf's rational map as one of the ingredients, the author showed the strange duality for sheaves on $\mathbb{P}^2$ when one of the moduli spaces appearing in the strange duality has height zero.

The aim of this paper is to pursue, in the case of the quadric surface $S = \mathbb{P}^1 \times \mathbb{P}^1$ under the assumption "with symmetric c_1," analogues to the above results.

1.1 Height and moduli spaces of height zero

First of all, we need to determine Chern classes of semistable sheaves with symmetric c_1 and define height for positive-dimensional moduli spaces of sheaves with symmetric c_1 on S. In [R94], Rudakov described Chern classes of semistable sheaves on S (under the condition $\Delta \neq 1/2$). His description involves *all* exceptional bundles. The set of all exceptional bundles on S is more complicated than the set of those on $\mathbb{P}^2$ (see [R89]). So it seems difficult to define, in the quadric surface case, an analogue to the function δ introduced by Drezet for $\mathbb{P}^2$, and to define height of the moduli space of semistable sheaves on S.

The essential observation of this paper (Proposition 3.6) says that if we restrict ourselves to considering semistable sheaves on S *having symmetric* c_1, then we can describe their Chern classes using only symmetric exceptional bundles. Although the proof of this observation is easy, we note that it still is not a trivial fact because the filters of the Harder–Narasimhan filtration of a sheaf with symmetric c_1 do not necessarily have symmetric c_1. The set of symmetric exceptional bundles on S is well understood (cf. [R89, Section 6]), so we can define an analogous function δ to Drezet's function δ in the $\mathbb{P}^2$ case, and define height of a moduli space of semistable sheaves on S having symmetric c_1 (Definition 3.8). Then we proceed to show that a moduli space of height zero is isomorphic to a moduli space of representations of a certain quiver (see Section 5.2 for the appearing quivers). The result has some overlap with the result in [Ka], [Ku]. In both [Ka] and [Ku], the authors do not impose the assumption "having symmetric c_1". Karpov [Ka] gave a sufficient condition for a moduli space of semistable sheaves on S to be isomorphic to a moduli space of representations of Kronecker quivers. In [Ka, Section 7], he considered a symmetric case, and

[Ka, Theorem 7.3] is a special case of Theorem 5.4 in this paper. Kuleshov gave some examples of moduli spaces of semistable sheaves on S that are isomorphic to moduli spaces of representations of quivers like Q^{α} and Q^{β} (see Section 5.2 for the quivers Q^{α}, Q^{β}).

1.2 Coskun, Huizenga and Woolf's rational map

After defining height, and describing the structure of moduli spaces of height zero, we move on to studying moduli spaces of positive height.

Let M be a moduli space of positive height of semistable sheaves with symmetric c_1 on S. What we do is the following:

(i) We construct a rational map of M to a moduli space of representations of quivers, which is an analogue of the rational map constructed by Coskun, Huizenga and Woolf in case $\mathbb{P}^2$ [CHW].
(ii) We determine some part of the effective cone of M. More precisely, we define a 2-dimensional subspace V in $\mathrm{NS}(M)_{\mathbb{R}}$, and determine the cone $\mathrm{Eff}(M) \cap V$.
(iii) We establish a strange duality for height-zero moduli spaces in the case S.

Actually, once we succeeded in defining height, doing these things is a more or less straight-forward task. Having said so, there are some technical issues to cope with. The technical difficulty is caused by the following two facts:

- the Picard group of S is not cyclic,
- there are two kinds of symmetric exceptional slopes: even ones and odd ones.

For example, to carry out (i), we need to show an analogous result to [CHW, Theorem 4.16], which concerns continued fraction expansions of exceptional slopes. This is Theorem 7.4, and its proof is messier than [CHW, Theorem 4.16] due to the above second fact. Also, to prove (iii), we need to show a dimension-estimate result analogous to [Le, Lemma 18.3.1]. This is Proposition 6.1, and its proof is more involved than [Le, Lemma 18.3.1] due to the above first fact.

Comment on the method of the proof of Theorem 5.4: in the proof of Theorem 5.4, it is essential to show that the semistability of a sheaf corresponds exactly to the semistability of the representation of a quiver. To show this, we take a different method from that in [D], [Ka], [Ku]; we employ the argument using the Bridgeland stability as in [ABCH], [Oh]. It is just a matter of the author's taste; the proof via the Bridgeland stability feels more conceptual.

Organization of the paper: In Section 2, we collect basic facts about symmetric exceptional bundles. In Section 3, we define the function δ, prove the existence theorem (Theorem 3.7), which is a counterpart of the second statement in [D, Theorem 1], and define the height of a moduli space. In Section 4, we study the Bridgeland stability for symmetric exceptional bundles. The Bridgeland stability of exceptional bundles on $\mathbb{P}^2$ was studied by Huizenga [H, Section 9], and we trace his argument. In Section 5, we study the relation between the usual semistability and the Bridgeland semistability. Then, after introducing notation of quivers, we show that the moduli space of height zero is isomorphic to a moduli space of representations of a quiver. Section 6 is devoted to proving Proposition 6.1, which gives a dimension estimate for the locus of nonsemistable sheaves in a complete family of torsion-free sheaves. In Section 7, we consider continued fraction expansions of symmetric exceptional slopes. This section is a counterpart of [H, Section 3] and [CHW, Section 4]. In Section 8, we consider resolutions of semistable sheaves by symmetric exceptional bundles. In Section 9, we define a rational map of a moduli space of sheaves to a moduli space of representations of a quiver, which is an analogue, in the quadric surface case, to Coskun, Huizenga and Woolf's rational map. In Section 10, we define a 2-dimensional subspace in the real Néron–Severi group of the moduli space of sheaves, and determine the intersection of the subspace and the effective cone. In Section 11, we state a strange duality for height-zero moduli spaces in the quadric surface case. In Appendix we give a Beilinson-type spectral sequence used in the paper for readers' convenience.

Notation and Convention. In this paper, the variety S denotes $\mathbb{P}^1 \times \mathbb{P}^1$.

The line bundle $\mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1}(b)$ is denoted by $\mathcal{O}(a, b)$, and $E \otimes \mathcal{O}(a, b)$ is abbreviated as $E(a, b)$ for a sheaf E. We identify $\operatorname{Pic} S$ with $\mathbb{Z}^2$ by the correspondence $\mathcal{O}(a, b) \leftrightarrow (a, b)$.

Unless otherwise stated, a semistable sheaf is a semistable torsion-free sheaf, and semistability means Gieseker–Maruyama semistability with respect to the ample line bundle $\mathcal{O}(1, 1)$. The rank of a coherent sheaf E is denoted by $\mathrm{r}(E)$. The reduced Hilbert polynomial of a sheaf E is denoted by p_E, that is, $p_E(n) = \chi(E(n, n))/\mathrm{r}(E)$ for $n \gg 0$.

For polynomials f and g, $f \succeq g$ (resp. $f \succ g$) means that $f(n) \geqslant g(n)$ (resp. $f(n) > g(n)$) for $n \gg 0$.

A class $\xi \in K(S)$ is said to be *semistable* if there exists a semistable sheaf F with $\xi = [F]$.

For $\xi \in K(S)$, we denote by $M(\xi)$ the moduli space of semistable sheaves F on S with $[F] = \xi$.

For a smooth projective variety X, $\mathrm{D}(X)$ denotes the bounded derived category of coherent sheaves on X.

$\mathrm{P}(x)$ denotes the polynomial $(x+1)^2$.

§2. Preliminaries

2.1 Numerical Invariants

The rank of $\xi \in K(S)$ is denoted by $\mathrm{r}(\xi)$. If ξ has rank $r > 0$, its discriminant is defined by

$$\Delta(\xi) = \frac{1}{r}\left(c_2(\xi) - \frac{r-1}{2r} c_1(\xi)^2\right).$$

If $c_1(\xi) = (a, b)$, then we put

$$\begin{array}{lll} \nu(\xi) = (a,b)/r, & \nu'(\xi) = a/r, & \nu''(\xi) = b/r, \\ \deg \xi = a+b, & \mu(\xi) = (a+b)/r, & \bar{\mu}(\xi) = \mu(\xi)/2. \end{array}$$

Here $\nu(\xi)$ is an element of $\operatorname{Pic} S \otimes \mathbb{Q} \simeq \mathbb{Q}^2$. We say that ξ has symmetric c_1 if $a = b$. We mainly use these notations when ξ is a class of a sheaf.

If $\xi, \eta \in K(S)$ have positive rank, then by the Riemann–Roch theorem we have

$$\chi(\xi,\eta) = \mathrm{r}(\xi)\mathrm{r}(\eta)\left\{\left(\nu'(\eta) - \nu'(\xi) + 1\right)\left(\nu''(\eta) - \nu''(\xi) + 1\right) - \Delta(\xi) - \Delta(\eta)\right\}.$$

If $\nu(\xi) = (\bar{\mu}(\xi) + t, \bar{\mu}(\xi) - t)$ and $\nu(\eta) = (\bar{\mu}(\eta) + s, \bar{\mu}(\eta) - s)$, then the above formula can also be expressed as

$$\chi(\xi,\eta) = \mathrm{r}(\xi)\mathrm{r}(\eta)\left\{\mathrm{P}(\bar{\mu}(\eta) - \bar{\mu}(\xi)) - (t-s)^2 - \Delta(\xi) - \Delta(\eta)\right\}. \tag{2.1}$$

In particular, if ξ and η have symmetric c_1, then

$$\chi(\xi,\eta) = \mathrm{r}(\xi)\mathrm{r}(\eta)\left\{\mathrm{P}(\bar{\mu}(\eta) - \bar{\mu}(\xi)) - \Delta(\xi) - \Delta(\eta)\right\}. \tag{2.2}$$

DEFINITION 2.1. A coherent sheaf E on a smooth projective variety is said to be *exceptional* if $\operatorname{End}(E) \simeq \mathbb{C}$ and $\operatorname{Ext}^i(E,E) = 0$ for $i > 0$.

All exceptional sheaves on S are locally free (cf. [KO, Proposition 2.9]). A general member C of the anti-canonical linear system $|-K_S|$ is an elliptic curve, and the restriction $E|_C$ of an exceptional bundle E to C is simple (cf. [KO, Lemma 3.6]). It follows from this that E is μ-stable (with respect to the polarization $\mathcal{O}(1,1)$), and that if $c_1(E)=(a,b)$ then $2(a+b)$ and $\mathrm{r}(E)$ are coprime. In particular, $\mathrm{r}(E)$ is odd.

2.2 Symmetric Exceptional bundles

$\iota : S \to S$ denotes the involution $(x,y)\mapsto(y,x)$. A coherent sheaf E on S is said to be *symmetric* if $\iota^*E \simeq E$. We recall the following result [R89, Proposition 6.1].

PROPOSITION 2.2. *Let E be a symmetric bundle that is a direct sum $E_1 \oplus E_2$ with E_1 and E_2 exceptional bundles. The following are equivalent.*

(1) $\chi(E,E)=2$.
(2) $\chi(E_1,E_2)=\chi(E_2,E_1)=0$.
(3) $\mathrm{Ext}^i(E_1,E_2)=\mathrm{Ext}^i(E_2,E_1)=0$.
(4) $\nu(E_1),\ \nu(E_2)\in\{(a,a\pm 1)/r \mid a,r\in\mathbb{Z}\}$.

After Rudakov [R89], we use the following terminology.

DEFINITION 2.3. A *symmetric exceptional bundle* is either an exceptional bundle which is symmetric or a symmetric bundle satisfying the equivalent conditions in Proposition 2.2. The former is called an *odd* symmetric exceptional bundle, and the latter is called an *even* symmetric exceptional bundle. (Note that an even symmetric exceptional bundle is NOT an exceptional bundle.)

The use of the adjective "even" and "odd" is justified by the fact that the rank of an even (resp. odd) symmetric exceptional bundle is even (resp. odd).

2.3 Mutation in a general context

We recall an operation called mutation in a general context. Our references are [Bo] and [BS].

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category. For a full triangulated subcategory $\mathcal{B}$, we put

$$\mathcal{B}^{\perp}=\{A\in\mathcal{D}\mid \mathrm{Hom}(B,A)=0 \text{ for any } B\in\mathcal{B}\},$$
$$^{\perp}\mathcal{B}=\{A\in\mathcal{D}\mid \mathrm{Hom}(A,B)=0 \text{ for any } B\in\mathcal{B}\}.$$

Then $\mathcal{B}$ is defined to be left (resp. right) admissible if the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{D}$ has a left (resp. right) adjoint. $\mathcal{B}$ is said to be admissible if it is left and right admissible. It is known (cf. [Bo, Lemma 3.1]) that $\mathcal{B}$ is left (resp. right) admissible if and only if the pair $(\mathcal{B}, {}^{\perp}\mathcal{B})$ (resp. $(\mathcal{B}^{\perp}, \mathcal{B})$) is a semi-orthogonal decomposition. Here, a pair $(\mathcal{B}, \mathcal{C})$ of full triangulated subcategories of $\mathcal{D}$ is a semi-orthogonal decomposition if

- for $B \in \mathcal{B}$ and $C \in \mathcal{C}$, we have $\mathrm{Hom}(C, B) = 0$,
- any $A \in \mathcal{D}$ fits in a triangle

$$C \to A \to B, \tag{2.3}$$

  with $C \in \mathcal{C}$ and $B \in \mathcal{B}$. (One can easily see that B and C are determined uniquely up to unique isomorphism.)

Assume that $\mathcal{B}$ is admissible. For $A \in {}^{\perp}\mathcal{B}$, we define $L_{\mathcal{B}}(A) \in \mathcal{B}^{\perp}$ to be the object determined by the triangle

$$B \to A \to L_{\mathcal{B}}(A)$$

with $B \in \mathcal{B}$. $L_{\mathcal{B}}(A)$ is called a mutation of A through $\mathcal{B}$. Similarly, for $A \in \mathcal{B}^{\perp}$, we define $R_{\mathcal{B}}(A) \in {}^{\perp}\mathcal{B}$ to be the object determined by the triangle

$$R_{\mathcal{B}}(A) \to A \to B$$

with $B \in \mathcal{B}$. $R_{\mathcal{B}}(A)$ is called a right mutation of A through $\mathcal{B}$.

If $\mathbb{E} = (E_1, \ldots, E_m)$ is an exceptional collection, the full subcategory $\langle \mathbb{E} \rangle$ generated by $\mathbb{E}$ is admissible (cf. [Bo, Theorem 3.2]). We write $L_{\mathbb{E}}$ and $R_{\mathbb{E}}$ for $L_{\langle \mathbb{E} \rangle}$ and $R_{\langle \mathbb{E} \rangle}$.

Definition 2.4. A *d-block exceptional collection* is an exceptional collection $\mathbb{E}$ together with a partition of $\mathbb{E}$ into d subcollections $\mathbb{E} = (\mathbb{E}_1, \ldots, \mathbb{E}_d)$ such that the objects in each block $\mathbb{E}_i$ are mutually orthogonal, that is, for any $E, E' \in \mathbb{E}_i$, we have $\mathrm{Hom}^k(E, E') = \mathrm{Hom}^k(E', E) = 0$ for any k.

Given a d-block exceptional collection $\mathbb{E} = (\mathbb{E}_1, \ldots, \mathbb{E}_d)$, we define, for $1 < i \leqslant d$, a d-block exceptional collection $\tau_i^L(\mathbb{E})$ to be

$$\big(\mathbb{E}_1, \ldots, \mathbb{E}_{i-2}, L_{\mathbb{E}_{i-1}}(\mathbb{E}_i)[-1], \mathbb{E}_{i-1}, \mathbb{E}_{i+1}, \ldots, \mathbb{E}_d\big).$$

Similarly for $1 \leqslant i < d$, we define $\tau_i^R(\mathbb{E})$ to be

$$\big(\mathbb{E}_1, \ldots, \mathbb{E}_{i-1}, \mathbb{E}_{i+1}, R_{\mathbb{E}_{i+1}}(\mathbb{E}_i)[1], \mathbb{E}_{i+2}, \ldots, \mathbb{E}_d\big).$$

We call τ_i^L and τ_i^R d-block mutation operators.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Now we specialize to the case $\mathcal{D} = \mathrm{D}(X)$ with X a smooth projective variety of dimension n. If $\mathbb{E} = (\mathbb{E}_1, \ldots, \mathbb{E}_{n+1})$ is a full (n+1)-block exceptional collection consisting of exceptional sheaves, then $\mathbb{E}$ is strong, and $\tau_i^L(\mathbb{E})$ and $\tau_i^R(\mathbb{E})$ also consists of exceptional sheaves (cf. [BS, Theorem 4.5]). It follows from this that if $\mathbb{E}_{i-1} = (E_{\alpha+1}, \ldots, E_\beta)$, $\mathbb{E}_i = (E_{\beta+1}, \ldots, E_\gamma)$ and $\mathbb{E}_{i+1} = (E_{\gamma+1}, \ldots, E_\delta)$, then for $\beta + 1 \leqslant k \leqslant \gamma$, the natural morphism

$$\bigoplus_{l=\alpha+1}^{\beta} \mathrm{Hom}(E_l, E_k) \otimes E_l \to E_k$$

is surjective, and $L_{\mathbb{E}_{i-1}}(E_k)[-1]$ is its kernel. Similarly, the natural morphism

$$E_k \to \bigoplus_{l=\gamma+1}^{\delta} \mathrm{Hom}(E_k, E_l)^* \otimes E_l$$

is injective, and $R_{\mathbb{E}_{i+1}}(E_k)[1]$ is its cokernel.

2.4 Mutation of Symmetric Exceptional bundles

A *symmetric exceptional triple* is a triple (E_1, E_2, E_3) of symmetric exceptional bundles on S such that one of $E_i's$ is even, and the other two are (necessarily) odd, and that $\mathrm{Ext}^k(E_i, E_j) = 0$ for any k if $i > j$.

If (E_1, E_2, E_3) is a symmetric exceptional triple, and E_i is an even symmetric exceptional bundle $E_i' \oplus E_i''$, then by substituting (E_i', E_i'') for E_i, we obtain a full 3-block exceptional collection. Conversely, if $(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3)$ is a full 3-block exceptional collection, then one of the blocks $\mathbb{E}_i$'s consists of two exceptional bundles and the other two blocks consist of a single exceptional bundle. Say $\mathbb{E}_1 = (E_1', E_1'')$ and $\mathbb{E}_j = \{E_j\}$, $j = 2, 3$. Then $(E_1' \oplus E_1'', E_2, E_3)$ is a symmetric exceptional triple if the sheaves in the triple are symmetric. By this correspondence between symmetric exceptional triples and full 3-block exceptional collections such that the direct sum of bundles in each block is symmetric, we can apply 3-block mutation operators τ_i^L, τ_i^R for symmetric exceptional triples. For example, for a symmetric exceptional triple $\mathcal{E} = (E_1, E_2, E_3)$,

$$\tau_3^L(\mathcal{E}) = (E_1, L_{E_2}(E_3)[-1], E_2), \qquad \tau_1^R(\mathcal{E}) = (E_2, R_{E_2}(E_1)[1], E_3),$$

where if E is an even symmetric exceptional bundle $E' \oplus E''$, then L_E and R_E are understood to be $L_{\langle E', E''\rangle}$ and $R_{\langle E', E''\rangle}$, respectively.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

2.5 Construction of Symmetric Exceptional bundles

Let $\mathcal{SE}$ be the set of all symmetric exceptional bundles. Put $\mathfrak{D} := \{p/2^q \mid p \in \mathbb{Z},\ q \in \mathbb{Z}_{\geqslant 0}\}$. We define a map $\eta : \mathfrak{D} \to \mathcal{SE}$, by induction on q, such that for odd $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$,

$$\left(\eta\left(\frac{p-1}{2^q}\right), \eta\left(\frac{p}{2^q}\right), \eta\left(\frac{p+1}{2^q}\right)\right)$$

is a symmetric exceptional triple. For $p \in \mathbb{Z}$, we define $\eta(p) = \mathcal{O}(p, p)$. For odd $p \in \mathbb{Z}$, we define

$$\eta(p/2) = \mathcal{O}\left(\frac{p-1}{2}, \frac{p+1}{2}\right) \oplus \mathcal{O}\left(\frac{p+1}{2}, \frac{p-1}{2}\right).$$

Let p be an odd integer and $q \geqslant 2$. If $p \equiv 1 \pmod 4$, then

$$\left(\eta\left(\frac{p-1}{2^q}\right), \eta\left(\frac{p+1}{2^q}\right), \eta\left(\frac{p+3}{2^q}\right)\right) \tag{2.4}$$

is a symmetric exceptional triple, so we define $\eta(p/2^q)$ by

$$\left(\eta\left(\frac{p-1}{2^q}\right), \eta\left(\frac{p}{2^q}\right), \eta\left(\frac{p+1}{2^q}\right)\right) = \tau_3^L \text{ (the triple (2.4))}.$$

If $p \equiv 3 \pmod 4$, then

$$\left(\eta\left(\frac{p-3}{2^q}\right), \eta\left(\frac{p-1}{2^q}\right), \eta\left(\frac{p+1}{2^q}\right)\right) \tag{2.5}$$

is a symmetric exceptional triple, so we define $\eta(p/2^q)$ by

$$\left(\eta\left(\frac{p-1}{2^q}\right), \eta\left(\frac{p}{2^q}\right), \eta\left(\frac{p+1}{2^q}\right)\right) = \tau_1^R \text{ (the triple (2.5))}.$$

Rudakov proved the following theorem (cf. [R89, Theorem 6.5]).

THEOREM 2.5. *The map η is bijective.*

2.6 Numerics of Symmetric Exceptional bundles

If $\alpha = a/r$ with $r > 0$ is an irreducible fraction, we call r the rank of α and denote it by r_α; we define the discriminant of α by

$$\Delta_\alpha = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{1}{r^2}\right) & \text{if } r \text{ is odd,} \\ \dfrac{1}{2}\left(1 - \dfrac{2}{r^2}\right) & \text{if } r \text{ is even.} \end{cases}$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

We can see that if E is a symmetric exceptional bundle, then $\Delta(E) = \Delta_{\bar{\mu}(E)}$. For α, $\beta \in \mathbb{Q}$ with $2 + \alpha - \beta \neq 0$, we define

$$\alpha.\beta = \frac{\alpha + \beta}{2} + \frac{\Delta_\beta - \Delta_\alpha}{2(2 + \alpha - \beta)}.$$

We define a map $\epsilon : \mathfrak{D} \to \mathbb{Q}$ inductively on q as follows. For $p \in \mathbb{Z}$, we define $\epsilon(p) = p$. For $p/2^q$ with p odd and $q > 0$, we define

$$\epsilon\left(\frac{p}{2^q}\right) = \epsilon\left(\frac{p-1}{2^q}\right).\epsilon\left(\frac{p+1}{2^q}\right).$$

From the lemma below, we see that ϵ is a strictly increasing function. For α, $\beta \in \mathbb{Q}$, we put

$$\chi(\alpha, \beta) := r_\alpha r_\beta\,(\mathrm{P}(\beta - \alpha) - \Delta_\alpha - \Delta_\beta).$$

Lemma 2.6. *Let α, $\beta \in \mathbb{Q}$ satisfy $\alpha < \beta < \alpha + 2$ and $\chi(\beta, \alpha) = 0$.*

(1) *If $\gamma \in \mathbb{Q}$ satisfies $\chi(\beta, \gamma) = \chi(\gamma, \alpha) = 0$, then $\gamma = \alpha.\beta$.*

(2) *We have*

$$\alpha.\beta - \alpha = \frac{1 - 2\Delta_\alpha}{2(2 + \alpha - \beta)}, \qquad \beta - \alpha.\beta = \frac{1 - 2\Delta_\beta}{2(2 + \alpha - \beta)}. \tag{2.6}$$

In particular, we have $\alpha < \alpha.\beta < \beta$.

(3) *We have*

$$1 - 2\Delta_{\alpha.\beta} = \frac{(1 - 2\Delta_\alpha)(1 - 2\Delta_\beta)}{2(2 + \alpha - \beta)^2}.$$

(4) *We have*

$$\frac{\alpha + \alpha.\beta}{2} + \frac{\Delta_\alpha - \Delta_{\alpha.\beta}}{2(\alpha.\beta - \alpha)} = \beta - 1 \tag{2.7}$$

$$\frac{\alpha.\beta + \beta}{2} + \frac{\Delta_{\alpha.\beta} - \Delta_\beta}{2(\beta - \alpha.\beta)} = \alpha + 1. \tag{2.8}$$

(5) *We have*

$$\left(\frac{\alpha - \alpha.\beta}{2}\right)^2 - \frac{\mathrm{P}(\alpha - \alpha.\beta)}{2} + \left(\frac{\Delta_\alpha - \Delta_{\alpha.\beta}}{2(\alpha.\beta - \alpha)}\right)^2 = \Delta_\beta \tag{2.9}$$

$$\left(\frac{\alpha.\beta - \beta}{2}\right)^2 - \frac{\mathrm{P}(\alpha.\beta - \beta)}{2} + \left(\frac{\Delta_{\alpha.\beta} - \Delta_\beta}{2(\beta - \alpha.\beta)}\right)^2 = \Delta_\alpha. \tag{2.10}$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Proof. (1) This follows from

$$\begin{aligned} \mathrm{P}(\gamma-\beta)-\Delta_\gamma-\Delta_\beta&=0,\\ \mathrm{P}(\alpha-\gamma)-\Delta_\gamma-\Delta_\alpha&=0. \end{aligned}$$

(2) Since $\chi(\beta,\alpha)=0$, we have

$$\begin{aligned} \Delta_\alpha-\Delta_\beta &= 2\Delta_\alpha-\mathrm{P}(\alpha-\beta)\\ &= 2\Delta_\alpha-1+(\beta-\alpha)(2+\alpha-\beta). \end{aligned}$$

Thus

$$\frac{1-2\Delta_\alpha}{2(2+\alpha-\beta)}=\frac{\beta-\alpha}{2}+\frac{\Delta_\beta-\Delta_\alpha}{2(2+\alpha-\beta)}=\alpha.\beta-\alpha.$$

The proof of the other equality is similar. Since the discriminants are less than $1/2$, the inequalities follow.

(3) We have

$$\begin{aligned} \Delta_{\alpha.\beta} &= \mathrm{P}(\alpha-\alpha.\beta)-\Delta_\alpha\\ &= \left(\frac{\alpha-\beta}{2}+1-\frac{\Delta_\beta-\Delta_\alpha}{2(2+\alpha-\beta)}\right)^2-\Delta_\alpha\\ &= \frac{1}{4}(\alpha-\beta+2)^2-\frac{\Delta_\alpha+\Delta_\beta}{2}+\left(\frac{\Delta_\beta-\Delta_\alpha}{2(2+\alpha-\beta)}\right)^2\\ &= \frac{1}{4}(\alpha-\beta+2)^2-\frac{1}{2}(\alpha-\beta+1)^2+\left(\frac{\Delta_\beta-\Delta_\alpha}{2(2+\alpha-\beta)}\right)^2\\ &= \frac{1}{2}-\frac{(\alpha-\beta)^2}{4}+\left(\frac{\Delta_\beta-\Delta_\alpha}{2(2+\alpha-\beta)}\right)^2\\ &= \frac{1}{2}-\left(\frac{\alpha-\beta}{2}-\frac{\Delta_\beta-\Delta_\alpha}{2(2+\alpha-\beta)}\right)\left(\frac{\alpha-\beta}{2}+\frac{\Delta_\beta-\Delta_\alpha}{2(2+\alpha-\beta)}\right)\\ &= \frac{1}{2}-\left(\frac{2\Delta_\alpha-1}{2(2+\alpha-\beta)}\right)\left(\frac{2\Delta_\beta-1}{2(2+\alpha-\beta)}\right). \end{aligned}$$

Here in the first equality we used the equality $\chi(\alpha.\beta,\alpha)=0$.

(4) We prove the first equality. From $\chi(\alpha.\beta,\alpha)=0$, we obtain

$$(\alpha-\alpha.\beta)(\alpha-\alpha.\beta+2)+\Delta_\alpha-\Delta_{\alpha.\beta}=2\Delta_\alpha-1.$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Then we have

$$\begin{aligned}\frac{\Delta_\alpha - \Delta_{\alpha.\beta}}{2(\alpha.\beta - \alpha)} &= \frac{2\Delta_\alpha - 1}{2(\alpha.\beta - \alpha)} + \frac{\alpha - \alpha.\beta + 2}{2} \\ &= \beta - 1 - \frac{\alpha + \alpha.\beta}{2},\end{aligned}$$

where we used (2.6) in the second equality.

(5) We prove the first equality. Put $x = \alpha - \alpha.\beta$. Using (2.7), we have

$$\begin{aligned}\text{LHS. of (2.9)} &= \left(\frac{x}{2}\right)^2 - \frac{(x+1)^2}{2} + \left(\beta - \alpha - 1 + \frac{x}{2}\right)^2 \\ &= (\beta - \alpha - 2)x + (\beta - \alpha - 1)^2 - \frac{1}{2} \\ &= -\Delta_\alpha + \mathrm{P}(\alpha - \beta) = \Delta_\beta,\end{aligned}$$

where we used (2.6) in the third equality. ∎

Proposition 2.7. *We have $\bar{\mu}\,(\eta(d)) = \epsilon(d)$ for $d \in \mathfrak{D}$.*

Proof. Let $d = p/2^q$ with p odd. When $q = 0, 1$, the equality holds because $\bar{\mu}\,(\mathcal{O}(p,p)) = p$, and

$$\bar{\mu}\left(\mathcal{O}\left(\frac{p-1}{2}, \frac{p+1}{2}\right) \oplus \mathcal{O}\left(\frac{p+1}{2}, \frac{p-1}{2}\right)\right) = \frac{p}{2}.$$

For $q \geqslant 2$, the equality follows by induction on q using Lemma 2.6(1). ∎

Notation 2.8. Put $\mathfrak{E} := \epsilon(\mathfrak{D})$. This is the set of $\bar{\mu}$'s of symmetric exceptional bundles. We call elements of $\mathfrak{E}$ *symmetric exceptional slopes*. A symmetric exceptional bundle is determined uniquely by its value of $\bar{\mu}$; for $\alpha \in \mathfrak{E}$, E_α denotes the symmetric exceptional bundle such that $\bar{\mu}(E_\alpha) = \alpha$. We say that a symmetric exceptional slope α is even (resp. odd) if r_α is even (resp. odd). (N.B. If $\alpha \in \mathfrak{E}$ is an integer, it is odd as a symmetric exceptional slope. This might be a little confusing because α can be an even integer. But in this paper we do not mention the parity as an integer, so it does not cause confusion.)

§3. Existence of Semistable sheaves

In this section, we determine the Chern classes of semistable sheaves with symmetric c_1 on S.

LEMMA 3.1. *Let F be a semistable sheaf with symmetric c_1. Assume that $\Delta(F) < 1/2$. Then $\bar{\mu}(F) \in \mathfrak{E}$ and F is a direct sum of $E_{\bar{\mu}(F)}$.*

Proof. We proceed by induction on the rank of F. If F is stable, then F is an odd symmetric exceptional bundle and we are done. If F is not stable, then let F_1 be the first filter of a Jordan–Hölder filtration of F. Let $\nu(F_1) = (\bar{\mu}(F) + t, \bar{\mu}(F) - t)$. Since $\chi(F_1)/\mathrm{r}(F_1) = \chi(F)/\mathrm{r}(F)$, we have, by (2.1),

$$t^2 + \Delta(F_1) = \Delta(F).$$

So we have $\Delta(F_1) < 1/2$, and F_1 is an exceptional bundle. If we let $c_1(F_1) = (a_1, a_2)$, then $t = \frac{a_1 - a_2}{2\mathrm{r}(F_1)}$. We have

$$\frac{1}{2} > \Delta(F) = \Delta(F_1) + t^2 = \frac{1}{2} - \frac{2 - (a_1 - a_2)^2}{4\mathrm{r}(F_1)^2}.$$

Therefore, $|a_1 - a_2| = 0$ or 1. If $|a_1 - a_2| = 0$, then F_1 is an odd symmetric exceptional bundle. F/F_1 is a semistable sheaf with symmetric c_1 such that $\bar{\mu}(F) = \bar{\mu}(F/F_1)$ and $\Delta(F) = \Delta(F/F_1)$. By applying the induction hypothesis to F/F_1, we obtain the result. If $|a_1 - a_2| = 1$, then F_1 is a direct summand of an even symmetric exceptional bundle. Note that

$$\frac{\chi(F_1, F)}{\mathrm{r}(F_1)\mathrm{r}(F)} = 1 - t^2 - \Delta(F_1) - \Delta(F) = 1 - 2\Delta(F) > 0.$$

By this we also have $\chi(\iota^* F_1, F) > 0$, so $\hom(\iota^* F_1, F) > 0$ because $\mathrm{Ext}^2(\iota^* F_1, F) = 0$. Then we can find a subbundle G of F that is S-equivalent to $F_1 \oplus \iota^* F_1$. Since $\mathrm{Ext}^1(F_1, \iota^* F_1) = 0$, we have $G \simeq F_1 \oplus \iota^* F_1$. Applying the induction hypothesis to F/G, we obtain the result. ∎

PROPOSITION 3.2. *Let F be a semistable sheaf on S with symmetric c_1. Let $\alpha \in \mathfrak{E}$ satisfy $|\bar{\mu}(F) - \alpha| < 2$. Then the inequality*

$$\mathrm{P}\left(-|\bar{\mu}(F) - \alpha|\right) - \Delta(F) - \Delta_\alpha \leqslant 0 \tag{3.1}$$

holds unless $\bar{\mu}(F) = \alpha$ and F is a direct sum of E_α.

Proof. If $\alpha - 2 < \bar{\mu}(F) < \alpha$, then by semistability of F and E_α we have $\mathrm{Hom}(E_\alpha, F) = \mathrm{Ext}^2(E_\alpha, F) = 0$. Thus $\chi(E_\alpha, F) \leqslant 0$, which is equivalent to (3.1). A similar argument shows (3.1) when $\alpha < \bar{\mu}(F) < \alpha + 2$. Suppose that $\alpha = \bar{\mu}(F)$ and (3.1) does not hold. We have

$$1 - \Delta(F) - \Delta_\alpha > 0. \tag{3.2}$$

We show that F is a direct sum of E_α. Since $\chi(E_\alpha, F) > 0$, we have a nonzero morphism $E_\alpha \to F$. If r_α is odd, then this is injective by μ-stability of E_α. By semistability of F, we have $\Delta_\alpha \geqslant \Delta(F)$. If r_α is even, then F has a subsheaf isomorphic to E'_α or E''_α. Again by semistability of F, we have $\Delta(F) \leqslant (1/r_\alpha)^2 + \Delta(E'_\alpha) = \Delta_\alpha$. In any case, $\Delta_\alpha \geqslant \Delta(F)$. From this and (3.2), we have $\Delta(F) < 1/2$. Then F is a direct sum of E_α by Lemma 3.1. ∎

REMARK 3.3. The inequality (3.1) holds for a μ-semistable sheaf F with $0 < |\bar{\mu}(F) - \alpha| < 2$. (Indeed, in the above proof, for F with $\bar{\mu}(F) \neq \alpha$, only the μ-semistability is used.) From this, we see that if a μ-semistable sheaf F with $0 < |\bar{\mu}(F) - \alpha| < 2$ satisfies

$$\mathrm{P}\left(-|\bar{\mu}(F) - \alpha|\right) - \Delta(F) - \Delta_\alpha = 0,$$

then it is locally free.

For $\alpha \in \mathfrak{E}$, put $x_\alpha := 1 - \sqrt{\Delta_\alpha + 1/2}$, which is the smaller solution of the quadratic equation

$$\mathrm{P}(-x) - \Delta_\alpha = \frac{1}{2}.$$

Put $I_\alpha := (\alpha - x_\alpha, \alpha + x_\alpha)$.

PROPOSITION 3.4. *$\mathbb{Q}$ is the disjoint union of $I_\alpha \cap \mathbb{Q}$ ($\alpha \in \mathfrak{E}$).*

Proof. The argument of the proof of [D, Théorème 1(1)] applies to our quadratic surface case. Details are left to the reader. ∎

By the above proposition, the union $\bigcup_{\alpha \in \mathfrak{E}} I_\alpha$ is a disjoint union, and contains the set $\mathbb{Q}$ of rational numbers.

We define a function $\delta : \bigcup_{\alpha \in \mathfrak{E}} I_\alpha \to \mathbb{R}$ so that $\delta(\bar{\mu}) = \mathrm{P}(-|\bar{\mu} - \alpha|) - \Delta_\alpha$ for $\bar{\mu} \in I_\alpha$ ($\alpha \in \mathfrak{E}$).

LEMMA 3.5. *Let $(\bar{\mu}, \Delta) \in \mathbb{Q}^2$ satisfy $\Delta \geqslant \delta(\bar{\mu})$. Then for any $\beta \in \mathfrak{E}$ with $|\beta - \bar{\mu}| \leqslant 1$, we have*

$$\mathrm{P}(-|\bar{\mu} - \beta|) - \Delta_\beta \leqslant \Delta. \tag{3.3}$$

Proof. Let $\alpha \in \mathfrak{E}$ be such that $\bar{\mu} \in I_\alpha$. If $\beta = \alpha$, the inequality (3.3) is nothing but $\Delta \geqslant \delta(\bar{\mu})$. Assume that $\beta \neq \alpha$. From the definition of x_β, it follows that for $x \in [\beta - 2 + x_\beta, \beta - x_\beta] \cup [\beta + x_\beta, \beta + 2 - x_\beta]$, we have $\mathrm{P}(-|x - \beta|) - \Delta_\beta \leqslant 1/2$. Since $\bar{\mu}$ lies in this union of intervals, and $\Delta \geqslant \delta(\bar{\mu}) > 1/2$, the inequality (3.3) holds. ∎

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

PROPOSITION 3.6. *Let $(\bar{\mu}, \Delta) \in \mathbb{Q}^2$ satisfy $\Delta \geqslant \delta(\bar{\mu})$. Let E be an exceptional bundle with $\nu(E) = (e_1, e_2)$. If $\bar{\mu} \leqslant \bar{\mu}(E) \leqslant \bar{\mu} + 1$, then*

$$(\bar{\mu} - e_1 + 1)(\bar{\mu} - e_2 + 1) - \Delta(E) \leqslant \Delta.$$

If $\bar{\mu} - 1 \leqslant \bar{\mu}(E) \leqslant \bar{\mu}$, then

$$(e_1 - \bar{\mu} + 1)(e_2 - \bar{\mu} + 1) - \Delta(E) \leqslant \Delta.$$

Proof. Assuming $\bar{\mu} \leqslant \bar{\mu}(E) \leqslant \bar{\mu} + 1$, we show the first inequality. The proof of the second inequality is similar. If E is an odd symmetric exceptional bundle, the inequality follows from Lemma 3.5. If E is a direct summand of an even symmetric exceptional bundle, then

$$(\bar{\mu} - e_1 + 1)(\bar{\mu} - e_2 + 1) - \Delta(E) = \mathrm{P}(\bar{\mu} - \bar{\mu}(E)) - \Delta(E \oplus \iota^* E) \leqslant \Delta$$

again by Lemma 3.5.

Suppose that E is neither an odd symmetric exceptional bundle nor a direct summand of an even symmetric exceptional bundle. We show that

$$(\bar{\mu} - e_1 + 1)(\bar{\mu} - e_2 + 1) - \Delta(E) \leqslant \frac{1}{2}. \tag{3.4}$$

The left-hand side of (3.4) attains the maximum for $\bar{\mu} = \bar{\mu}(E)$. Thus

$$\begin{aligned} \text{LHS of (3.4)} &\leqslant (\bar{\mu}(E) - e_1 + 1)(\bar{\mu}(E) - e_2 + 1) - \Delta(E) \\ &= \frac{1}{2} + \frac{2 - (a_1 - a_2)^2}{4\mathrm{r}(E)^2}, \end{aligned}$$

where $c_1(E) = (a_1, a_2)$. Since we are assuming that E is neither an odd symmetric exceptional bundle nor a direct summand of an even symmetric exceptional bundle, we have $|a_1 - a_2| \geqslant 2$, so (3.4) holds. □

Now we come to the main theorem of this section.

THEOREM 3.7.

(1) *If F is a semistable sheaf with symmetric c_1, then either $\Delta \geqslant \delta(\bar{\mu}(F))$, or $\bar{\mu}(F) \in \mathfrak{E}$ and F is a direct sum of $E_{\bar{\mu}(F)}$.*
(2) *Assume that $(r, \bar{\mu}, \Delta) \in \mathbb{Z}_{>0} \times \mathbb{Q}^2$ satisfies the following conditions:*

 (a) *$r(\mathrm{P}(\bar{\mu}) - \Delta) \in \mathbb{Z}$ and $r\bar{\mu} \in \mathbb{Z}$,*
 (b) *$\Delta \geqslant \delta(\bar{\mu})$.*

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Then there exists a μ-stable sheaf F with symmetric c_1 such that $\mathrm{r}(F) = r$, $\bar{\mu}(F) = \bar{\mu}$ *and* $\Delta(F) = \Delta$.

Proof. (1) This is a consequence of Proposition 3.2.

(2) *Step* 1. We show the existence of stable sheaves.

By Proposition 3.6, we see that the condition (D-L) in [R94, Theorem] holds. Thus by the theorem in [R94], we have a desired stable sheaf F. For reader's convenience, we reproduce a sketch of proof of Rudakov's theorem (adapted for the symmetric c_1 case).

Let t be the smallest integer such that $\mathrm{P}(t+\bar{\mu}) - \Delta \geqslant 0$ and $t+\bar{\mu}+1 \geqslant 0$. Put $A := r(\mathrm{P}(t+\bar{\mu}) - \Delta)$, $B := r((t+\bar{\mu}+1)(t+\bar{\mu}) - \Delta)$ and $C := r(\mathrm{P}(t+\bar{\mu}-1) - \Delta)$. We have $A \geqslant 0$ and $C < 0$. If $B \geqslant 0$, then we set

$$
\begin{aligned}
F^{-1} &:= \mathcal{O}(-t-1,-t-1)^{-C} \oplus (\mathcal{O}(-t-1,-t) \oplus \mathcal{O}(-t,-t-1))^{B},\\
F^{0} &:= \mathcal{O}(-t,-t)^{A}.
\end{aligned}
$$

If $B < 0$, then set

$$
\begin{aligned}
F^{-1} &:= \mathcal{O}(-t-1,-t-1)^{-C},\\
F^{0} &:= \mathcal{O}(-t,-t)^{A} \oplus (\mathcal{O}(-t-1,-t) \oplus \mathcal{O}(-t,-t-1))^{-B}.
\end{aligned}
$$

Put $\mathbb{H} := \mathrm{Hom}(F^{-1}, F^{0})$. Fix a smooth rational curve in the linear system $|\mathcal{O}(1,1)|$. Then we can find a nonempty Zariski open subset $U \subset \mathbb{H}$ such that for any point $[f] \in U$, the following hold:

(a) f is injective and $G := \mathrm{Coker}\, f$ is a torsion-free sheaf with symmetric c_1 such that $\mathrm{r}(G) = r$, $\bar{\mu}(G) = \bar{\mu}$ and $\Delta(G) = \Delta$, and that G is locally free along D and $G|_D$ is a rigid bundle,

(b) the Kodaira–Spencer map $T_{[f]}U \to \mathrm{Ext}^1(G, G)$ is surjective,

(c) $\mathrm{Ext}^2(G, G) = 0$.

Suppose that for $[f] \in U$, $G := \mathrm{Coker}\, f$ is not semistable. Let

$$
0 = G_0 \subset G_1 \subset \cdots \subset G_l = G \tag{3.5}
$$

be the Harder–Narasimhan filtration of G. By condition (a), we have $\mu(G_1) - \mu(G_l/G_{l-1}) \leqslant 1$. If G_1 is not stable, then we can find (cf. [KO, Proposition 4.4]) a subsheaf $G_1' \subset G_1$ such that

- G_1 and G_1' have the same reduced Hilbert polynomial,
- G_1 is S-equivalent to H^a for some stable sheaf H,

- $\operatorname{Hom}(G_1', G_1/G_1') = 0$.

If G_1 is stable, let $G_1' = G_1$. Similarly, if G_l/G_{l-1} is not stable, then we can find a subsheaf $G_{l-1} \subset G_l' \subset G_l$ such that

- G_l/G_{l-1} and G_l/G_l' have the same reduced Hilbert polynomial,
- G_l/G_l' is S-equivalent to H'^b for some stable sheaf H',
- $\operatorname{Hom}(G_l'/G_{l-1}, G_l/G_l') = 0$.

If G_l/G_{l-1} is stable, let $G_l' = G_{l-1}$. We write

$$0 = \bar{G}_0 \subset \cdots \subset \bar{G}_m = G$$

for the filtration

$$0 = G_0 \subset G_0' \subset G_1 \subset \ldots G_{l-1} \subset G_l' \subset G_l = G.$$

Then we have $\operatorname{Ext}^2(\bar{G}_i/\bar{G}_{i-1}, \bar{G}_j/\bar{G}_{j-1}) = 0$ for $i \leqslant j$, and $\operatorname{Hom}(\bar{G}_i/\bar{G}_{i-1}, \bar{G}_j/\bar{G}_{j-1}) = 0$ for $i < j$. Moreover, $\bar{G}_1$ and $\bar{G}_m/\bar{G}_{m-1}$ are S-equivalent to H^a and H'^b for some stable sheaves H and H'.

Claim 3.7.1. $\dim \operatorname{Ext}^1_{\bar{G}_\bullet,+}(G, G) > 0$.

Proof of Claim. If $\dim \operatorname{Ext}^1_{\bar{G}_\bullet,+}(G, G) = 0$, then $\operatorname{Ext}^1(gr_i(G), gr_j(G)) = 0$ for $i < j$, hence we have

$$\chi(gr_i(G), gr_j(G)) = 0 \text{ for } i < j. \tag{3.6}$$

If we let $\nu(gr_i(G)) = (\bar{\mu}_i + t_i, \bar{\mu}_i - t_i)$, then we have

$$0 = (\bar{\mu}_m - \bar{\mu}_1 + 1)^2 - (t_1 - t_m)^2 - \Delta(gr_1(G)) - \Delta(gr_m(G)).$$

We have $\Delta(gr_1(G)) + \Delta(gr_m(G)) < 1$ unless $\bar{\mu}_m - \bar{\mu}_1 = 0$ and $t_1 = t_m$. If $\bar{\mu}_m - \bar{\mu}_1 = 0$ and $t_1 = t_m$, then $\Delta(gr_1(G)) + \Delta(gr_m(G)) = 1$. If $\Delta(gr_1(G)) = \Delta(gr_m(G)) = 1/2$, then

$$\begin{aligned} \chi(gr_m(G))/\mathrm{r}(gr_m(G)) &= (\bar{\mu}_m + 1)^2 - t_m^2 - \Delta(gr_m(G)) \\ &= (\bar{\mu}_1 + 1)^2 - t_1^2 - \Delta(gr_1(G)) \\ &= \chi(gr_1(G))/\mathrm{r}(gr_1(G)). \end{aligned}$$

This is a contradiction. Therefore in any case, we have $\Delta(gr_1(G)) < 1/2$ or $\Delta(gr_m(G)) < 1/2$. Suppose $\Delta(gr_1(G)) < 1/2$. Recall that $gr_1(G)$ is S-equivalent to H^a for some stable sheaf H. By $\Delta(H) < 1/2$, H is an

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

exceptional bundle. Moreover,

$$\chi(H, G) = \frac{1}{a}\chi(gr_1(G), G) = \frac{1}{a}\chi(gr_1(G), gr_1(G)) = a\chi(H, H) > 0,$$

where the second equality follows from (3.6). This is a contradiction. In the case $\Delta(gr_m(G)) < 1/2$, we lead to a contradiction by a similar argument. This is the end of proof of the claim. □

Using the claim, we can see (cf. the proof of [DL, Théorème 4.7]) that there exists a nonempty Zariski open subset $U' \subset U$ such that for any $[f] \in U'$, Coker f is semistable.

Suppose that for $[f] \in U'$, $G :=$ Coker f is not stable. Then we can find a filtration

$$0 = G_0 \subset G_1 \subset G_2 = G$$

such that $gr_1(G)$ and $gr_2(G)$ have the same reduced Hilbert polynomial. If we let $\nu(gr_i(G)) = (\bar{\mu}_i + t_i, \bar{\mu}_i - t_i)$, then

$$\begin{aligned} \dim \operatorname{Ext}^1_{G_\bullet,+}(G, G) &= \dim \operatorname{Ext}^1(gr_1(G), gr_2(G)) \\ &\geqslant -\chi(gr_1(G), gr_2(G)) \\ &\geqslant r_1 r_2\{-1 + (t_2 - t_1)^2 + \Delta(gr_1(G)) + \Delta(gr_2(G))\} \\ &= r_1 r_2\{-1 - 2t_1 t_2 + 2\Delta\}. \end{aligned}$$

Here $r_i := \mathrm{r}(gr_i(G))$, and we used $\Delta(gr_i(G)) + t_i^2 = \Delta$. By assumption, we have $\Delta > 1/2$. Since G has symmetric c_1, we have $t_1 t_2 \leqslant 0$. Hence we have $\dim \operatorname{Ext}^1_{G_\bullet,+}(G, G) > 0$. This implies (cf. the proof of [DL, Théorème 4.10]) that there exists a nonempty Zariski open subset $U'' \subset U'$ such that for any $[f] \in U''$, Coker f is stable.

Step 2. We show the existence of μ-stable sheaves.

If $r = 1$, we are done, so assume that $r \geqslant 2$. Then there exists a nonempty Zariski open subset $U''' \subset U''$ such that for any $[f] \in U'''$, $G :=$ Coker f is locally free (cf. [Le, Section 17.1]). If G is not μ-stable, then the G^* is not semistable. But repeating the (first half of) argument in Step 1 for the family $\{(\text{Coker } f)^*\}_{[f] \in U'''}$ of dual sheaves, we see that for general $[f] \in U'''$, $(\text{Coker } f)^*$ is semistable. So for general $[f] \in U'''$, Coker f is μ-stable. □

The theorem allows us to define an invariant *height*.

Definition 3.8. Let $\xi \in K(S)$ be a semistable class with symmetric c_1. When dim $M(\xi) > 0$, we define the height of the moduli space $M(\xi)$ to be the nonnegative integer

$$\mathrm{r}(E_\gamma)\mathrm{r}(\xi)\,\{\Delta(\xi) - \delta(\bar{\mu}(\xi))\},$$

where γ is the unique symmetric exceptional slope such that $\bar{\mu}(\xi) \in I_\gamma$.

§4. Bridgeland stability of symmetric exceptional bundles

In Section 4.1, we recall the Bridgeland semistability and walls. In Section 4.2, we consider Bridgeland semistability of symmetric exceptional bundles.

4.1 Abelian category $\mathcal{A}_s$

Following [AB], we consider a particular kind of Bridgeland semistability. The presentation in this section follows that in [ABCH, Sections 5, 6].

If $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ is the Harder–Narasimhan filtration of a torsion-free sheaf E for μ-stability, that is, E_i/E_{i-1}, $1 \leqslant i \leqslant n$, are μ-semistable and $\mu(E_1/E_0) > \cdots > \mu(E_n/E_{n-1})$, then we put $\bar{\mu}_{\min}(E) := \bar{\mu}(E_n/E_{n-1})$ and $\bar{\mu}_{\max}(E) := \bar{\mu}(E_1/E_0)$.

For $s \in \mathbb{R}$, we let $\mathcal{Q}_s$ be the full subcategory of $\mathrm{coh}(S)$ consisting of coherent sheaves Q with $\bar{\mu}_{\min}(Q/\mathrm{tor}(Q)) > s$, and we let $\mathcal{F}_s$ be the full subcategory of $\mathrm{coh}(S)$ consisting of torsion-free coherent sheaves F with $\bar{\mu}_{\max}(F) \leqslant s$. We define a full subcategory $\mathcal{A}_s$ of $\mathrm{D}(S)$ by

$$\mathcal{A}_s = \left\{E^\bullet \,\middle|\, \mathrm{H}^0(E^\bullet) \in \mathcal{Q}_s, \mathrm{H}^{-1}(E^\bullet) \in \mathcal{F}_s \text{ and } \mathrm{H}^i(E^\bullet) = 0 \text{ for } i \neq 0, -1\right\}.$$

Then $\mathcal{A}_s$ is an abelian category.

For $s, t \in \mathbb{R}$, we define the map $Z_{(s,t)} : \mathrm{D}(S) \to \mathbb{C}$ by

$$Z_{(s,t)}(E) := -\int_S e^{(s+ti)c_1(L)}\mathrm{ch}(E),$$

where $L = \mathcal{O}(1,1)$. Explicitly we have

$$\begin{aligned} Z_{(s,t)}(E) = &\left(sc_1(E)c_1(L) - \mathrm{r}(E)(s^2 - t^2) - \mathrm{ch}_2(E)\right) \\ &+ t\left(c_1(E)c_1(L) - 2\mathrm{r}(E)s\right) i. \end{aligned}$$

Now assume that $t > 0$, then the pair $(\mathcal{A}_s, Z_{(s,t)})$ is a Bridgeland stability condition (cf. [AB]).

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

For $E \in \mathcal{A}_s$, we put

$$\mu_{s,t}(E) = -\frac{\operatorname{Re} Z_{(s,t)}(E)}{\operatorname{Im} Z_{(s,t)}(E)},$$

where it is understood as $+\infty$ if the denominator is zero. Explicitly we have

$$\mu_{s,t}(E) = \frac{\mathrm{r}(E)(s^2 - t^2) - sc_1(E)c_1(L) + \mathrm{ch}_2(E)}{t(c_1(E)c_1(L) - 2\mathrm{r}(E)s)}.$$

An object $E \in \mathcal{A}_s$ is said to be (s, t)-semistable if for any nonzero subobject $F \subset E$ in $\mathcal{A}_s$, the inequality $\mu_{s,t}(E) \geqslant \mu_{s,t}(F)$ holds.

For $(r, c, d) \in \mathbb{R}^3$, we define

$$\mu_{s,t}(r, c, d) = \frac{r(s^2 - t^2) - sc + d}{t(c - 2rs)}.$$

For (r, c, d) and (r', c', d'), we define the wall $W_{(r,c,d),(r',c',d')}$ by

$$W_{(r,c,d),(r',c',d')} = \left\{(s, t) \in \mathbb{R} \times \mathbb{R}_{>0} \,\middle|\, \mu_{s,t}(r, c, d) = \mu_{s',t'}(r', c', d')\right\}. \tag{4.1}$$

The equality in the condition of the definition of the wall is equivalent to

$$(r'c - rc')s^2 - 2(r'd - rd')s + (r'c - rc')t^2 + c'd - cd' = 0. \tag{4.2}$$

If (r, c, d) and (r', c', d') are not proportional, then we have the following cases.

Case (1). $(r, r') = (0, 0)$. In this case, the wall $W_{(r,c,d),(r',c',d')}$ is empty.

Case (2). $(r, r') \neq (0, 0)$ and $r'c = rc'$. In this case, the wall is the vertical line

$$s = \frac{c'd - cd'}{2(r'd - rd')}.$$

So if $r \neq 0$, then it is $s = \frac{c}{2r}$.

Case (3). $r'c \neq rc'$. In this case, the wall is a semicircle in the (s, t)-half plane $\mathbb{R} \times \mathbb{R}_{>0}$ with center

$$\left(\frac{r'd - rd'}{r'c - rc'}, 0\right)$$

and radius

$$\sqrt{\left(\frac{r'd - rd'}{r'c - rc'}\right)^2 - \frac{c'd - cd'}{r'c - rc'}}.$$

For $E, E' \in \mathrm{D}(S)$, we write $W_{E,(r',c',d')}$ for $W_{(\mathrm{r}(E),c_1(E)c_1(L),\mathrm{ch}_2(E)),(r',c',d')}$, and $W_{E,E'}$ for $W_{(\mathrm{r}(E),c_1(E)c_1(L),\mathrm{ch}_2(E)),(\mathrm{r}(E'),c_1(E')c_1(L),\mathrm{ch}_2(E'))}$. If E is a semistable sheaf, then $W_{E,(r',c',d')}$ is either a vertical line $s=\bar{\mu}(E)$ or a semicircle with the center $(x,0)$, where

$$x = \frac{\mathrm{r}(E)d' - r'\mathrm{ch}_2(E)}{\mathrm{r}(E)(c' - 2r'\bar{\mu}(E))}, \tag{4.3}$$

and the radius

$$\sqrt{(x-\bar{\mu}(E))^2 - \Delta(E) + \frac{c_1(E)^2}{2\mathrm{r}(E)^2} - \bar{\mu}(E)^2} \leqslant |x - \bar{\mu}(E)|. \tag{4.4}$$

From this, we can see that for each point $(s,t) \in \mathbb{R} \times \mathbb{R}_{>0}$, there exists a unique wall $W_{E,*}$ passing through the point.

Remark 4.1. It can happen that the radius (4.4) of the wall is zero. Although, strictly speaking, such a wall is empty (because we consider walls in the region $t>0$), we call it a wall with radius zero.

4.2 (s,t)-semistability of symmetric exceptional bundles

We consider (s,t)-semistability of symmetric exceptional bundles. We follow closely the argument in [H, Section 9]. The argument goes as follows. Suppose that E, F and G are symmetric exceptional bundles (or their shifts), and that there exists a triangle

$$E \to F \to G \to E[1]$$

in $\mathrm{D}(S)$. Then $W_{E,F} = W_{F,G} = W_{E,G} (=: W)$. Suppose, moreover, that for a point (s,t) on the wall W, E, F and G belong to $\mathcal{A}_s$. Then the above triangle gives rise to an exact sequence

$$0 \to E \to F \to G \to 0$$

in the abelian category $\mathcal{A}_s$. If two of E, F and G are (s,t)-semistable, then the remaining one is also (s,t)-semistable since their values of $\mu_{s.t}$ are equal.

Since all symmetric exceptional bundles are obtained from the symmetric exceptional triple $(E_0, E_{1/2}, E_1)$, the following lemma is the first step of the argument.

Lemma 4.2. *Let α be an integer or a half integer. Then the symmetric exceptional bundle E_α is (s,t)-semistable for any $s<\alpha$. The shift $E_\alpha[1]$ is (s,t)-semistable for any $\alpha \leqslant s$.*

Proof. If α is an integer, then E_α is a line bundle. The (s,t)-semistability of E_α and $E_\alpha[1]$ follows from [AM, Theorem 1,1] (or we can argue as in the proof of Proposition 6.2 (d) in [ABCH]). If α is a half integer, then E_α is a direct sum of two line bundles with the same $\mu_{s,t}$. So the result follows. ∎

LEMMA 4.3. *Let $\alpha,\beta,\eta\in\mathfrak{E}$ satisfy $\alpha<\beta<\eta<\alpha+2$ and $\chi(\eta,\alpha)=\chi(\beta,\alpha)=\chi(\eta,\beta)=0$. Then the center of the semicircular wall W_{E_α,E_β} (resp. W_{E_β,E_η}) is $(\eta-1,0)$ (resp. $(\alpha+1,0)$), and its radius is $\sqrt{\Delta_\eta}$ (resp. $\sqrt{\Delta_\alpha}$).*

Proof. We prove the lemma for W_{E_α,E_β}. By (4.3), the center of W_{E_α,E_β} is $(x,0)$, where

$$\begin{aligned}x&=\frac{\mathrm{ch}_2(E_\alpha)/\mathrm{r}(E_\alpha)-\mathrm{ch}_2(E_\beta)/\mathrm{r}(E_\beta)}{2(\bar\mu(E_\alpha)-\bar\mu(E_\beta))}\\&=\frac{\alpha^2-\Delta_\alpha-(\beta^2-\Delta_\beta)}{2(\alpha-\beta)}=\frac{\alpha+\beta}{2}+\frac{\Delta_\beta-\Delta_\alpha}{2(\alpha-\beta)}=\eta-1,\end{aligned}$$

where we used Lemma 2.6(4) in the last equality. By (4.4), the square of the radius of W_{E_α,E_β} is

$$\begin{aligned}\left(\frac{\alpha-\beta}{2}+\frac{\Delta_\beta-\Delta_\alpha}{2(\alpha-\beta)}\right)^2-\Delta_\beta&=\left(\frac{\alpha-\beta}{2}\right)^2-\frac{\Delta_\alpha+\Delta_\beta}{2}+\left(\frac{\Delta_\beta-\Delta_\alpha}{2(\alpha-\beta)}\right)^2\\&=\left(\frac{\alpha-\beta}{2}\right)^2-\frac{\mathrm{P}(\alpha-\beta)}{2}+\left(\frac{\Delta_\beta-\Delta_\alpha}{2(\alpha-\beta)}\right)^2\\&=\Delta_\eta,\end{aligned}$$

where we used $\chi(\beta,\alpha)=0$ in the second equality, and Lemma 2.6(5) in the last equality. ∎

Now we come to the main theorem of this section.

THEOREM 4.4. *Consider symmetric exceptional slopes α,β and η given by*

$$\alpha=\epsilon\left(\frac{p}{2^q}\right),\qquad\beta=\epsilon\left(\frac{p+1}{2^q}\right),\qquad\eta=\epsilon\left(\frac{p+2}{2^q}\right),$$

where p is even and $q\geqslant 1$. Then E_β (resp. $E_\beta[1]$) is (s,t)-semistable if (s,t) with $s<\beta$ (resp. $s>\beta$) lies on or outside the semicircular wall W_{E_α,E_β} (resp. W_{E_β,E_η}). When the radius of the wall is zero, this sentence should be understood as "E_β (resp. $E_\beta[1]$) is (s,t)-semistable for any (s,t) with $\beta>s$ (resp. $s>\beta$)."

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

The proof of Theorem 4.4 goes as that of [H, Theorem 9.1], but occasionally gets more involved due to the appearance of walls with radius zero.

LEMMA 4.5. *Let E be a semistable sheaf.*

(1) *Assume that E is $(s_0.t_0)$-semistable with $s_0 < \bar{\mu}(E)$. If (s,t) with $s < \bar{\mu}(E)$ is outside the wall $W_{E,*}$ passing through (s_0,t_0), then E is (s,t)-semistable.*

(2) *Assume that $E[1]$ is $(s_0.t_0)$-semistable with $s_0 > \bar{\mu}(E)$. If (s,t) with $s > \bar{\mu}(E)$ is outside the wall $W_{E,*}$ passing through (s_0,t_0), then $E[1]$ is (s,t)-semistable.*

Proof. We prove (1). Let W be the unique wall $W_{E,*}$ passing through (s,t). Let (s_0,t') be the point on W. If E is not (s,t)-semistable, then it is not (s_0,t')-semistable. (In fact, if $M \to E$ is a (s,t)-destabilizing subobject in $\mathcal{A}_s$ with minimum rank, then the argument of the proof of [ABCH, Lemma 6.3] shows that $M \to E$ is also a (s_0,t')-destabilizing subobject in $\mathcal{A}_{s_0}$.) Suppose that a subobject $M \subset E$ in $\mathcal{A}_{s_0}$ (s_0,t')-destabilizes E. The inequality $\mu_{s_0,t'}(M) > \mu_{s_0,t'}(E)$ is equivalent to

$$\frac{(1/2)(s_0^2 - t'^2) - s_0\bar{\mu}(M) + (\mathrm{ch}_2(M)/2\mathrm{r}(M))}{\bar{\mu}(M) - s_0} > \frac{(1/2)(s_0^2 - t'^2) - s_0\bar{\mu}(E) + (\mathrm{ch}_2(E)/2\mathrm{r}(E))}{\bar{\mu}(E) - s_0}.$$

If $\bar{\mu}(M) = \bar{\mu}(E)$, then $\mathrm{ch}_2(M)/2\mathrm{r}(M) > \mathrm{ch}_2(E)/2\mathrm{r}(E)$, which contradicts the semistability of E. Thus $\bar{\mu}(M) < \bar{\mu}(E)$. Then we have $\mu_{s_0,t''}(M) < \mu_{s_0,t''}(E)$ for $t'' \gg 0$. This shows that the wall $W_{E,M}$ lies outside W. On the other hand, we have $\mu_{s_0,t_0}(M) \leqslant \mu_{s_0,t_0}(E)$ because E is (s_0,t_0)-semistable. This shows that the wall $W_{E,M}$ lies inside W. This is absurd. $\square$

We put

$$\alpha = \epsilon\left(\frac{p}{2^q}\right), \qquad \beta = \epsilon\left(\frac{p+1}{2^q}\right), \quad \eta = \epsilon\left(\frac{p+2}{2^q}\right)$$
$$\zeta_2 = \epsilon\left(\frac{p-2}{2^q}\right), \qquad \omega_2 = \epsilon\left(\frac{p-2}{2^q} + 2\right),$$
$$\omega_0 = \epsilon\left(\frac{p+4}{2^q}\right), \qquad \zeta_0 = \epsilon\left(\frac{p+4}{2^q} - 2\right),$$

where p is even and $q \geqslant 2$.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

NOTATION 4.6. If E is a symmetric exceptional bundle, we put

$$\tilde{\chi}(E, F) = \begin{cases} \chi(E, F)/2 & \text{if } E \text{ is even,} \\ \chi(E, F) & \text{if } E \text{ is odd.} \end{cases}$$

If F is a symmetric exceptional bundle, we put

$$\tilde{\chi}^*(E, F) = \begin{cases} \chi(E, F)/2 & \text{if } F \text{ is even,} \\ \chi(E, F) & \text{if } F \text{ is odd.} \end{cases}$$

With this notation, we have the following lemma.

LEMMA 4.7. *Let* $i \in \{0, 2\}$. *For* $p \equiv i \pmod 4$, *there are exact sequences*

$$0 \to E_{\zeta_i} \to E_\alpha^{\tilde{\chi}(E_\alpha, E_\beta)} \to E_\beta \to 0, \tag{4.5}$$

$$0 \to E_\beta \to E_\eta^{\tilde{\chi}^*(E_\beta, E_\eta)} \to E_{\omega_i} \to 0. \tag{4.6}$$

Proof. We prove the case $i = 0$. By the construction of symmetric exceptional bundles, we have $E_\beta = L_{E_\eta}(E_{\omega_0})[-1]$. Since right and left mutations are inverses to each other, we have $E_{\omega_0} = R_{E_\eta}(E_\beta)[1]$, which shows (4.5). By [BS, Lemma 5.2], we have $E_\beta = R_{E_\alpha}(E_{\zeta_0})[1]$. So we have $E_{\zeta_0} = L_{E_\alpha}(E_\beta)[1]$, which show (4.6). □

Proof of Theorem 4.4. We prove the theorem by induction on q. When $q = 1$, the theorem holds because of Lemma 4.2. Suppose $q \geqslant 2$. We show that E_β is (s, t)-semistable along and outside the semicircular wall W_{E_α, E_β}. The verification for $E_\beta[1]$ is similar and left to the reader.

Case (i) $p \equiv 0 \pmod 4$. The center of W_{E_α, E_β} is $(\eta - 1, 0)$ by Lemma 4.3. Since $\zeta_0 < \eta - 1 < \alpha$, the wall $W_{E_\alpha, E_\beta} = W_{E_\alpha, E_{\zeta_0}}$ lies between the vertical lines $s = \zeta_0$ and $s = \alpha$. For any point (s, t) on W_{E_α, E_β}, a shift of (4.5)

$$0 \to E_\alpha^{\tilde{\chi}(E_\alpha, E_\beta)} \to E_\beta \to E_{\zeta_0}[1] \to 0$$

is an exact sequence in $\mathcal{A}_s$. In order to show the (s, t)-semistability of E_β for $(s, t) \in W_{E_\alpha, E_\beta}$, we show that E_α and $E_{\zeta_0}[1]$ are (s, t)-semistable. Here note that E_α and $E_{\zeta_0}[1]$ have the same value of $\mu_{s,t}$ because (s, t) is on $W_{E_\alpha, E_\beta} = W_{E_\alpha, E_{\zeta_0}}$.

If α is an integer or a half integer, then E_α is (s, t)-semistable by Lemma 4.2. Otherwise, we have

$$u := \epsilon\left(\frac{p'}{2^{q'}}\right), \qquad \alpha = \left(\frac{p'+1}{2^{q'}}\right), \qquad v := \epsilon\left(\frac{p'+2}{2^{q'}}\right),$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

where p' is even and $q-2 \geqslant q' \geqslant 2$. By the induction hypothesis, E_α is (s,t)-semistable on the wall W_{E_u,E_α}. To show that E_α is (s,t)-semistable on W_{E_α,E_β}, it suffices to show that W_{E_u,E_α} lies inside W_{E_α,E_β}. The center of W_{E_u,E_α} is $(v-1,0)$ by Lemma 4.3. Note that $v-1<\alpha$ and $\eta-1<v-1$. This shows that both W_{E_u,E_α} and W_{E_α,E_β} lie to the left of the vertical line $s=\alpha$, and the center of W_{E_u,E_α} is right to the center of W_{E_α,E_β}. So W_{E_u,E_α} lies inside W_{E_α,E_β}.

Next we show that $E_{\zeta_0}[1]$ is (s,t)-semistable on the wall W_{E_α,E_β}. If ζ_0 is an integer or a half integer, this follows from Lemma 4.2. Otherwise, we have

$$\sigma' := \epsilon\left(\frac{p'}{2^{q'}}\right), \qquad \zeta_0 = \left(\frac{p'+1}{2^{q'}}\right), \qquad \tau' := \epsilon\left(\frac{p'+2}{2^{q'}}\right),$$

where p' is even and $q-2 \geqslant q' \geqslant 2$. Note that $\sigma' \leqslant \alpha-2$. By the induction hypothesis, $E_{\zeta_0}[1]$ is (s,t)-semistable on $W_{E_{\zeta_0},E_{\tau'}}$. The center of $W_{E_{\zeta_0},E_{\tau'}}$ is $(\sigma'+1,0)$. Since $\zeta_0<\sigma'+1<\eta-1$, $W_{E_{\zeta_0},E_{\tau'}}$ lies inside W_{E_α,E_β}.

Case (ii) $p \equiv 2 \pmod 4$. In this case, η can be an integer. So we consider two cases.

Case (ii-a) *η is not an integer.* In this case the semicircle W_{E_α,E_β} has a positive radius with center $(\eta-1,0)$. We have $\eta-1<\zeta_2$. For $(s,t) \in W_{E_\alpha,E_\beta}$, the exact sequence

$$0 \to E_{\zeta_2} \to E_\alpha^{\tilde{\chi}(E_\alpha,E_\beta)} \to E_\beta \to 0 \tag{4.7}$$

in Lemma 4.7 is an exact sequence in $\mathcal{A}_s$. In order to show the (s,t)-semistability of E_β, we show E_{ζ_2} and E_α are (s,t)-semistable on W_{E_α,E_β}.

If α is a half integer, then this follows from Lemma 4.2. Otherwise, $\alpha=\zeta_2.\eta$, and E_α is (s,t)-semistable on $W_{E_{\zeta_2},E_\alpha}=W_{E_\alpha,E_\beta}$ by the induction hypothesis.

If ζ_2 is an integer or a half integer, then E_{ζ_2} is (s,t)-semistable on W_{E_α,E_β} by Lemma 4.2. Otherwise, we have

$$\sigma := \epsilon\left(\frac{p'}{2^{q'}}\right), \qquad \zeta_2 = \left(\frac{p'+1}{2^{q'}}\right), \qquad \tau := \epsilon\left(\frac{p'+2}{2^{q'}}\right),$$

where p' is even and $q-2 \geqslant q' \geqslant 2$. Then E_{ζ_2} is (s,t)-semistable on $W_{E_\sigma,E_{\zeta_2}}$ by the induction hypothesis. The center of $W_{E_\sigma,E_{\zeta_2}}$ is $(\tau-1,0)$. Since $\eta \leqslant \tau$, $W_{E_\sigma,E_{\zeta_2}}$ lies inside $W_{E_\alpha,E_{\zeta_2}}=W_{E_\alpha,E_\beta}$.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Case (ii-b) *η is an integer.* In this case W_{E_α,E_β} has radius 0. We need to show that E_β is (s,t)-semistable for any $s<\beta$. Suppose that E_β is not (s,t)-semistable for some (s,t) with $s<\beta$. Then using the argument in the proof of Lemma 4.5, we can see that E_β is not $(\eta-1,t_0)$-semistable for some $t_0>0$. Consider a destabilizing subobject $M\subset E_\beta$ in $\mathcal{A}_{\eta-1}$, that is, $\mu_{\eta-1,t_0}(M)>\mu_{\eta-1,t_0}(E_\beta)$. Using the semistability of E_β, we can see that $\mu_{\eta-1,t'}(M)<\mu_{\eta-1,t'}(E_\beta)$ for $t'\gg 0$. Hence the point $(\eta-1,t_0)$ lies inside the wall $W_{E_\beta,M}$. It follows from this that

$$\mu_{\eta-1,t'}(M)>\mu_{\eta-1,t'}(E_\beta)$$

for any $0<t'<t_0$. Moreover, in the limit $t'\to 0$, we have

$$\lim_{t'\to 0} t'\mu_{\eta-1,t'}(M)>\lim_{t'\to 0} t'\mu_{\eta-1,t'}(E_\beta).$$

In order to obtain a contradiction, we derive the opposite inequality

$$\lim_{t'\to 0} t'\mu_{\eta-1,t'}(M)\leqslant\lim_{t'\to 0} t'\mu_{\eta-1,t'}(E_\beta). \tag{4.8}$$

We first consider the case $q\geqslant 3$. In this case, we have $\eta-1<\zeta_2$. So the sequence (4.7) is an exact sequence in $\mathcal{A}_{\eta-1}$. By the induction hypothesis, E_{ζ_2} and E_α are $(\eta-1,t')$-semistable for any $t'>0$. Thus we have

$$\mu_{\eta-1,t'}(M)\leqslant\max\left\{\mu_{\eta-1,t'}(E_{\zeta_2}),\mu_{\eta-1,t'}(E_\alpha)\right\}.$$

Taking the limit, we obtain

$$\begin{aligned}\lim_{t'\to 0} t'\mu_{\eta-1,t'}(M) &\leqslant \lim_{t'\to 0}\max\left\{t'\mu_{\eta-1,t'}(E_{\zeta_2}),t'\mu_{\eta-1,t'}(E_\alpha)\right\}\\ &=\lim_{t'\to 0} t'\mu_{\eta-1,t'}(E_\beta),\end{aligned}$$

where the last equality holds because $(\eta-1,0)$ is the center of the wall $W_{E_\alpha,E_\beta}=W_{E_\alpha,E_{\zeta_2}}$ (of radius zero).

Finally we consider the case $q=2$. In this case, we have $\eta-1=\zeta_2$ and $E_{\zeta_2}=\mathcal{O}(\zeta_2,\zeta_2)$. So the shift of (4.7)

$$0\to E_\alpha^{\tilde{\chi}(E_\alpha,E_\beta)}\to E_\beta\xrightarrow{f}E_{\zeta_2}[1]\to 0$$

is an exact sequence in $\mathcal{A}_{\eta-1}$. By Lemma 4.8, the subobjects of $E_{\zeta_2}[1]$ in $\mathcal{A}_{\eta-1}$ are 0 and $E_{\zeta_2}[1]$. If $f(M)=0$, then $\mu_{\eta-1,t'}(M)\leqslant\mu_{\eta-1,t'}(E_\alpha)$.

By taking the limit, we obtain (4.8). If $f(M)=E_{\zeta_2}[1]$, then

$$Z_{(\eta-1,t')}(M)=Z_{(\eta-1,t')}(E_{\zeta_2}[1])+Z_{(\eta-1,t')}(\mathrm{Ker}(f|_M)).$$

By an easy calculation, we see that $\lim_{t'\to 0} Z_{(\eta-1,t')}(E_{\zeta_2}[1])=0$. So

$$\begin{aligned}\lim_{t'\to 0} t'\mu_{\eta-1,t'}(M) &= \lim_{t'\to 0} t'\mu_{\eta-1,t'}(\mathrm{Ker}(f|_M))\\ &\leqslant \lim_{t'\to 0} t'\mu_{\eta-1,t'}(E_\alpha)=\lim_{t'\to 0} t'\mu_{\eta-1,t'}(E_\beta).\end{aligned}$$

This is the end of the proof of Theorem 4.4. ∎

LEMMA 4.8. *Let a be an integer and V a finite-dimensional $\mathbb{C}$-vector space. The subobjects of $V\otimes E_a[1]$ in $\mathcal{A}_a$ are $W\otimes E_a[1]$ for subspaces $W\subset V$.*

To prove this lemma, we use the following lemma whose proof is left to the reader.

LEMMA 4.9. *Let Y be a projective variety of dimension d. If F is a μ-semistable sheaf with respect to an ample line bundle H such that $c_1(F)\cdot c_1(H)^{d-1}=0$, then $h^0(Y,F)\leqslant \mathrm{r}(F)$, where equality holds if and only if $F\simeq \mathcal{O}_Y^{\mathrm{r}(F)}$.*

Proof of Lemma 4.8. We may assume that $a=0$. Then $E_a=\mathcal{O}_S$. Let $U\subset V\otimes\mathcal{O}_S[1]$ be a subobject in $\mathcal{A}_0$. Then we have an exact sequence

$$0\to \mathrm{H}^{-1}(U)\to V\otimes\mathcal{O}_S\xrightarrow{g} F\to \mathrm{H}^0(U)\to 0$$

of sheaves with $F\in\mathcal{F}_0$. Since $\bar{\mu}_{\max}(F)\leqslant 0$, we have $\bar{\mu}(\mathrm{H}^{-1}(U))=\bar{\mu}(\mathrm{Im}\, g)=0$, and $\mathrm{H}^{-1}(U)$ and $\mathrm{Im}\, g$ are μ-semistable sheaves. By Lemma 4.9, $\mathrm{Im}\, g$ is a trivial vector bundle. Now it is easy to show that $\mathrm{H}^0(U)=0$ and $\mathrm{H}^{-1}(U)$ is $W\otimes E_a[1]$ for a subspace $W\subset V$. ∎

§5. Height-zero moduli spaces

Fix $\xi\in K(S)$ such that $\mathrm{r}(\xi)=r$, $\nu(\xi)=(\bar{\mu},\bar{\mu})$ and $\Delta(\xi)=\Delta$.

Throughout this section, we assume that the height of the moduli space $M(\xi)$ is zero. We show that $M(\xi)$ is isomorphic to a moduli space of representations of a quiver.

There is a unique $\gamma\in\mathfrak{E}$ with $\bar{\mu}\in I_\gamma$. We have either (i) $\bar{\mu}\in(\gamma-x_\gamma,\gamma]$, or (ii) $\bar{\mu}\in(\gamma,\gamma+x_\gamma)$. We only consider the case (i). If we are in the case (ii), then by taking a dual of sheaves we will be in the case (i). (Note that in the case (ii), any $F\in M(\xi)$ is locally free by Remark 3.3.) So in the rest of this section, we assume that $\gamma-x_\gamma<\bar{\mu}\leqslant\gamma$.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

5.1 Bridgeland semistability of semistable sheaves

In this section, we study for which (s,t) a sheaf $F\in M(\xi)$ is (s,t)-semistable.

Express $\gamma=\epsilon(p/2^q)$ where p is odd if $q\geqslant 1$. If $q\geqslant 1$, then we set

$$\alpha=\epsilon\left(\frac{p-1}{2^q}\right)\qquad\text{and}\qquad\beta=\epsilon\left(\frac{p+1}{2^q}\right).$$

If $q=0$, then we set

$$\alpha=\gamma-\frac{1}{2}\qquad\text{and}\qquad\beta=\gamma+1.$$

Then $\gamma=\alpha.\beta$ and $(E_{\beta-2},E_\alpha,E_\gamma)$ is a symmetric exceptional triple. Put

$$\begin{aligned}E^{(1)}&=E_{\beta-2}, & E^{(2)}&=E_\alpha, & E^{(3)}&=E_\gamma,\\ G^{(1)}&=E_\beta, & G^{(2)}&=E_{\gamma.\beta}, & G^{(3)}&=E_\gamma.\end{aligned}$$

Then for $F\in \mathrm{D}(S)$, there exists a spectral sequence

$$E_1^{p,q}=\mathrm{Ext}^q(G^{(p+3)},F)\hat{\otimes}E^{(p+3)} \tag{5.1}$$

converging to $H^{p+q}(F)$, where $E_1^{p,q}=0$ unless $-2\leqslant p\leqslant 0$ (see (A5) for this spectral sequence and the notation $\hat{\otimes}$).

We consider the semicircular walls $W_{E_{\beta-2},E_\alpha}$, $W_{E_{\beta-2},E_\gamma}$ and W_{E_α,E_γ} in the upper half (s,t)-plane.

The center of the walls $W_{E_{\beta-2},E_\alpha}$, $W_{E_{\beta-2},E_\gamma}$ and W_{E_α,E_γ} are $(\gamma-1,0)$, $(\gamma.\beta-1,0)$ and $(\beta-1,0)$, respectively. When $q=0$, the walls $W_{E_{\beta-2},E_\alpha}$ and W_{E_α,E_γ} have radii 0, and $W_{E_{\beta-2},E_\gamma}$ has positive radius. When $q=1$, the wall W_{E_α,E_γ} has radius 0, the other two walls have positive radii; the wall $W_{E_{\beta-2},E_\alpha}$ lies inside $W_{E_{\beta-2},E_\gamma}$. When $q\geqslant 2$, W_{E_α,E_γ} lies inside $W_{E_{\beta-2},E_\alpha}$, and $W_{E_{\beta-2},E_\alpha}$ lies inside $W_{E_{\beta-2},E_\gamma}$. (See Figures 1–3.)

LEMMA 5.1. *If a point (s_0,t_0) with $\beta-2<s_0<\alpha$ in the upper half (s,t)-plane lies on or outside the wall $W_{E_{\beta-2},E_\alpha}$, then E_α, E_γ and $E_{\beta-2}[1]$ are (s_0,t_0)-semistable.*

Proof. Since the proof is similar, we prove the lemma only for $E_{\beta-2}[1]$. If β is an integer or a half integer, this is clear by Lemma 4.2. Otherwise, we can write $\beta=\sigma.\tau$, where σ, $\tau\in\mathfrak{E}$ with $\sigma\leqslant\alpha$. By Theorem 4.4, $E_{\beta-2}[1]$ is (s,t)-semistable outside the wall $W_{E_{\beta-2},E_{\tau-2}}$ whose center is $(\sigma-1,0)$. Since the center of $W_{E_{\beta-2},E_\alpha}$ is $(\gamma-1,0)$, which is to the right of $(\sigma-1,0)$, $W_{E_{\beta-2},E_\alpha}$ lies outside $W_{E_{\beta-2},E_{\tau-2}}$. ∎

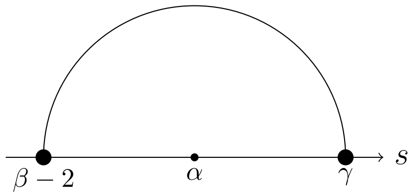

Figure 1.
The case $q = 0$. The semicircle is $W_{E_{\beta-2},E_\gamma}$; the dot at $\beta - 2$ is $W_{E_{\beta-2},E_\alpha}$; the dot at γ is W_{E_α,E_γ}.

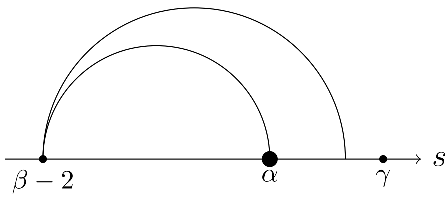

Figure 2.
The case $q = 1$. The outer semicircle is $W_{E_{\beta-2},E_\gamma}$; the inner semicircle is $W_{E_{\beta-2},E_\alpha}$; the dot at α is W_{E_α,E_γ}.

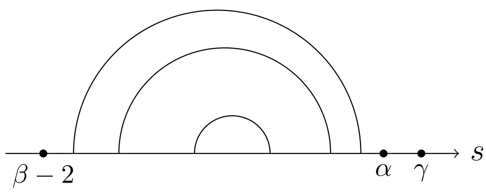

Figure 3.
The case $q \geqslant 2$. The outermost semicircle is $W_{E_{\beta-2},E_\gamma}$; the innermost semicircle is W_{E_α,E_γ}; the semicircle in between is $W_{E_{\beta-2},E_\alpha}$.

PROPOSITION 5.2. *A semistable (resp. stable) sheaf $F \in M(\xi)$ is (s_0, t_0)-semistable (resp. (s_0, t_0)-stable) if the point (s_0, t_0) with $\beta - 2 < s_0 < \alpha$ in the upper half (s, t)-plane lies outside the wall $W_{E_{\beta-2},E_\alpha}$.*

Proof. First we consider a semistable sheaf F. Note that $F \in \mathcal{A}_{s_0}$ since $s_0 < \bar{\mu}$ and F is semistable. Suppose that F is not (s_0, t_0)-semistable. Then there exists a subobject $S \subset F$ in $\mathcal{A}_{s_0}$ such that

$$\mu_{s_0,t_0}(S) > \mu_{s_0,t_0}(F). \tag{5.2}$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Then S is a sheaf and there exists an exact sequence

$$0 \to K \to S \to F \to Q \to 0 \tag{5.3}$$

of sheaves. We choose such an S with minimal rank. Then for any (s,t) on the wall $W := W_{F,*}$ passing through (s_0,t_0), S is a subobject of F and $\mu_{s,t}(S) > \mu_{s,t}(F)$ (cf. the argument of the proof of [ABCH, Lemma 6.3]).

Claim. The wall W lies outside $W_{E_{\beta-2},E_\alpha}$.

Proof of Claim. Using the spectral sequence (5.1), we see that there exists a short exact sequence

$$0 \to E_{\beta-2}^m \to E_\alpha^n \to F \to 0, \tag{5.4}$$

where

$$m = -\tilde{\chi}(E_\beta, F) \qquad \text{and} \qquad n = -\tilde{\chi}(E_{\gamma.\beta}, F). \tag{5.5}$$

So $W_{E_{\beta-2},E_\alpha} = W_{F,E_\alpha}$. Both W and W_{F,E_α} are walls of the form $W_{F,*}$, so they are disjoint. Since the point (s_0,t_0) is outside W_{F,E_α}, W is outside W_{F,E_α}. This is the end of the proof of the claim. ∎

Put $s_1 = \gamma - 1$. Then $(s_1, 0)$ be the center of $W_{E_{\beta-2},E_\alpha}$. Let (s_1,t_1) and (s_1,t_2) be the point on W and $W_{E_{\beta-2},E_\alpha}$, respectively. (When γ is an integer, we understand that $t_2 = 0$.) The exact sequence (5.3) and the semistability of F imply either

$$\bar{\mu}(S) < \bar{\mu}(F) \tag{5.6}$$

or

$$K = 0 \quad \text{and} \quad \bar{\mu}(S) = \bar{\mu}(F) \quad \text{and} \quad \frac{\mathrm{ch}_2(S)}{\mathrm{r}(S)} \leqslant \frac{\mathrm{ch}_2(F)}{r}. \tag{5.7}$$

If (5.7) holds, then $\mu_{s,t}(S) \leqslant \mu_{s,t}(F)$ for any $s \leqslant \bar{\mu}(F)$, this contradicts the inequality (5.2). If (5.6) holds, then $\mu_{s_1,t}(S) < \mu_{s_1,t}(F)$ for $t \gg 0$, so there exists $t_3 > t_1$ such that $\mu_{s_1,t_3}(S) = \mu_{s_1,t_3}(F)$. This shows that

$$\mu_{s_1,t}(S) > \mu_{s_1,t}(F) \tag{5.8}$$

for any $0 < t < t_3$. Moreover, we have

$$\lim_{t\to+0} t\mu_{s_1,t}(S) > \lim_{t\to+0} t\mu_{s_1,t}(F). \tag{5.9}$$

In order to obtain a contradiction, we show the opposite inequality.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Consider first the case when γ is not an integer. Then $\beta-2<s_1<\alpha$ and $t_2>0$. By Lemma 5.1, $E_{\beta-2}[1]$ and E_α are (s_1,t_2)-semistable and from (5.4) we obtain a short exact sequence

$$0\to E_\alpha^n\to F\xrightarrow{f} E_{\beta-2}[1]^m\to 0 \tag{5.10}$$

in $\mathcal{A}_{s_1}$. So we have $\mu_{s_1,t_2}(S)\leqslant\mu_{s_1,t_2}(F)$. This contradicts the inequality (5.8).

Finally consider the case when γ is an integer. In this case, we have $s_1=\beta-2$ and $t_2=0$. By Lemma 4.2, $E_{\beta-2}[1]$ and E_α are (s_1,t)-semistable for any $t>0$, and (5.10) is a short exact sequence in $\mathcal{A}_{s_1}$. By Lemma 4.8, the subobject $f(S)\subset E_{\beta-2}[1]^m$ is isomorphic to $E_{\beta-2}[1]^l$ for some $l\geqslant 0$. We have

$$Z_{s_1,t}(S)=lZ_{s_1,t}(E_{\beta-2}[1])+Z(\mathrm{Ker}\,(f|_S))$$

and

$$Z_{s_1,t}(E_{\beta-2}[1])=-t^2.$$

This implies that

$$\begin{aligned}\lim_{t\to+0}t\mu_{s_1,t}(S)&=\lim_{t\to+0}t\mu_{s_1,t}(\mathrm{Ker}\,(f|_S))\\&\leqslant\lim_{t\to+0}t\mu_{s_1,t}(E_\alpha)=\lim_{t\to+0}t\mu_{s_1,t}(F).\end{aligned}$$

This contradicts (5.9).

Finally we have to show that if F is stable, then it is (s_0,t_0)-stable. But this can be proved by a similar argument. In fact, we already know that F is (s_0,t_0)-semistable. If F is not (s_0,t_0)-stable, then there exists a subobject $0\neq S\subsetneq F$ in $\mathcal{A}_{s_0}$ with $\mu_{s_0,t_0}(S)=\mu_{s_0,t_0}(F)$. Arguing in the semistable case, we obtain a contradiction. ∎

Conversely (s,t)-semistability implies usual semistability:

PROPOSITION 5.3. *Assume that the point (s_0,t_0) with $\beta-2<s_0<\alpha$ in the upper half (s,t)-plane lies outside the wall $W_{E_{\beta-2},E_\alpha}$. Let $F^\bullet$ be an object in $\mathcal{A}_{s_0}$ with $[F^\bullet]=\xi$ in $K(S)$. Assume that $F^\bullet$ is (s_0,t_0)-semistable (resp. (s_0,t_0)-stable). Then $F^\bullet$ is a semistable (resp. stable) sheaf.*

Proof. We show the semistable case first.

Step 1. We show that $F^\bullet$ is quasi-isomorphic to a 2-term complex $E_{\beta-2}^m\to E_\alpha^n$, where E_α^n and $E_{\beta-2}^m$ are in degree 0 and -1, respectively, and

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

m, n are those in (5.5). We use the spectral sequence (5.1). Since $\mathrm{H}^i(F^\bullet)=0$ except $i=0,-1$, the term $E_1^{p,q}$ in the spectral sequence is zero unless $-1\leqslant q\leqslant 2$. If $u\in\{\beta,\gamma.\beta,\gamma\}$, then $\mathrm{Ext}^2(E_u,F^\bullet)=0$ because

$$\mathrm{Ext}^2(E_u,\mathrm{H}^0(F^\bullet))\simeq \mathrm{Hom}(\mathrm{H}^0(F^\bullet),E_{u-2})=0$$

(note that $\bar{\mu}_{\min}(\mathrm{H}^0(F^\bullet)/tor)>s_0>u-2$). We can also show that $\mathrm{Ext}^{-1}(E_u,F^\bullet)=0$ by a similar argument.

Claim. We have $E_1^{0,0}=0$.

Proof of Claim. We show that $\mathrm{Hom}(E_\gamma,F^\bullet)=0$. By the (s_0,t_0)-semistability of $F^\bullet$, it suffices to show that $\mu_{s_0,t_0}(E_\gamma)>\mu_{s_0,t_0}(F^\bullet)$. Since

$$Z_{s_0,t_0}(F^\bullet)=nZ_{s_0,t_0}(E_\alpha)+mZ_{s_0,t_0}(E_{\beta-2}^m[1])$$

by the condition on the K-class of $F^\bullet$, and $\mu_{s_0,t_0}(E_{\beta-2}[1])>\mu_{s_0,t_0}(E_\alpha)$, we have $\mu_{s_0,t_0}(F^\bullet)<\mu_{s_0,t_0}(E_{\beta-2}[1])$. If (s_0,t_0) lies inside the wall $W_{E_{\beta-2},E_\gamma}$, then $\mu_{s_0,t_0}(E_\gamma)>\mu_{s_0,t_0}(E_{\beta-2}[1])$, thus $\mu_{s_0,t_0}(E_\gamma)>\mu_{s_0,t_0}(F^\bullet)$. For $t\gg 0$, we have $\mu_{s_0,t}(E_\gamma)>\mu_{s_0,t}(F^\bullet)$ because $\bar{\mu}(F^\bullet)<\gamma$, or $\bar{\mu}(F^\bullet)=\gamma$ and $\mathrm{ch}_2(F^\bullet)/r<\mathrm{ch}_2(E_\gamma)/\mathrm{r}(E_\gamma)$. These show that $\mu_{s_0,t_0}(E_\gamma)>\mu_{s_0,t_0}(F^\bullet)$ for any (s_0,t_0) in the proposition. This is the end of the proof of the claim. $\square$

Since $\chi(E_\gamma,F^\bullet)=0$, we have $E_1^{0,1}=0$. Now we have shown that $E_1^{p,q}$ is zero except $E_1^{-2,0}$, $E_1^{-2,1}$, $E_1^{-1,0}$, $E_1^{-1,1}$. Since $\mathrm{H}^{-2}(F^\bullet)=0$, the kernel of the map $d_1^{-2,0}:E_1^{-2,0}\to E_1^{-1,0}$ is zero. The cokernel of $d_1^{-2,0}$ is a subsheaf of $H^{-1}(F^\bullet)$. Since $\mathrm{Hom}(E_\alpha,\mathrm{H}^{-1}(F^\bullet))=0$ by the inequality $\bar{\mu}_{\max}(\mathrm{H}^{-1}(F^\bullet))\leqslant s_0<\alpha$, we have $\mathrm{Coker}\, d_1^{-2,0}=0$. Hence $d_1^{-2,0}$ is an isomorphism. This occurs only when $E_1^{-2,0}=E_1^{-1,0}=0$.

Step 2. We let

$$F^\bullet=[E_{\beta-2}^m\xrightarrow{f}E_\alpha^n].$$

We show that if f is not injective, or if f is injective and Coker is not semistable, then there exists a surjection $F^\bullet\to V$ in $\mathcal{A}_{s_0}$, where V is a semistable sheaf such that the following condition (a) or (b) holds:

(a) $\alpha\leqslant\bar{\mu}(V)<\bar{\mu}$,
(b) $\bar{\mu}(V)=\bar{\mu}$ and $\mathrm{ch}_2(F)/r>\mathrm{ch}_2(V)/\mathrm{r}(V)$.

To prove this, we consider 3 cases:

(i) f is not injective,

(ii) f is injective and Coker f is not torsion-free,
(iii) f is injective, Coker f is torsion-free, and Coker f is not semistable.

Case (i): We have

$$\begin{aligned}\bar{\mu}(\operatorname{Coker} f) &= \frac{\deg E^n_\alpha - \deg \operatorname{Im} f}{2(\mathrm{r}(E^n_\alpha) - \mathrm{r}(\operatorname{Im} f))} \\ &\leqslant \frac{\deg E^n_\alpha - \mu(E_{\beta-2})\mathrm{r}(\operatorname{Im} f)}{2(\mathrm{r}(E^n_\alpha) - \mathrm{r}(\operatorname{Im} f))} < \frac{\deg E^n_\alpha - \deg E^m_{\beta-2}}{2(\mathrm{r}(E^n_\alpha) - \mathrm{r}(E^m_{\beta-2}))} \\ &= \bar{\mu}(F^\bullet) = \bar{\mu}.\end{aligned}$$

Let Coker $f/tor \twoheadrightarrow V$ be the quotient with minimum value of $\bar{\mu}$. Then the composite of morphisms

$$F^\bullet \to \mathrm{H}^0(F^\bullet) = \operatorname{Coker} f \to \operatorname{Coker} f/tor \twoheadrightarrow V \tag{5.11}$$

is a surjection in $\mathcal{A}_{s_0}$ and V is a semistable sheaf satisfying (a).

Case (ii): We have $\bar{\mu}(\operatorname{Coker} f/tor) < \bar{\mu}(F^\bullet)$. We define V and $F^\bullet \to V$ as in Case (i). Then V satisfies (a).

Case (iii): There exists a surjective morphism Coker $f \to V$ of sheaves such that V is a semistable sheaf satisfying (a) or (b). The composite (5.11) (with tor $= 0$) is what we want.

Step 3. Let $F^\bullet \to V$ be the surjection in $\mathcal{A}_{s_0}$ which we obtained in Step 2. By the (s_0, t_0)-semistability of $F^\bullet$, we have

$$\mu_{s_0,t_0}(F^\bullet) \leqslant \mu_{s_0,t_0}(V). \tag{5.12}$$

We have

$$\mu_{s_0,t}(F^\bullet) > \mu_{s_0,t}(V)$$

for $t \gg 0$ because of the conditions (a) and (b). So we have

$$\mu_{s_0,t_1}(F^\bullet) = \mu_{s_0,t_1}(V) \tag{5.13}$$

for some $t_1 \geqslant t_0$.

Suppose that the condition (b) in Step 2 holds. Then the wall $W_{F^\bullet,V}$ is a vertical wall and we see that $\mu_{s_0,t}(F^\bullet) > \mu_{s_0,t}(V)$ for any $t > 0$. This is a contradiction.

Suppose that the condition (a) in Step 2 holds. The center of the semicircular wall $W_{F^\bullet,V}$ is $(x, 0)$, where

$$x = \frac{\mathrm{ch}_2(F^\bullet)/r - \mathrm{ch}_2(V)/\mathrm{r}(V)}{\mu(F^\bullet) - \mu(V)}.$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Since the moduli space $M(\xi)$ has height zero, we have

$$0=\frac{\chi(E_\gamma,F^\bullet)}{\mathrm{r}(E_\gamma)r}=\chi(\mathcal{O}_S)+\frac{\mathrm{ch}_2(F^\bullet)}{r}$$
$$-\frac{c_1(E_\gamma)c_1(F)}{\mathrm{r}(E_\gamma)r}+\frac{\mathrm{ch}_2(E_\gamma)}{\mathrm{r}(E_\gamma)}+\mu(F^\bullet)-\mu(E_\gamma). \tag{5.14}$$

Since the semistable sheaf V satisfies (a), we have

$$0\geqslant\frac{\chi(E_\gamma,V)}{\mathrm{r}(E_\gamma)\mathrm{r}(V)}=\chi(\mathcal{O}_S)+\frac{\mathrm{ch}_2(V)}{\mathrm{r}(V)}$$
$$-\frac{c_1(E_\gamma)c_1(V)}{\mathrm{r}(E_\gamma)\mathrm{r}(V)}+\frac{\mathrm{ch}_2(E_\gamma)}{\mathrm{r}(E_\gamma)}+\mu(V)-\mu(E_\gamma). \tag{5.15}$$

From (5.14) and (5.15), we obtain

$$x=-\frac{\chi(E_\gamma,V)}{\mathrm{r}(E_\gamma)\mathrm{r}(V)(\mu(F^\bullet)-\mu(V))}+\gamma-1\geqslant\gamma-1.$$

Since the center of the wall $W_{E_{\beta-2},E_\alpha}=W_{F^\bullet,E_\alpha}$ is $(\gamma-1,0)$, the wall $W_{F^\bullet,V}$ is equal to or lies inside $W_{E_{\beta-2},E_\alpha}$. This contradicts (5.13).

Finally we consider the stable case. If $F^\bullet$ is (s_0,t_0)-stable, we already know it is a semistable sheaf F. If F is not stable, then there is a subsheaf $0\neq G\subsetneq F$ with $p_G=p_F$. Then G is a subobject of F in $\mathcal{A}_{s_0}$ with $\mu_{s_0,t_0}(G)=\mu_{s_0,t_0}(F)$. This contradicts the (s_0,t_0)-stability of F. ∎

5.2 Isomorphism to moduli of quiver representations

In this section, we see that $M(\xi)$ is isomorphic to a moduli space of quiver representations.

A quiver is a quadruple $Q=(Q_0,Q_1,s,t)$, where Q_0 is the set of vertexes, Q_1 is the set of arrows, and the maps $s,t:Q_1\to Q_0$ send an arrow to its source and target respectively. A path of the quiver Q is a sequence of arrows $(\alpha_n,\dots,\alpha_1)$ such that $t(\alpha_i)=s(\alpha_{i+1})$. For a path $f=(\alpha_n,\dots,\alpha_1)$, we define $s(f):=s(\alpha_1)$ and $t(f):=t(\alpha_n)$. The path algebra $A(Q)$ of a quiver is the algebra whose elements are $\mathbb{C}$-linear combinations of paths of Q. If f and g are paths of Q such that $s(f)=t(g)$, then fg is the path obtained by connecting g and f.

Giving a left $A(Q)$-module is equivalent to giving data

$$\left(\{V_v\}_{v\in Q_0},\{\rho_\alpha\}_{\alpha\in Q_1}\right), \tag{5.16}$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

where V_v is a $\mathbb{C}$-vector space, and $\rho_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ is a $\mathbb{C}$-linear map.

We call the data (5.16) a representation of the quiver Q. The map $\delta : Q_0 \to \mathbb{Z}_{\geqslant 0}$ given by $\delta(v) = \dim V_v$ is called the dimension vector of the representation. Fix a map $\theta : Q_0 \to \mathbb{Q}$, called a weight vector, such that $\sum_{v \in Q_0} \delta(v)\theta(v) = 0$. The representation (5.16) of the quiver is said to be θ-semistable if for all subrepresentations

$$\left(\left\{V'_v\right\}_{v\in Q_0}, \left\{\rho_\alpha|_{V'_{s(\alpha)}}\right\}_{\alpha\in Q_1}\right),$$

of (5.16), the inequality $\sum_{v\in Q_0} \theta(v) \dim V'_v \geqslant 0$ holds.

In the following, in order to describe a quiver, we use the notation

$$Q = (Q_0, \{N_{v,v'}\}_{(v,v')\in Q_0\times Q_0}),$$

where $N_{v,v'}$ means the number of arrows α with $s(\alpha) = v$ and $t(\alpha) = v'$.

Recall α, β and γ defined at the beginning of Section 5.1. When α, β or γ is even respectively, we define quivers Q^α, Q^β and Q^γ as follows.

The quiver $Q^\alpha = (Q_0^\alpha, \{N_{v,v'}\})$ is defined by $Q_0^\alpha = \{v_1, v_2, v_3\}$ and $N_{v_1,v_2} = N_{v_1,v_3} = \tilde{\chi}(E_{\gamma.\beta}, E_\beta)$, and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j) (see Notation 4.6 for the notation $\tilde{\chi}$).

The quiver $Q^\beta = (Q_0^\beta, \{N_{v,v'}\})$ is defined by $Q_0^\beta = \{v_1, v_2, v_3\}$ and $N_{v_2,v_1} = N_{v_3,v_1} = \tilde{\chi}^*(E_{\gamma.\beta}, E_\beta)$, and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j).

The quiver $Q^\gamma = (Q_0^\gamma, \{N_{v,v'}\})$ is defined by $Q_0^\gamma = \{v_1, v_2\}$ and $N_{v_1,v_2} = \chi(E_{\gamma.\beta}, E_\beta)$, and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j). (See Figures 4–6.)

We define moduli spaces N^α, N^β and N^γ of representations of quivers as follows. N^α (resp. N^β) is the coarse moduli space of θ-semistable representations of the quiver Q^α (resp. Q^β) with dimension vector $(\delta(v_1), \delta(v_2), \delta(v_3))$ equal to (m, n, n) (resp. (n, m, m)), where the weight vector $(\theta(v_1), \theta(v_2), \theta(v_3))$ is $(-2n, m, m)$ (resp. $(2m, -n, -n)$). N^γ is the

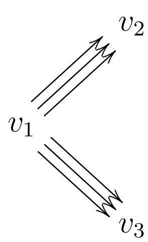

Figure 4. Q^α.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

$$\begin{tikzcd}
v_2 \arrow[dr, shift left=1.5] \arrow[dr] \arrow[dr, shift right=1.5] & \\
& v_1 \\
v_3 \arrow[ur, shift left=1.5] \arrow[ur] \arrow[ur, shift right=1.5] &
\end{tikzcd}$$

Figure 5.
Q^{β}.

$$\begin{tikzcd}
v_1 \arrow[r, shift left=1.5] \arrow[r] \arrow[r, shift right=1.5] & v_2
\end{tikzcd}$$

Figure 6.
Q^{γ}.

coarse moduli space of θ-semistable representations of the quiver Q^{γ} with dimension vector $(\delta(v_1), \delta(v_2))$ equal to (m, n), where the weight vector $(\theta(v_1), \theta(v_2))$ is $(-n, m)$.

THEOREM 5.4. *If the symmetric exceptional bundle E_α (resp. E_β or E_γ) is even, then the moduli space $M(\xi)$ is isomorphic to N^α (resp. N^β or N^γ).*

For the proof of the theorem, we imitate the argument in [ABCH, Sections 7, 8], [Oh].

Fix a point with $\beta - 2 < s < \alpha$ in the upper half (s, t)-plane such that (s, t) lies outside the wall $W_{E_{\beta-2}, E_\alpha}$ and inside $W_{E_{\beta-2}, E_\gamma}$. Then we have

$$\mu_{s,t}(E_\gamma) > \mu_{s,t}(E_{\beta-2}[1]) > \mu(E_\alpha).$$

Fix $0 < \phi < 1$ such that

$$\arg Z_{s,t}(E_\gamma) > \phi > \arg Z_{s,t}(E_{\beta-2}[1]).$$

Put

$$\begin{aligned}
\mathcal{Q}_\phi &:= \langle Q \in \mathcal{A}_s \mid Q \text{ is } (s,t)\text{-semistable}, \arg Z_{s,t}(Q) > \phi\pi \rangle \\
\mathcal{F}_\phi &:= \langle F \in \mathcal{A}_s \mid F \text{ is } (s,t)\text{-semistable}, \arg Z_{s,t}(Q) \leqslant \phi\pi \rangle,
\end{aligned}$$

and define $\mathcal{A}[\phi] := \langle \mathcal{Q}_\phi, \mathcal{F}_\phi[1] \rangle$ and $Z[\phi](E) := e^{-i\pi\phi} Z_{s,t}(E)$. Then the pair $(\mathcal{A}[\phi], Z[\phi])$ is also a Bridgeland stability condition of $\mathrm{D}(S)$. Since the operation "$[\phi]$" does not change the Bridgeland semistable objects, the moduli

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

space of (s, t)-semistable objects with K-class equal to ξ is isomorphic to the moduli space of $Z[\phi]$-semistable objects in $\mathcal{A}[\phi]$ with K-class $-\xi$ by the correspondence $F^\bullet$ to $F^\bullet[1]$.

Put

$$E := E_\gamma \oplus E_{\gamma.\beta} \oplus E_\beta \quad \text{and} \quad A := \mathrm{End}(E).$$

We define a functor

$$\Phi : D(S) \to D(A^{op} - \mathrm{mod})$$

by $\Phi(-) = \mathbf{R}\mathrm{Hom}(E, -)$, where $D(A^{op} - \mathrm{mod})$ is the bounded derived category of right A-modules with finite-dimensional cohomology. This gives an equivalence of triangulated categories (cf. [Bo, Theorem 6.2]).

The algebra A^{op} is isomorphic to the path algebra of a quiver modulo a certain ideal. We define quivers $\tilde{Q}^\alpha$, $\tilde{Q}^\beta$ and $\tilde{Q}^\gamma$ as follows. $\tilde{Q}^\alpha = (\tilde{Q}^\alpha_0, \{N_{v,v'}\})$ is defined by $\tilde{Q}^\alpha_0 = \{v_1, v_2, v_3, v_4\}$ and $N_{v_1,v_2} = N_{v_1,v_3} = \tilde{\chi}(E_{\gamma.\beta}, E_\beta)$, $N_{v_2,v_4} = N_{v_3,v_4} = \tilde{\chi}^*(E_\gamma, E_{\gamma.\beta})$ and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j). $\tilde{Q}^\beta = (\tilde{Q}^\alpha_0, \{N_{v,v'}\})$ is defined by $\tilde{Q}^\beta_0 = \{v_1, v_2, v_3, v_4\}$ and $N_{v_2,v_1} = N_{v_3,v_1} = \tilde{\chi}^*(E_{\gamma.\beta}, E_\beta)$, $N_{v_1,v_4} = \chi(E_\gamma, E_{\gamma.\beta})$ and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j). $\tilde{Q}^\gamma = (\tilde{Q}^\alpha_0, \{N_{v,v'}\})$ is defined by $\tilde{Q}^\gamma_0 = \{v_1, v_2, v_3, v_4\}$ and $N_{v_1,v_2} = \chi(E_{\gamma.\beta}, E_\beta)$, $N_{v_2,v_3} = N_{v_2,v_4} = \tilde{\chi}(E_\gamma, E_{\gamma.\beta})$ and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j). (See Figures 7–9.)

When E_α (resp. E_β) is even, the algebra A^{op} is isomorphic to the path algebra $A(\tilde{Q}^\alpha)$ (resp. $A(\tilde{Q}^\beta)$) modulo an ideal generated by linear combinations of paths from v_1 to v_4. When E_γ is even, the algebra A^{op} is isomorphic to the path algebra $A(\tilde{Q}^\gamma)$ modulo an ideal generated by linear combinations of paths from v_1 to v_4, and from v_1 to v_3.

Proof of Theorem 5.4. We give the proof of Theorem 5.4 in case E_α is even. The other cases can be handled similarly.

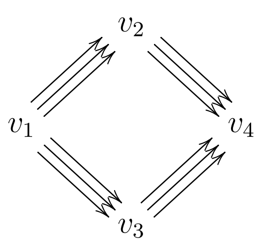

Figure 7.
$\tilde{Q}^\alpha$.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

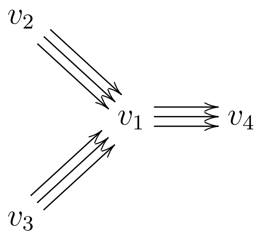

Figure 8.
$\tilde{Q}^{\beta}$.

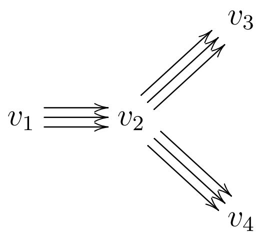

Figure 9.
$\tilde{Q}^{\gamma}$.

The objects $E_{\beta-2}[2]$, $E'_{\alpha}[1]$, $E''_{\alpha}[1]$ and E_{γ} in $\mathcal{A}[\phi]$ are mapped, via Φ, to simple $A(\tilde{Q}^{\alpha})$-modules with dimension vector $(\delta(v_1), \ldots, \delta(v_4))$ equal to $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, respectively. These simple modules generate the abelian category $\mathrm{mod}(A^{op})$ of finite- dimensional A^{op}-modules. We have $\Phi(\mathcal{A}[\phi]) \supset \mathrm{mod}(A^{op})$. Since both abelian categories $\mathcal{A}[\phi]$ and $\mathrm{mod}(A^{op})$ are hearts of a t-structure, we have $\Phi(\mathcal{A}[\phi]) = \mathrm{mod}(A^{op})$. So the $Z[\phi]$-semistable objects in $\mathcal{A}[\phi]$ correspond to $(Z[\phi] \circ \Phi^{-1})$-semistable objects in $\mathrm{mod}(A^{op})$.

Consider a complex $F^{\bullet} = [E^{m}_{\beta-2} \to E^{n}_{\alpha}]$, where m, n are those in (5.5), and E^{n}_{α} is in degree 0. Then $F^{\bullet}[1] \in \mathcal{A}[\phi]$ and $\Phi(F^{\bullet}[1])$ is a representation of the quiver $\tilde{Q}^{\alpha}$ with dimension vector $(\delta(v_1), \ldots, \delta(v_4))$ equal to $(m, n, n, 0)$, which can be regarded as a representation of the quiver Q^{α} with dimension vector (m, n, n).

Put

$$\zeta_{\alpha} := Z_{(s,t)}(E_{\alpha}), \qquad \zeta_{\beta} := Z_{(s,t)}(E_{\beta-2}[1]), \qquad \zeta_{\gamma} := Z_{(s,t)}(E_{\gamma}).$$

Then we have

$$Z[\phi](E'_{\alpha}[1]) = Z[\phi](E''_{\alpha}[1]) = -\tfrac{1}{2}e^{-i\pi\phi}\zeta_{\alpha}$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

$$Z[\phi](E_{\beta-2}[2]) = -e^{i\pi\phi}\zeta_\beta, \qquad Z[\phi](E_\gamma) = e^{-i\pi\phi}\zeta_\gamma.$$

The representation $\Phi(F^\bullet)$ of the quiver Q^α is $(Z[\phi]\circ\Phi^{-1})$-semistable if and only if for any subrepresentation $\{V_{v_i}\}_{1\leqslant i\leqslant 3}$ of $\Phi(F^\bullet)$, the following inequality holds

$$\operatorname{Im}\frac{-me^{-i\pi\phi}\zeta_\beta - ne^{-i\pi\phi}\zeta_\alpha}{-(\dim V_{v_1})e^{-i\pi\phi}\zeta_\beta - \frac{1}{2}(\dim V_{v_2} + \dim V_{v_3})e^{-i\pi\phi}\zeta_\alpha} \geqslant 0. \tag{5.17}$$

Using $\operatorname{Im}\zeta_\beta\bar{\zeta_\alpha} > 0$, one can see that the inequality (5.17) is equivalent to

$$m(\dim V_{v_2} + \dim V_{v_3}) - 2n\dim V_{v_1} \geqslant 0.$$

This shows that $(Z[\phi]\circ\Phi^{-1})$-semistability is equivalent to $(-2n, m, m)$-semistability of a representation of the quiver Q^α.

All in all, by associating $\Phi(F^\bullet[1])$ to $F^\bullet$, we have an isomorphism $M(\xi)\simeq N^\alpha$. This completes the proof of Theorem 5.4. ∎

REMARK 5.5. To $F^\bullet = [E^m_{\beta-2} \xrightarrow{f} E^n_\alpha]$, we associated $\Phi(F^\bullet[1])$, a representation of a quiver. Here is another way to associate the representation of the quiver. Assume, for example, that α is even, and $E_\alpha = E'_\alpha \oplus E''_\alpha$. We can regard f as an element of

$$\operatorname{Hom}(\mathbb{C}^m, \mathbb{C}^n) \otimes \big(\operatorname{Hom}(E_{\beta-2}, E'_\alpha) \oplus \operatorname{Hom}(E_{\beta-2}, E''_\alpha)\big).$$

We can see that there is a canonical isomorphism $\operatorname{Hom}(E_{\beta-2}, E_\alpha)^* \to \operatorname{Hom}(E_{\gamma.\beta}, E_\beta)$ compatible with the direct sum decompositions $E_\alpha = E'_\alpha \oplus E''_\alpha$ and $E_{\gamma.\beta} = E'_{\gamma.\beta} \oplus E''_{\gamma.\beta}$. Therefore, f gives a representation of the quiver Q^α with dimension vector (m, n, n). We can verify that this representation is isomorphic to $\Phi(F^\bullet[1])$.

5.3 Complements

Some arguments in this section remain valid even for moduli spaces of positive height. Fix $\gamma \in \mathfrak{E}$, and define α and β as at the beginning of Section 5.1 so that $\gamma = \alpha.\beta$. Let $\xi \in K(S)$ with symmetric c_1 and $\mathrm{r}(\xi) > 0$. Assume that

$$\gamma - x_\gamma < \bar{\mu}(\xi) \qquad \text{and} \qquad \chi(E_\gamma, \xi) = 0. \tag{5.18}$$

We can easily check the condition in Theorem 3.7(2), so ξ is a semistable class. If moreover $\bar{\mu}(\xi) \leqslant \gamma$, then the moduli space $M(\xi)$ is of height zero and isomorphic to one of N^{α}, N^{β}, N^{γ}. But even if $\gamma < \bar{\mu}(\xi)$, we can see that $M(\xi)$ is birational to one of N^{α}, N^{β}, N^{γ} as follows.

Under the condition (5.18), we have

$$m := -\tilde{\chi}(E_{\beta}, \xi) > 0 \qquad \text{and} \qquad n := -\tilde{\chi}(E_{\gamma.\beta}, \xi) > 0.$$

(This follows by verifying that the $\bar{\mu}$-coordinate of the intersection of the parabolas $\Delta = \mathrm{P}(\bar{\mu} - \gamma) - \Delta_{\gamma}$ and $\Delta = \mathrm{P}(\bar{\mu} - \beta) - \Delta_{\beta}$ (resp. $\Delta = \mathrm{P}(\bar{\mu} - \gamma.\beta) - \Delta_{\gamma.\beta}$) is smaller than $\gamma - x_{\gamma}$.) Put $\mathbf{H} := \mathrm{Hom}(E_{\beta-2}^{m}, E_{\alpha}^{n})$, and define $U \subset \mathbf{H}$ to be the open subset consisting of such $\varphi : E_{\beta-2}^{m} \to E_{\alpha}^{n}$ that φ is injective and Coker φ is torsion-free. By a standard dimension estimate, we can see that $U \neq \emptyset$. We can check that the family $\{F_{\varphi} := \mathrm{Coker}\,\varphi\}_{\varphi \in U}$ is a complete family such that $\mathrm{Ext}^2(F_{\varphi}, F_{\varphi}(-1,-1)) = 0$. If we put

$$U \supset U' := \{\varphi \in U \mid F_{\varphi} \text{ is stable}\},$$

then we can see that $U' \neq \emptyset$ (repeat the argument of the proof of Theorem 3.7). In the proof of Proposition 5.2, we only used the fact

- $\alpha < \bar{\mu}(F)$,
- F is a (semi)stable sheaf fitting in a short exact sequence (5.4).

So by the same proof, we obtain the following.

PROPOSITION 5.6. *If a point (s_0, t_0) with $\beta - 2 < s_0 < \alpha$ in the upper half (s,t)-plane lies outside the wall $W_{E_{\beta-2}, E_{\alpha}}$, then F_{φ} is (s_0, t_0)-stable for $\varphi \in U'$.*

Denote by $M(\xi)^{\diamond}$ the open subscheme of $M(\xi)$ parametrizing stable sheaves fitting in a short exact sequence (5.4). By the same proof of Theorem 5.4, we have the following.

PROPOSITION 5.7. *Let $\star \in \{\alpha, \beta, \gamma\}$. If the symmetric exceptional bundle $E_{\star}$ is even, then $M(\xi)^{\diamond}$ is isomorphic to an open subscheme of $N^{\star s}$, where $N^{\star s}$ is the subscheme of $N^{\star}$ consisting of θ-stable representations. In particular, $N^{\star s}$ is nonempty.*

5.4 Torsion-sheaves

We can also apply the same arguments in the preceding section to torsion-sheaves. Fix a K-class $\xi \in K(S)$ with symmetric c_1 such that

$$\text{(5.19)} \quad \mathrm{r}(\xi) = 0, \qquad \deg \xi := c_1(\xi) \cdot c_1(L) > 0 \qquad \text{and} \qquad \gamma := \frac{\chi(\xi)}{\deg \xi} \in \mathfrak{E}.$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Define α and β as at the beginning of Section 5.1. We have

$$m := -\tilde{\chi}(E_\beta, \xi) > 0 \qquad \text{and} \qquad n := -\tilde{\chi}(E_{\gamma.\beta}, \xi) > 0.$$

As before we put $\mathbf{H} := \mathrm{Hom}(E_{\beta-2}^m, E_\alpha^n)$, and define $U \subset \mathbf{H}$ to be the open subset consisting of such $\varphi : E_{\beta-2}^m \to E_\alpha^n$ that φ is injective. Then for any $\varphi \in U$, $F_\varphi := \mathrm{Coker}\,\varphi$ is a semistable pure 1-dimensional sheaf. Indeed, since $F := F_\varphi$ fits in a short exact sequence (5.4), we have $\mathrm{Ext}^i(E_\gamma, F) = 0$ for any i. If F has a subsheaf F' with $\chi(F')/\deg\, F' > \gamma$, then $\chi(E_\gamma, F') > 0$, thus $\mathrm{Hom}(E_\gamma, F') \neq 0$. This contradicts $\mathrm{Hom}(E_\gamma, F) = 0$. Put

$$U \supset U' := \{\varphi \in U \mid F_\varphi \text{ is stable}\}.$$

We claim that $U' \neq \emptyset$. If $F := F_\varphi$, $\varphi \in U$, is not stable, then F has a filtration

$$0 = F_0 \subset \cdots \subset F_l = F, \quad l \geqslant 2$$

such that $\chi(G_i)/\deg\, G_i = \gamma$ for any i, where $G_i = F_i/F_{i-1}$. We can see, by calculation, that $\dim \mathrm{Ext}^1_+(F, F) > 0$ for this filtration. This implies that $U' \neq \emptyset$ (see the argument in the proof of [DL, Theorem 4.10]). Adapting the proof of Proposition 5.2, we can obtain the following.

Proposition 5.8. *If a point (s_0, t_0) with $\beta - 2 < s_0 < \alpha$ in the upper half (s, t)-plane lies outside the wall $W_{E_{\beta-2}, E_\alpha}$, then F_φ is (s_0, t_0)-stable for $\varphi \in U'$.*

Denote by $M(\xi)^\diamond$ the moduli space of stable pure 1-dimensional sheaves with K-class ξ fitting in a short exact sequence (5.4). By arguing as in the proof of Theorem 5.4, we have the following.

Proposition 5.9. *Let $\star \in \{\alpha, \beta, \gamma\}$. If the symmetric exceptional bundle $E_\star$ is even, then $M(\xi)^\diamond$ is isomorphic to an open subscheme of $N^{\star s}$, where $N^{\star s}$ is the subscheme of $N^\star$ consisting of θ-stable representations. In particular, $N^{\star s}$ is nonempty.*

§6. Dimension Estimate

The following proposition is the quadric surface counterpart of [Le, Lemma 18.3.1].

Proposition 6.1. *Let $\xi \in K(S)$ be a semistable class with symmetric c_1 such that the height of the moduli space $M(\xi)$ is positive. Consider a complete family $\{F_t\}$ of torsion-free coherent sheaves on S with K-class ξ parametrized by a smooth variety T. Assume that $\mathrm{Ext}^2(F_t, F_t(-1,-1)) = 0$ for any $t \in T$.*

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

(1) *The set of points $t \in T$ such that F_t is not stable forms a closed subset of at least codimension* 2.
(2) *If* $\mathrm{r}(\xi) \geqslant 3$, *then the set of points $t \in T$ such that F_t is not a μ-stable bundle forms a closed subset of at least codimension* 2.

Only in the proof of the proposition, we use the terminology "multistable": a torsion-free sheaf F is said to be *multistable* if F is semistable and S-equivalent to G^a with G a stable sheaf.

Proof. (1) By a similar argument as in the proof of [Le, Corollary 15.4.4], we can see that the set of points $t \in T$ such that $\mu_{\max}(F_t) - \mu_{\min}(F_t) > 2$ forms a closed subset of codimension at least 2. So we may assume that $\mu_{\max}(F_t) - \mu_{\min}(F_t) \leqslant 2$ for any $t \in T$.

We first show that the set of points $t \in T$ such that F_t is not semistable is of codimension at least 2. To this end, it suffices to show that if $F = F_t$ is not semistable, then there exists a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = F \tag{6.1}$$

satisfying the following conditions (i), (ii) and (iii):

(i) $G_i := F_i/F_{i-1}$ is semistable, $p_{G_1} \succeq \cdots \succeq p_{G_l}$, $p_{G_1} \succ p_{G_l}$, and $\bar{\mu}(G_1) - \bar{\mu}(G_l) \leqslant 1$;
(ii) $\mathrm{Hom}(G_i, G_j) = 0$ for $i < j$;
(iii) $\sum_{i<j} \chi(G_i, G_j) < -1$.

In fact, if the filtration (6.1) satisfies (i), (ii) and (iii), then

$$\dim \mathrm{Ext}^1_+(F, F) = -\sum_{i<j} \chi(G_i, G_j) \geqslant 2.$$

Using [DL, Propositions 1.5 and 1.7], we see that the subset of sheaves having a filtration satisfying (i), (ii) and (iii) is of codimension at least 2.

In the following, we shall use repeatedly the fact that for the filtration (6.1) satisfying (i) and (ii), we have $\chi(G_i, G_j) \leqslant 0$ for $i < j$.

Step 1. We show that if $F = F_t$ has a filtration (6.1) satisfying the conditions (i), (ii) and the following condition (iv), then (iii) or (v) below holds.

(iv) G_1 and G_l are multistable.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

(v) $\chi(G_1, G_l) = -1$, and $\chi(G_i, G_j) = 0$ for $i < j$ with $(i, j) \neq (1, l)$. Moreover, G_i is not multistable for $i \neq 1, l$.

Suppose that the filtration (6.1) satisfies (i), (ii) and (iv), and the inequality

$$\sum_{i<j} \chi(G_i, G_j) \geqslant -1 \tag{6.2}$$

holds. Let us show that (v) holds.

Claim 6.1.1. If G_i is multistable, then $\Delta(G_i) \geqslant 1/2$. In particular, $\Delta(G_1) \geqslant 1/2$ and $\Delta(G_l) \geqslant 1/2$.

Proof of Claim 6.1.1. Suppose that G_i is multistable and $\Delta(G_i) < 1/2$. Then G_i is S-equivalent to E^a with E an exceptional bundle. If $\bar{\mu}(G_i) \geqslant \bar{\mu}(F)$, then we have

$$\begin{aligned}
0 > \chi(G_i, F) &= \sum_{j<i} \chi(G_i, G_j) + \chi(G_i, G_i) + \sum_{i<j} \chi(G_i, G_j) \\
&\geqslant \sum_{j<i} \chi(G_j, G_i) + \chi(G_i, G_i) + \sum_{i<j} \chi(G_i, G_j) \\
&\geqslant -1 + \chi(G_i, G_i) = -1 + a^2 \geqslant 0,
\end{aligned}$$

which is a contradiction. Here the first inequality follows from the assumption that the height of $M(\xi)$ is positive, and Proposition 3.6. If $\bar{\mu}(G_i) < \bar{\mu}(F)$, then by considering $\chi(F, G_i)$, we obtain a contradiction as well. $\square$

Claim 6.1.2. If $\chi(G_i, G_j) = 0$ for some $i < j$, then $\Delta(G_i) + \Delta(G_j) \leqslant 1$, where the equality holds if and only if $(\nu'(G_i), \nu''(G_i)) = (\nu'(G_j), \nu''(G_j))$.

Proof of Claim 6.1.2. By $\chi(G_i, G_j) = 0$, we have

$$\big(\nu'(G_j) - \nu'(G_i) + 1\big)\big(\nu''(G_j) - \nu''(G_i) + 1\big) = \Delta(G_i) + \Delta(G_j). \tag{6.3}$$

Since $-2 + \nu'(G_i) + \nu''(G_i) \leqslant \nu'(G_j) + \nu''(G_j) \leqslant \nu'(G_i) + \nu''(G_i)$, we see, by calculation, that the maximum of left-hand side of (6.3) is 1, and the maximum is attained only when $(\nu'(G_i), \nu''(G_i)) = (\nu'(G_j), \nu''(G_j))$. $\square$

We have $\chi(G_1, G_l) < 0$. Indeed, if $\chi(G_1, G_l) = 0$, then Claims 6.1.1 and 6.1.2 imply that $\Delta(G_1) = \Delta(G_2) = 1/2$ and $(\nu'(G_1), \nu''(G_1)) = (\nu'(G_l), \nu''(G_l))$. This shows that $p_{G_1} = p_{G_l}$, which contradicts (i).

By the inequality (6.2), we have $\chi(G_1, G_l) = -1$ and $\chi(G_i, G_j) = 0$ for $i < j$ with $(i, j) \neq (1, l)$. If G_i is multistable for $i \neq 1, l$, then by $\chi(G_1, G_i) = \chi(G_i, G_l) = 0$ and $\Delta(G_i) \geqslant 1/2$, as in the preceding paragraph, we obtain $p_{G_1} = p_{G_i} = p_{G_l}$, which contradicts (i).

Step 2. We show that if $F = F_t$ is not semistable, then either F has a filtration (6.1) satisfying (i), (ii) and (iii), or F has a filtration

$$0 = F_0 \subset F_1 \subset F_2 = F \tag{6.4}$$

such that $G_0 = F_1/F_0$ and $G_2 = F_2/F_1$ are stable, $p_{G_1} \succ p_{G_2}$ and $\chi(G_1, G_2) = -1$.

Let

$$0 = F^{(0)} \subset F^{(1)} \subset \cdots \subset F^{(k)} = F \tag{6.5}$$

be the Harder–Narasimhan filtration of F. We have $k \geqslant 2$ because F is not semistable. If $F^{(i)}/F^{(i-1)}$, $1 \leqslant i \leqslant k$, is not multistable, then insert a filter $F^{(i-1)} \subset \bar{F}^{(i)} \subset F^{(i)}$ such that $p_{\bar{F}^{(i)}/F^{(i-1)}} = p_{F^{(i)}/\bar{F}^{(i)}}$, $\mathrm{Hom}(\bar{F}^{(i)}/F^{(i-1)}, F^{(i)}/\bar{F}^{(i)}) = 0$, and moreover for $1 \leqslant i \leqslant k-1$ (resp. for $i = k$) $\bar{F}^{(i)}/F^{(i-1)}$ (resp. $F^{(k)}/\bar{F}^{(k)}$) is multistable (cf. [KO, Proposition 4.4]). The resulting filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = F \tag{6.6}$$

satisfies (i), (ii) and (iv).

If $k \geqslant 3$, then at least 3 of the graded sheaves of the resulting filtration are multistable. So the filtration satisfies (iii) by Step 1.

Next we consider the case $k = 2$, and at least one of $F^{(1)}/F^{(0)}$, $F^{(2)}/F^{(1)}$ is not multistable. We treat the case where $F^{(1)}/F^{(0)}$ is not multistable. (The case where $F^{(2)}/F^{(1)}$ is not multistable can be handled similarly.) The filtration (6.6) in this case is either

$$\begin{aligned} &0 = F^{(0)} \subset \bar{F}^{(1)} \subset F^{(1)} \subset F^{(2)} = F \\ \text{or}\quad &0 = F^{(0)} \subset \bar{F}^{(1)} \subset F^{(1)} \subset \bar{F}^{(2)} \subset F^{(2)} = F. \end{aligned} \tag{6.7}$$

If the filtration (6.7) satisfies (iii), then we are done. If not, then the filtration satisfies (v) by Step 1. Then $\bar{F}^{(1)}$ must be stable because $\chi(G_1, G_l) = -1$. Since $F^{(1)}/\bar{F}^{(1)}$ is not multistable, we can find a subsheaf $\bar{F}^{(1)} \subset D \subset F^{(1)}$ such that $p_{F^{(1)}/D} = p_{D/\bar{F}^{(1)}}$, $\mathrm{Hom}(D/\bar{F}^{(1)}, F^{(1)}/D) = 0$,

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

and $F^{(1)}/D$ is multistable. If $\mathrm{Hom}(\bar{F}^{(1)}, F^{(1)}/D) \neq 0$, then $F^{(1)}/D$ is S-equivalent to a direct sum of some $\bar{F}^{(1)}$'s. Then we have $\chi(F^{(1)}/D, G_l) < 0$ since $\chi(G_1, G_l) = -1$. On the other hand, we have $\chi(F^{(1)}/\bar{F}^{(1)}, G_l) = 0$, so we have $\chi(F^{(1)}/D, G_l) = 0$. This is a contradiction. Therefore, we have $\mathrm{Hom}(\bar{F}^{(1)}, F^{(1)}/D) = 0$. Then the filtration obtained from (6.7) by inserting D as a filter satisfies (i), (ii) and (iv), and at least 3 of the graded sheaves are multistable. So it satisfies (iii) by Step 1.

Finally we consider the case where $k = 2$, and both $F^{(1)}/F^{(0)}$ and $F^{(2)}/F^{(1)}$ are multistable. The filtration (6.5) satisfies (i), (ii) and (iv). If (6.5) satisfies (iii), then we are done. If not, it satisfies (v) by Step 1. Since $\chi(F^{(1)}, F^{(2)}/F^{(1)}) = -1$, both $F^{(1)}$ and $F^{(2)}/F^{(1)}$ must be stable. So we obtain a desired filtration.

Step 3. To conclude that $F = F_t$ that is not semistable has a filtration (6.1) satisfying (i), (ii) and (iii), we have only to show that F does not have a filtration (6.4) such that $G_0 = F_1/F_0$ and $G_2 = F_2/F_1$ are stable, $p_{G_1} \succ p_{G_2}$ and $\chi(G_1, G_2) = -1$.

Suppose that such a filtration exists. For $i = 1, 2$, put

$$r_i = \mathrm{r}(G_i), \qquad \Delta_i = \Delta(G_i), \qquad \bar{\mu}_i = \bar{\mu}(G_i),$$

and define δ_i by

$$\big(\nu'(G_i), \nu''(G_i)\big) = (\bar{\mu}_i + \delta_i, \bar{\mu}_i - \delta_i).$$

From $\chi(G_1, G_2) = -1$, we have

$$(1 + \bar{\mu}_2 - \bar{\mu}_1)^2 - (\delta_2 - \delta_1)^2 = \Delta_1 + \Delta_2 - \frac{1}{r_1 r_2}. \tag{6.8}$$

Define a nonnegative integer k by

$$2\bar{\mu}_1 = 2\bar{\mu}_2 + \frac{k}{r_1 r_2}.$$

Since $\bar{\mu}_1 - \bar{\mu}_2 \leqslant 1$, we have $k \leqslant 2r_1 r_2$. By (6.8), we have

$$\left(1 - \frac{k}{2r_1 r_2}\right)^2 \geqslant \Delta_1 + \Delta_2 - \frac{1}{r_1 r_2} \geqslant 1 - \frac{1}{r_1 r_2},$$

where the last inequality follows from Claim 6.1.1. From this, we have

$$1 \geqslant k\left(1 - \frac{k}{4r_1 r_2}\right) \geqslant \frac{k}{2}. \tag{6.9}$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Thus $k \leqslant 2$. If $k = 2$, then the inequalities in (6.9) are all equality. Thus $r_1 = r_2 = 1$ and $\Delta_1 = \Delta_2 = 1/2$. This is absurd because the discriminant of a rank one sheaf is an integer.

Before we consider the case $k = 0, 1$, we note that we have

$$r_1\delta_1 + r_2\delta_2 = 0 \tag{6.10}$$

because $c_1(\xi)$ is symmetric.

Consider the case $k = 0$. We have $\bar{\mu}_1 = \bar{\mu}_2$. From (6.8), we have

$$\begin{aligned}\frac{1}{r_1 r_2} &= \Delta_1 + \Delta_2 - 1 + (\delta_1 - \delta_2)^2 \\ &\geqslant (\delta_1 - \delta_2)^2 = \left(\frac{r_1 + r_2}{r_2}\right)^2 \delta_1^2. \end{aligned} \tag{6.11}$$

If $\delta_1 \neq 0$, then $|\delta_1| \geqslant 1/(2r_1)$ because $2r_1\delta_1$ is an integer. We have $0 \geqslant (r_1 - r_2)^2$ from (6.11). Then $r_1 = r_2$, and the inequality in (6.11) is an equality, thus $\Delta_1 = \Delta_2 = 1/2$. Then we have $\Delta_1 + \delta_1^2 = \Delta_2 + \delta_2^2$, which implies $p_{G_1} = p_{G_2}$, a contradiction.

If $\delta_1 = 0$, then both $c_1(G_1)$ and $c_1(G_2)$ are symmetric. Thus $\Delta_i > 1/2$. More precisely, we have

$$\Delta_i \geqslant \frac{1}{2} + \frac{1}{2r_i^2}.$$

From (6.8), we have

$$\frac{1}{r_1 r_2} \geqslant \frac{1}{2r_1^2} + \frac{1}{2r_2^2},$$

thus $0 \geqslant (r_1 - r_2)^2$. Then we have $r_1 = r_2$ and $\Delta_1 = \Delta_2$, which implies a contradiction as in the preceding paragraph.

Consider the case $k = 1$. From (6.8), we have

$$\left(1 - \frac{1}{2r_1 r_2}\right)^2 - (\delta_2 - \delta_1)^2 = \Delta_1 + \Delta_2 - \frac{1}{r_1 r_2}.$$

From this, we have

$$\begin{aligned} 1 - (\delta_1 - \delta_2)^2 &= \Delta_1 + \Delta_2 - \frac{1}{4r_1^2 r_2^2} \\ &\geqslant 1 - \frac{1}{4r_1^2 r_2^2}. \end{aligned} \tag{6.12}$$

Therefore, $|\delta_1 - \delta_2| \leqslant 1/(2r_1 r_2)$.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

If $\delta_1 = \delta_2$, then $\delta_1 = \delta_2 = 0$ by (6.10). Since both $c_1(G_1)$ and $c_1(G_2)$ are symmetric, we have

$$\Delta_i \geqslant \frac{1}{2} + \frac{1}{2r_i^2}. \tag{6.13}$$

From (6.12) and (6.13), we have $1 \geqslant 2r_1^2 + 2r_2^2$, which is absurd.

If $|\delta_1 - \delta_2| = 1/(2r_1r_2)$, then we have $2r_2\delta_2 = \pm 1/(r_1 + r_2)$ by (6.10). This is a contradiction because $2r_2\delta_2$ is an integer.

From Steps 1 to 3, we can conclude that the set of points $t \in T$ such that F_t is not semistable is of codimension at least 2.

Step 4. We claim that the set of points $t \in T$ such that F_t is not stable is of codimension at least 2.

If $F = F_t$ is semistable, but not stable, then there is a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = F, \quad l \geqslant 2 \tag{6.14}$$

such that each $G_i := F_i/F_{i-1}$ is stable, and its reduced Hilbert polynomial is equal to p_F. Put $r_i = \mathrm{r}(G_i)$, $\Delta_i := \Delta(G_i)$ and define δ_i by $(\nu'(G_i), \nu''(G_i)) = (\bar{\mu} + \delta_i, \bar{\mu} - \delta_i)$, where $\bar{\mu} := \bar{\mu}(\xi)$. Put $\Delta := \Delta(\xi)$. To prove the claim in Step 4, we shall show that $\dim \mathrm{Ext}^1_+(F, F) \geqslant 2$ for this filtration.

We have

$$\sum_i (-1)^i \dim \mathrm{Ext}^i_+(F, F) = \sum_{i<j} \chi(G_i, G_j).$$

By $\mathrm{Ext}^2_+(F, F) = 0$, we have

$$\begin{aligned} \dim \mathrm{Ext}^1_+(F, F) \geqslant -\sum_{i<j} \chi(G_i, G_j) &= \sum_{i<j} r_ir_j(-1 - 2\delta_i\delta_j + 2\Delta) \\ &= \sum_{i<j} r_ir_j(2\Delta - 1) + \sum_k r_k^2\delta_k^2, \end{aligned} \tag{6.15}$$

where we used $\sum_i r_i\delta_i = 0$. This implies $\dim \mathrm{Ext}^1_+(F, F) > 0$. Assuming the value (6.15) is equal to 1, we shall obtain a contradiction.

Since each $2r_i\delta_i$ is an integer, we have $r_ir_j(2\Delta - 1) \in \frac{1}{2}\mathbb{Z}$. So we must have $l = 2$. There are two cases:

$$(r_1r_2(2\Delta - 1), r_1\delta_1, r_2\delta_2) = (1, 0, 0) \qquad \text{or} \qquad \left(\frac{1}{2}, \pm\frac{1}{2}, \mp\frac{1}{2}\right).$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

Case $(r_1r_2(2\Delta-1), r_1\delta_1, r_2\delta_2) = (1, 0, 0)$. In this case, both G_1 and G_2 have symmetric c_1 and

$$\Delta = \Delta_1 = \Delta_2 = \frac{1}{2}\left(1 + \frac{1}{r_1r_2}\right).$$

Since $\chi(G_1, G_2) = -1$, we have $r_1 = r_2$, so we have $\chi(G_1, G_1) = -1$. Then $\dim M(r_1, \bar{\mu}, \Delta) = 2$. This does not occur by Lemma 6.2.

Case $(r_1r_2(2\Delta-1), r_1\delta_1, r_2\delta_2) = \left(\frac{1}{2}, \pm\frac{1}{2}, \mp\frac{1}{2}\right)$. We have

$$\Delta = \frac{1}{2}\left(1 + \frac{1}{2r_1r_2}\right) = \Delta_i + \frac{1}{4r_i^2}. \tag{6.16}$$

We first consider the case $r_1 \neq r_2$, say $r_1 < r_2$. Then $\Delta_1 < 1/2$, so G_1 is rigid and we have $\Delta_1 = (1 - 1/{r_1}^2)/2$. Combining this and (6.16), we have $r_1 + r_2 = 0$, which is absurd. Finally consider the case $r_1 = r_2 = r/2$. Then $\Delta_1 = \Delta_2 = 1/2$. We have $\chi(G_i, G_i) = 0$. Then

$$\chi(F, F) = \chi(G_1, G_2) + \chi(G_2, G_1) = -2.$$

We have $\dim M(\xi) = 3$. By Lemma 6.2, $\bar{\mu} \in \mathfrak{E}$. Since the height of $M(\xi)$ is positive, we have

$$\Delta > \delta(\bar{\mu}) = \frac{1}{2}\left(1 + \frac{e}{r_{\bar{\mu}}^2}\right), \tag{6.17}$$

where e is 1 or 2, depending on whether $r_{\bar{\mu}}$ is odd or even. From (6.16) and (6.17), we have

$$r_{\bar{\mu}} > r\sqrt{e/2}. \tag{6.18}$$

Since $r\bar{\mu}$ is an integer, and since $r_{\bar{\mu}}$ is the denominator of the irreducible fraction expression of $\bar{\mu}$, we have $r \geqslant r_{\bar{\mu}}$. Moreover, when $r_{\bar{\mu}}$ is odd, $r \geqslant 2r_{\bar{\mu}}$ since r is even. This contradicts (6.18).

(2) By (1), we may assume that the complete family $\{F_t\}$ consists of stable sheaves. Since $\mathrm{r}(\xi) \geqslant 3$, the set of points $t \in T$ with F_t not locally free has codimension at least 2 (see [Le, Section 17.1]). Thus we may assume that F_t is locally free for all $t \in T$.

If $F = F_t$ is not μ-stable, then there exists a surjection $F \twoheadrightarrow G$ such that G is a torsion-free sheaf such that $\mu(F) = \mu(G)$ and $\chi(F)/\mathrm{r}(F) < \chi(G)/\mathrm{r}(G)$. Consider the dual map $G^* \hookrightarrow F^*$. We have $\mu(F^*) = \mu(G^*)$ and $\chi(F^*)/\mathrm{r}(F) < \chi(G^*)/\mathrm{r}(G)$. So F^* is not stable. But applying (1) to the family $\{F_t^*\}$ of dual bundles, we see that the subset of $t \in T$ such that F_t^* is not stable has codimension at least 2. This proves (2). □

LEMMA 6.2. *Let $\xi \in K(S)$ be a semistable class with symmetric c_1 such that the height of the moduli space $M(\xi)$ is positive. Then* $\dim M(\xi) \geqslant 3$. *If moreover $\bar{\mu}(\xi) \notin \mathfrak{E}$, then* $\dim M(\xi) \geqslant 4$.

Proof. We imitate the proof of [D, Proposition 33].

Put $(r, \bar{\mu}, \Delta) := (\mathrm{r}(\xi), \bar{\mu}(\xi), \Delta(\xi))$ and $d := \dim M(\xi) > 0$. We have

$$1 - d = \chi(\xi, \xi) = r^2(1 - 2\Delta),$$

thus

$$\Delta = \frac{1}{2}\left(1 + \frac{d-1}{r^2}\right). \tag{6.19}$$

Take $\alpha \in \mathfrak{E}$ such that $\bar{\mu} \in I_\alpha$. We may assume that $\alpha - x_\alpha < \bar{\mu} \leqslant \alpha$. (If $\alpha < \bar{\mu} < \alpha + x_\alpha$, then consider the dual of sheaves.)

Claim. If $d \leqslant 2$, then we have $\bar{\mu} \neq \alpha$.

Proof of the Claim. If $\bar{\mu} = \alpha$, then

$$\Delta > 1 - \Delta_\alpha = \frac{1}{2}\left(1 + \frac{e}{r_\alpha^2}\right), \tag{6.20}$$

where $e = 1$ or 2 depending on whether r_α is odd or even. By (6.19) and (6.20), we have $\sqrt{d-1}\, r_\alpha > r$, thus $r_\alpha > r$. This is absurd because r_α is the denominator of the irreducible fraction expression of α. This is the end of the proof of the claim. □

To prove the lemma, we claim that if $\alpha - x_\alpha < \bar{\mu} < \alpha$, then $d \geqslant 4$. Indeed, when $\alpha - x_\alpha < \bar{\mu} < \alpha$, we have $x_\alpha > \frac{1}{r r_\alpha}$. Denoting by h the height of the moduli space, we have

$$\begin{aligned} 0 < h &= r r_\alpha \left(\Delta + \Delta_\alpha - \mathrm{P}(\bar{\mu} - \alpha)\right) < r r_\alpha \left(\Delta - \frac{1}{2}\right) \\ &= \frac{(d-1) r_\alpha}{2r} \leqslant \frac{(d-1) r_\alpha^2 x_\alpha}{2} \\ &= \frac{(d-1)e}{4\left(1 + \sqrt{1 - \frac{e}{2r_\alpha^2}}\right)} \leqslant \frac{(d-1)e}{4}, \end{aligned}$$

where the inequality in the second line follows from $\mathrm{P}(\bar{\mu} - \alpha) - \Delta_\alpha > 1/2$. We have

$$d > \frac{4h}{e} + 1 \geqslant 3.$$

□

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

§7. Properties of the set $\mathfrak{E}$

The goal of this section is Theorem 7.4, a counterpart of [CHW, Theorem 4.16].

7.1 Notation of continued fraction expansion

We use the same notation as in [H, Section 3], and follow the presentation there.

For real numbers $a_0, \dots, a_k$, we define the number $[a_0; a_1, \dots, a_k]$ by

$$[a_0; a_1, \dots, a_k] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}},$$

when it makes sense.

Any rational number $0 \leqslant \alpha < 1$ has a unique continued fraction expansion $\alpha = [0; a_1, \dots, a_k]$ where a_i are positive integers and k is even. This is called the even length continued fraction expansion of α. If p_n and q_n are the numerator and denominator of $[0; a_1, \dots, a_n]$, the nth convergent of α, then we have the relation

$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.1}$$

From this, it follows that $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$. Moreover, taking the transpose of the above equation, we obtain the following fact [H, Lemma 3.1].

LEMMA 7.1. *A continued fraction expansion $[0; a_1, \dots, a_k]$ is palindromic, that is, $a_i = a_{k+1-i}$, if and only if $p_k = q_{k-1}$.*

By the same reasoning, we have the following.

LEMMA 7.2. *Assume that for continued fraction expansions $\alpha = [0; a_1, \dots, a_k]$ and $\alpha' = [0; a'_1, \dots, a'_k]$, the inequalities $a_i = a'_{k+1-i}$ hold. Let p_n/q_n and p'_n/q'_n be the nth convergents of α and α', respectively. Then we have $p_n = q'_{n-1}$.*

7.2 Continued fraction expansion of symmetric exceptional slopes

In the case of the projective plane, the even length continued fraction expansion of an exceptional slope $0 \leqslant \alpha < 1$ is palindromic (cf. [H, Theorem 3.2]). In our quadric surface case, the behavior of the even length continued fraction expansion of a symmetric exceptional slope $0 \leqslant \alpha < 1/2$ depends on the parity of α.

THEOREM 7.3. *Take $\gamma \in \mathfrak{E}$ with $0 \leqslant \gamma < 1/2$. Let $[0; c_1, \ldots, c_n]$ and $[0; c'_1, \ldots, c'_{n'}]$ be the even length continued fraction expansions of γ and 2γ, respectively. (If $\gamma = 0$, we understand $n = n' = 0$.)*

(1) *We have $c_1 = 2$ or 3, and $c'_1 = 1$.*
(2) *If the symmetric exceptional slope γ is even, then $n' = n + 2$, and the continued fraction expansions are palindromic.*
(3) *If the symmetric exceptional slope γ is odd, then $n = n'$ and $c_i = c'_{n+1-i}$.*

Proof. We imitate the proof of [H, Theorem 3.2].

If $\gamma = 0$, then the theorem holds obviously.

Consider $\gamma \in \mathfrak{E}$ with $0 < \gamma < 1/2$. It is expressed as

$$\gamma = \epsilon\,(p/2^q) \tag{7.2}$$

with p odd and $q \geqslant 2$. We proceed by induction on q. But we first treat the case $\gamma = \epsilon\left(\frac{1}{2} - \frac{1}{2^q}\right)$ separately. This γ is an odd symmetric exceptional slope. Since

$$\epsilon\left(\frac{1}{2} - \frac{1}{2^{q+1}}\right) = \gamma \cdot \frac{1}{2},$$

we have, by Lemma 2.6(2),

$$\epsilon\left(\frac{1}{2} - \frac{1}{2^{q+1}}\right) = \frac{1}{2} - \frac{1}{6 + 4\gamma}.$$

From this, by an easy calculation, we find

$$\epsilon\left(\frac{1}{2} - \frac{1}{2^{q+1}}\right) = [0; 2, 1 + \gamma], \qquad 2\epsilon\left(\frac{1}{2} - \frac{1}{2^{q+1}}\right) = [0; 1, 2 + 2\gamma].$$

Since $\epsilon(0) = [0; \emptyset]$, we have

$$\epsilon\left(\frac{1}{2} - \frac{1}{2^q}\right) = [0; (2, 1)^{q-1}], \qquad 2\epsilon\left(\frac{1}{2} - \frac{1}{2^q}\right) = [0; (1, 2)^{q-1}]$$

for $q \geqslant 2$. Thus the theorem holds in this case.

Now let us consider $0 < \gamma < 1/2$ expressed as in (7.2). The case $q = 2$ is covered by the above consideration, so we let $q \geqslant 3$. Let

$$\alpha = \epsilon\left(\frac{p-1}{2^q}\right), \qquad \beta = \epsilon\left(\frac{p+1}{2^q}\right)$$

so that $\gamma = \alpha.\beta$. We put the even length continued fraction expansions as follows:

$$\begin{aligned} \alpha &= [0; a_1, \ldots, a_m], & 2\alpha &= [0; a'_1, \ldots, a'_{m'}], \\ \beta &= [0; b_1, \ldots, b_l], & 2\beta &= [0; b'_1, \ldots, b'_{l'}]. \end{aligned}$$

Claim 7.3.1. The even length continued fraction expansions of γ and 2γ are given as follows.

(1) If α is even, then

$$\gamma = [0; a_1, \ldots, a_m, 1, 1, b_1 - 1, b_2, \ldots, b_l] \tag{7.3}$$

$$2\gamma = [0; a'_1, \ldots, a'_{m'}, 3, b'_2 + 1, b'_3, \ldots, b'_{l'}]. \tag{7.4}$$

(2) If α is odd, then

$$\gamma = [0; a_1, \ldots, a_m, 3, b'_2 + 1, b'_3, \ldots, b'_{l'}] \tag{7.5}$$

$$2\gamma = [0; a'_1, \ldots, a'_{m'}, 1, 1, , b_1 - 1, b_2, \ldots, b_l]. \tag{7.6}$$

(3) If β is even, then

$$\gamma = [0; b_1, \ldots, b_{l-1}, b_l - 1, 1, 1, a_1, \ldots, a_m] \tag{7.7}$$

$$2\gamma = [0; b'_1, \ldots, b'_{l'-2}, b'_{l'-1} + 1, 3, a'_1, \ldots, a'_{m'}]. \tag{7.8}$$

(4) If β is odd, then

$$\gamma = [0; b_1, \ldots, b_{l-2}, b_{l-1} + 1, 3, a'_1, \ldots, a'_{m'}] \tag{7.9}$$

$$2\gamma = [0; b'_1, \ldots, b'_{l'-1}, b'_{l'} - 1, 1, 1, a_1, \ldots, a_m]. \tag{7.10}$$

We first see that the claim implies the theorem. If γ is even, then both α and β are odd. Comparing the two expressions (7.5) and (7.9) of the continued fraction expansion of γ, we infer that the expansion is palindromic, using the induction hypothesis. For the expansion of 2γ, we can argue similarly. If γ is odd, then one of α, β is odd, and the other

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

is even. Comparing (7.3), (7.4) and (7.9), (7.10), or (7.5), (7.6) and (7.7), (7.8), we can see that the theorem holds in this case as well.

It remains to prove the claim. We give a proof for the equalities (7.4) and (7.9); the other equalities can be proved similarly.

Assume that α is even. Let p'_k/q'_k be the kth convergent of 2α, that is, $p'_k/q'_k = [0; a'_1, \ldots, a'_k]$. We have $q'_{m'} = r_\alpha/2$ and $p'_{m'} = \alpha r_\alpha$. Since the continued fraction expansion of 2α is palindromic, we have $q'_{m'-1} = p'_{m'}$ by Lemma 7.1. By $q'_{m'}p'_{m'-1} - p'_{m'}q'_{m'-1} = 1$, we have $p'_{m'-1} = 2\alpha^2 r_\alpha + 2/r_\alpha$. Using by Lemma 2.6(2), we have

$$
\begin{aligned}
2\gamma = 2\alpha + \frac{2}{r_\alpha^2(2+\alpha-\beta)} &= \frac{(4-2\beta)\alpha r_\alpha + 2\alpha^2 r_\alpha + 2/r_\alpha}{(4-2\beta)r_\alpha/2 + \alpha r_\alpha} \\
&= \frac{(4-2\beta)p'_{m'} + p'_{m'-1}}{(4-2\beta)q'_{m'} + q'_{m'-1}} \\
&= [0; a'_1, \ldots, a'_{m'}, 4-2\beta]. \qquad (7.11)
\end{aligned}
$$

Using $4-2\beta = [3; 1, -1, 2\beta]$, the continued fraction expansion (7.11) can be rewritten as (7.4), using $b'_1 = 1$.

Finally assume that β is odd. Let p_k/q_k and p'_k/q'_k be the kth convergents of β and 2β, respectively. We have $q_l = r_\beta$ and $p_l = \beta r_\beta$. Since $l = l'$ and $b_i = b'_{l+1-i}$, we have $q_{l-1} = p'_l = 2\beta r_\beta$ by Lemma 7.2. So we have $p_{l-1} = 2\beta^2 r_\beta + 1/r_\beta$.

$$
\begin{aligned}
\gamma = \beta - \frac{1}{2r_\beta^2(2+\alpha-\beta)} &= \frac{-(4+2\alpha)\beta r_\beta + 2\beta^2 r_\beta + 1/r_\beta}{-(4+2\alpha)r_\beta + 2\beta r_\beta} \\
&= \frac{-(4+2\alpha)p_l + p_{l-1}}{-(4+2\alpha)q_l + q_{l-1}} \\
&= [0; b_1, \ldots, b_l, -(4+2\alpha)] \\
&= [0; b_1, \ldots, b_{l-1}, 1, -(4+2\alpha)]. \qquad (7.12)
\end{aligned}
$$

Using $[0; 1; -(4+2\alpha)] = [1; 3+2\alpha]$, we can rewrite (7.12) as (7.9). ∎

Put $C := \mathbb{R} \setminus \bigcup_{\alpha\in\mathfrak{E}} I_\alpha$. Any $x \in C$ is irrational.

THEOREM 7.4. *If $x \in C$ is not an endpoint of an interval I_α, then it is not a quadratic irrational number.*

To prove the theorem, we follow the argument in [CHW, Section 4]. In our case, the argument becomes messier due to the parity of symmetric exceptional slopes.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

To show that $x \in C$ in the theorem is not a quadratic irrational number, we use the following fact : if $[a_0; a_1, a_2, \dots]$ is a continued fraction expansion of an irrational number, then it is a quadratic irrational number if and only if the continued fraction expansion is eventually periodic, that is, there exist positive integers p and c such that $a_i = a_{i+p}$ for all $i > c$. So, we will study the continued fraction expansion of x.

We need some preparation.

Put $C' = C \cap (0, 1/2)$ and

$$C_n = (0, 1/2) \Big\backslash \bigcup_{\alpha \in \mathfrak{E},\ \mathrm{ord}\ \alpha \leqslant n} I_\alpha.$$

Here the order of a symmetric exceptional slope $\alpha = \epsilon(p/2^q)$, where p is odd if $q > 0$, is defined by ord $\alpha = q$.

To each $x \in C'$, we associate an infinite sequence σ_x of letters L and R as follows. We have $C_2 = [x_0, 1/3 - x_{1/3}] \sqcup [1/3 + x_{1/3}, 1/2 - x_{1/2}]$, and x lies in one of the two intervals. If it lies in the left (resp. right) interval, then the first term of σ_x is L (resp. R). If we denote by $I_{2,x}$ the interval of C_2 containing x, then $I_{2,x} \cap C_3$ consists of two disjoint intervals, and x lies in one of them. If it lies in the left (resp. right) interval, then the second term of σ_x is L (resp. R). We can proceed inductively. More precisely, if we denote by $I_{n,x}$ the interval of C_n containing x, then $I_{n,x} \cap C_{n+1}$ consists of two disjoint intervals, and x lies in one of them. If it lies in the left (resp. right) interval, then the nth term of σ_x is L (resp. R).

The correspondence $x \to \sigma_x$ gives a bijection between the set C' and the set $\{L, R\}^{\mathbb{N}}$ of infinite sequences of letters L and R. The following lemma is clear from the construction of σ_x.

LEMMA 7.5. *For $x \in C'$, x is an endpoint of an interval I_α, $\alpha \in \mathfrak{E}$, if and only if the infinite sequence σ_x is eventually constant, that is, there exists an $N > 0$ such that all letters in σ_x are the same after the Nth term.*

Consider a symmetric exceptional slope $\gamma \in \mathfrak{E}$ expressed as $\gamma = \epsilon(p/2^q)$ where $q > 0$ and p is odd. Let $\alpha = \epsilon((p-1)/2^q)$ and $\beta = \epsilon((p+1)/2^q)$ so that $\gamma = \alpha.\beta$. We define the symmetric exceptional slopes $\gamma \cdot L$ and $\gamma \cdot R$ by $\gamma \cdot L = \alpha.\gamma$ and $\gamma \cdot R = \gamma.\beta$. If $\sigma = (S_1, \dots, S_n)$ is a finite sequence of letters L and R, then we define the symmetric exceptional slope $\gamma \cdot \sigma$ by $(\dots((\gamma \cdot S_1) \cdot S_2) \dots) \cdot S_n$.

Let $\mathfrak{E}'$ be the set of symmetric exceptional slopes α with $0 < \alpha < 1/2$. If we note $1/3 = \epsilon(1/4)$, the following is clear.

LEMMA 7.6. *Denote by* $\{L, R\}^*$ *the set of finite sequences of letters* L *and* R *(including the empty sequence). The map* $\{L, R\}^* \to \mathfrak{E}'$ *given by* $\sigma \to \frac{1}{3} \cdot \sigma$ *is a bijection.*

Continued fractions Take $\alpha \in \mathfrak{E}$, and let β be $\alpha \cdot L$ or $\alpha \cdot R$. Let $[0; a_1, \ldots, a_m]$ and $[0; b_1, \ldots, b_n]$ be the even length continued fraction expressions of α and β respectively. Then from Claim 7.3.1, we see that $a_i = b_i$ for $1 \leqslant i \leqslant m-2$.

Take $x \in C'$ and consider the corresponding infinite sequence σ_x. Let $\sigma_x^{\leqslant n}$ be the finite sequence of initial n terms. Put $\phi_x^n = \frac{1}{3} \cdot \sigma_x^{\leqslant n}$. We have $\lim_{n\to\infty} \phi_x^n = x$. From the observation above, the following lemma is clear.

LEMMA 7.7. *If* $[0; a_1, \ldots, a_m]$ *is the even length continued fraction expansion of* ϕ_x^n, *and* $[0; c_1, c_2, \ldots]$ *is the continued fraction expansion of* x, *then* $a_i = c_i$ *for* $i \leqslant m-2$.

Now assume that $x \in C'$ *is not an endpoint of* I_α, $\alpha \in \mathfrak{E}$.

By Lemma 7.5, σ_x is not eventually constant. Then we can easily see that there are infinitely many even symmetric exceptional slopes in the sequence $\{\phi_x^n\}_{n\geqslant 0}$. We say that an even symmetric exceptional slope ϕ_x^n is of Type A if $\sigma_x^{n<}$ starts as $RL^kR\ldots$ for some $k > 0$; of type B if $\sigma_x^{n<}$ starts as $LR^kL\ldots$ for some $k > 0$; and of Type C if $\sigma_x^{n<}$ starts as $RRLL\ldots$, where $\sigma_x^{n<}$ is the infinite sequence obtained from σ_x by deleting the initial n terms. We can see that for at least one type of A, B or C, an infinite number of even ϕ_x^n of such type appear.

In the next lemma, we use the following notation. For a sequence $\mathbf{a} = (a_1, \ldots, a_m)$ of integers, we let

$$
\begin{aligned}
{}^{+}\mathbf{a} &= (3, a_2+1, a_3, \ldots, a_m), & {}^{-}\mathbf{a} &= (a_1-1, a_2, \ldots, a_m),\\
\mathbf{a}^{+} &= (a_1, \ldots, a_{m-2}, a_{m-1}+1, 3), & \mathbf{a}^{-} &= (a_1, \ldots, a_{m-1}, a_m-1).
\end{aligned}
$$

We define ${}^{+}\mathbf{a}^{+}$, ${}^{+}\mathbf{a}^{-}$, and so forth obviously.

LEMMA 7.8. *Assume that* ϕ_x^n *is an even symmetric exceptional slope. Put* $\gamma = \phi_x^n$. *Express* $\gamma = \epsilon(p/2^q)$ *with* p *odd, and let* $\alpha = \epsilon((p-1)/2^q)$ *and* $\beta = \epsilon((p+1)/2^q)$. *Let* $[0; \mathbf{c}]$, $[0; \mathbf{a}]$ *and* $[0; \mathbf{b}]$ *be the even length continued fraction expansions of* γ, α *and* β, *respectively, and let* $[0; \mathbf{c}']$, $[0; \mathbf{a}']$ *and* $[0; \mathbf{b}']$ *be the even length continued fraction expansions of* 2γ, 2α *and* 2β, *respectively.*

(1) *If γ is of Type A, and $\sigma^{n<}$ starts as $RL^kR\ldots$, then the even length continued fraction expansion of $\gamma \cdot RL^kR(=\phi_x^{n+k+2})$ is*

$$[0; \mathbf{c}, 1, 1,^- \mathbf{b}^+, (\mathbf{c}'^+)^{k-1}, \mathbf{c}',^+ \mathbf{c}',^+ \mathbf{b}'^-, (1, 1, \mathbf{c}^-)^{k-2}, 1, 1, \mathbf{c}], \tag{7.13}$$

where when $k = 1$, the last part $^+\mathbf{b}'^-, (1, 1, \mathbf{c}^-)^{k-2}, 1, 1, \mathbf{c}$ reads $^+\mathbf{b}'$.

(2) *If γ is of Type B, and $\sigma^{n<}$ starts as $LR^kL\ldots$, then the even length continued fraction expansion of $\gamma \cdot LR^kL(=\phi_x^{n+k+2})$ is*

$$[0; \mathbf{c}^-, 1, 1, \mathbf{a}, (^+\mathbf{c}')^{k-1}, ^+\mathbf{c}'^+, \mathbf{c}'^+, \mathbf{a}', (1, 1,^- \mathbf{c})^{k-1}]. \tag{7.14}$$

(3) *If γ is of Type C, then the even length continued fraction expansion of $\gamma \cdot RRLL(=\phi_x^{n+4})$ is*

$$[0; \mathbf{c}, 1, 1,^- \mathbf{b},^+ \mathbf{c}',^+ \mathbf{b}', 1, 1,^- \mathbf{b}^+, \mathbf{c}',^+ \mathbf{b}']. \tag{7.15}$$

Proof. (1) By (7.3) and (7.4), we have

$$\gamma \cdot R = [0; \mathbf{c}, 1, 1,^- \mathbf{b}], \qquad 2(\gamma \cdot R) = [0; \mathbf{c}',^+ \mathbf{b}'].$$

Using (7.9) and (7.10), we have

$$\begin{aligned} \gamma \cdot RL^k &= [0; \mathbf{c}, 1, 1,^- \mathbf{b}^+, (\mathbf{c}'^+)^{k-1}, \mathbf{c}'], \\ 2(\gamma \cdot RL^k) &= [0; \mathbf{c}',^+ \mathbf{b}'^-, (1, 1, \mathbf{c}^-)^{k-1}, 1, 1, \mathbf{c}]. \end{aligned}$$

Using (7.5), we obtain the result.

(2) The proof is similar as in (1).

(3) Using (7.3) and (7.4), we have

$$\gamma \cdot R = [0; \mathbf{c}, 1, 1,^- \mathbf{b}], \qquad 2(\gamma \cdot R) = [0; \mathbf{c}',^+ \mathbf{b}'].$$

Using (7.5) and (7.6), we have

$$\gamma \cdot RR = [0; \mathbf{c}, 1, 1,^- \mathbf{b},^+ \mathbf{b}'], \qquad 2(\gamma \cdot RR) = [0; \mathbf{c}',^+ \mathbf{b}', 1, 1,^- \mathbf{b}].$$

Using (7.7) and (7.8), we have

$$\begin{aligned} \gamma \cdot RRL &= [0; \mathbf{c}, 1, 1,^- \mathbf{b},^+ \mathbf{b}'^-, 1, 1, \mathbf{c}, 1, 1,^- \mathbf{b}] \\ 2(\gamma \cdot RRL) &= [0; \mathbf{c}',^+ \mathbf{b}', 1, 1,^- \mathbf{b}^+, \mathbf{c}',^+ \mathbf{b}']. \end{aligned}$$

Using (7.5), we have the result. $\square$

We say that a sequence $a_1, a_2, \dots, a_l$ is *periodic with period* p if $a_i = a_{i+p}$ for $1 \leqslant i \leqslant l-p$.

LEMMA 7.9. *Let A, B and M be sequences. Assume that the concatenated sequence $AMBM$ is periodic, and let d be the minimum period. Assume moreover that $|M| \geqslant d$. (Here $|M|$ denotes the length of the sequence M.) Then d divides $|MB|$. In particular, the last term of A and B are the same.*

Proof. Let $M = (z_1, \dots, z_m)$. Suppose that d does not divide $|MB|$. Then considering the initial term of the latter M in $AMBM$, we have $z_1 = z_l$ with $2 \leqslant l \leqslant d$. Then $AMBM$ has period $l-1$, which contradicts the minimality of d. □

Proof of Theorem 7.4. Take $x \in C'$ that is not an endpoint of an interval I_α, and suppose that the continued fraction expansion $[0; x_1, x_2, \dots]$ of x is eventually periodic with period p, that is, there exists $c > 0$ such that $x_i = x_{i+p}$ for all $fi > c$.

Consider the case that infinitely many ϕ_x^n of Type A appear in $\{\phi_x^n\}_{n \geqslant 0}$. Take a ϕ_x^n of Type A for $n \gg 0$. We use the same notation as in Lemma 7.8 (for example $\gamma = \phi_x^n = [0; \mathbf{c}]$). Since n is sufficiently large, we may assume $m' := |\mathbf{c}'| \gg \max\{c, p\}$. By Lemma 7.8, the even length continued fraction expansion of ϕ_x^{n+k+2} is given by (7.13). If we delete the initial c terms and the last two terms from (7.13), then the remaining sequence is periodic with period p (see Lemma 7.7). In particular, the subsequence $(\mathbf{c}',^+ \mathbf{c}')$ is periodic with period p. Put $A = (c_1', c_2')$, $M = (c_3', \dots, c_{m'}')$ and $B = (3, c_2' + 1)$, where $\mathbf{c}' = (c_1', \dots, c_{m'}')$. Applying Lemma 7.9, we see $c_2' = c_2' + 1$, which is a contradiction.

Other cases can be handled similarly.

When infinitely many ϕ_x^n of Type B appear, by arguing as in Type A case, the subsequence $({}^+\mathbf{c}'^+, \mathbf{c}'^+)$ in (7.14) is periodic. We put $A = (3, c_2' + 1)$, $M = (c_3', \dots, c_{m'-2}')$ and $B = (c_{m'-1} + 1, 3, c_1', c_2')$.

When infinitely many ϕ_x^n of Type C appear, we see that the subsequence $({}^+\mathbf{c}', {}^+\mathbf{b}', 1, 1,^- \mathbf{b}^+, \mathbf{c}')$ is periodic. We put $A = (3, c_2' + 1)$, $M = (c_3', \dots, c_{m'}')$ and $B = ({}^+\mathbf{b}', 1, 1, {}^-\mathbf{b}^+, c_1', c_2')$. □

§8. Resolution of sheaves

In this section, imitating the argument in [CHW, Section 5], we construct a resolution of a semistable sheaf with symmetric c_1 on S.

Fix a semistable element $\xi \in K(S)$ such that the height of the moduli space $M(\xi)$ is positive. In the $(\bar{\mu}, \Delta)$-plane, the parabola $\Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}(\xi)) - \Delta(\xi)$ intersects the line $\Delta = 1/2$ at two points. Let $(\bar{\mu}_0, 1/2)$ be the intersection point with the larger $\bar{\mu}$ coordinate.

LEMMA 8.1. *There exists a unique $\gamma \in \mathfrak{E}$ such that $\bar{\mu}_0 \in I_\gamma$.*

Proof. Once we have Theorem 7.4, the same proof of [CHW, Theorem 3.1] applies. ∎

There are 3 cases:
Case (1) $\mathrm{P}(\gamma + \bar{\mu}(\xi)) - \Delta(\xi) > \Delta_\gamma$.
Case (2) $\mathrm{P}(\gamma + \bar{\mu}(\xi)) - \Delta(\xi) = \Delta_\gamma$.
Case (3) $\mathrm{P}(\gamma + \bar{\mu}(\xi)) - \Delta(\xi) < \Delta_\gamma$.

We express $\gamma = \epsilon(p/2^q)$, where $q \geqslant 0$, and p is odd if $q \geqslant 1$. If $q = 0$, then put $\alpha = \gamma - 1/2$ and $\beta = \gamma + 1$. If $q > 0$, then put

$$\alpha = \epsilon\left(\frac{p-1}{2^q}\right), \qquad \beta = \epsilon\left(\frac{p+1}{2^q}\right).$$

Then we have $\gamma = \alpha.\beta$.

In Case (1) and Case (2), we put

$$m_1 := \tilde{\chi}(E_{-\gamma}, \xi), \qquad m_2 := -\tilde{\chi}(E_{-\alpha.\gamma}, \xi), \qquad m_3 := -\tilde{\chi}(E_{-\alpha}, \xi),$$

and $U := E_{-\alpha-2}^{m_3}$ and $V := E_{-\beta}^{m_2} \oplus E_{-\gamma}^{m_1}$. Here we have $m_2 > 0$, $m_3 > 0$; and $m_1 > 0$ in Case (1) and $m_1 = 0$ in Case (2). (See Notation 4.6 for the notation $\tilde{\chi}$.)

In Case (3), we put

$$m_1 := -\tilde{\chi}(E_{-\gamma}, \xi) > 0, \qquad m_2 := \tilde{\chi}(E_{-\beta}, \xi) > 0, \qquad m_3 := \tilde{\chi}(E_{-\gamma.\beta}, \xi) > 0,$$

and $U = E_{-\gamma-2}^{m_1} \oplus E_{-\alpha-2}^{m_3}$ and $V := E_{-\beta}^{m_2}$. Using Lemma 8.3 below, we can see that $\mathcal{H}om(U, V)$ is globally generated. By a standard dimension estimate, we can see that there is a closed subset Z of codimension at least 2 in the affine space $\mathrm{Hom}(U, V)$ such that for any point $[\phi : U \to V] \in \mathrm{Hom}(U, V) \setminus Z$, ϕ is injective and $\mathrm{Coker}\,\phi$ is torsion-free. Put $Y := \mathrm{Hom}(U, V) \setminus Z$, and $F_\phi := \mathrm{Coker}\,\phi$ for $\phi \in Y$.

One can see that the family $\{F_\phi\}$ parametrized by Y is a complete family of sheaves with K-class ξ (cf. the proof of [CHW, Proposition 5.3]). Moreover, using Lemma 8.3(2) below, we can see that $\mathrm{Ext}^2(F_\phi, F_\phi(-1, -1)) = 0$ for any $\phi \in Y$. Let Y' be the open subset of Y consisting of ϕ with F_ϕ stable. Then by Proposition 6.1 (1), the codimension of $Y \setminus Y'$ in Y is at least 2.

REMARK 8.2. When $\mathrm{r}(\xi) \geqslant 3$, if we let Y'' be the open subset of Y consisting of ϕ such that F_ϕ is a μ-stable bundle, then by Proposition 6.1(2), the codimension $Y \setminus Y''$ in Y is at least 2.

LEMMA 8.3. *Let*

$$\alpha = \epsilon\left(\frac{p}{2^q}\right), \qquad \beta = \left(\frac{p+1}{2^q}\right)$$

with $q \geqslant 0$.

(1) *The sheaf* $\mathcal{H}om(E_{\beta-2}, E_\alpha)$ *is globally generated.*
(2) *We have* $\mathrm{Ext}^1(E_{\beta-1}, E_\alpha) = 0$.

Proof. (1) The proof is identical to that of [H, Lemma 5.4]. For $q = 0$, $E_\alpha = \mathcal{O}(p, p)$ and $E_{\beta-2} = \mathcal{O}(p-1, p-1)$, thus $\mathcal{H}om(E_{\beta-2}, E_\alpha) \simeq \mathcal{O}(1, 1)$. The lemma holds.

Assume $q \geqslant 1$. When p is odd, put $\eta = \epsilon((p-1)/2^q)$, and

$$\delta = \begin{cases} \epsilon\left(\dfrac{p+3}{2^q} - 2\right) & \text{if } p \equiv 1 \pmod 4, \\ \epsilon\left(\dfrac{p-3}{2^q}\right) & \text{if } p \equiv -1 \pmod 4. \end{cases}$$

Then we have an exact sequence (cf. Lemma 4.7)

$$0 \to E_\delta \to E_\eta^{\tilde{\chi}(E_\eta, E_\alpha)} \to E_\alpha \to 0. \tag{8.1}$$

So we have a surjective map $\mathcal{H}om(E_{\beta-2}, E_\eta)^{\tilde{\chi}(E_\eta, E_\alpha)} \to \mathcal{H}om(E_{\beta-2}, E_\alpha)$ and $\mathcal{H}om(E_{\beta-2}, E_\eta)$ is globally generated by the induction hypothesis, so $\mathcal{H}om(E_{\beta-2}, E_\alpha)$ is globally generated. When p is even, considering the isomorphism $\mathcal{H}om(E_{\beta-2}, E_\alpha) \simeq \mathcal{H}om(E_{-\alpha-2}, E_{-\beta})$, the proof is reduced to the case p odd.

(2) For $q = 0, 1$, the vanishing of Ext^1 can be checked directly. Assume $q \geqslant 2$. As in (1), we may consider only the case where p is odd. We have a short exact sequence (8.1). From this, we obtain an exact sequence

$$\mathrm{Ext}^1(E_{\beta-1}, E_\eta)^{\tilde{\chi}(E_\eta, E_\alpha)} \to \mathrm{Ext}^1(E_{\beta-1}, E_\alpha) \to \mathrm{Ext}^2(E_{\beta-1}, E_\delta).$$

By the induction hypothesis, we have $\mathrm{Ext}^1(E_{\beta-1}, E_\eta) = 0$. We have $\mathrm{Ext}^2(E_{\beta-1}, E_\delta) \simeq \mathrm{Hom}(E_\delta, E_{\beta-3})^* = 0$ since $\delta > \beta - 3$. The result follows from the above exact sequence. □

§9. Rational map to moduli of quiver representations

Retain the notation ξ, α, β and γ in the previous section. In this section, we define three kinds of quivers, and construct a rational map of $M(\xi)$ to the moduli space of representations of a quiver.

9.1 Quivers

To describe a quiver Q, we use the notation

$$Q = (Q_0, \{N_{v,v'}\}_{(v,v')\in Q_0\times Q_0}),$$

where Q_0 is the set of vertexes of the quiver, and $N_{v,v'}$ is the number of arrows having v as the source and v' as the target.

We define quivers R^α, R^β and R^γ as follows.

When α is even, the quiver $R^\alpha = (R^\alpha_0, \{N_{v,v'}\})$ is defined by $R^\alpha_0 = \{v_1, v_2, v_3\}$ and $N_{v_2,v_1} = N_{v_3,v_1} = \tilde{\chi}(E_{-\alpha-2}, E_{-\beta})$ and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j).

When β is even, the quiver $R^\beta = (R^\beta_0, \{N_{v,v'}\})$ is defined by $R^\beta_0 = \{v_1, v_2, v_3\}$ and $N_{v_1,v_2} = N_{v_1,v_3} = \tilde{\chi}^*(E_{-\alpha-2}, E_{-\beta})$ and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j).

When γ is even, the quiver $R^\gamma = (R^\gamma_0, \{N_{v,v'}\})$ is defined by $R^\gamma_0 = \{v_1, v_2\}$ and $N_{v_2,v_1} = \chi(E_{-\alpha-2}, E_{-\beta})$ and $N_{v_i,v_j} = 0$ for other pairs (v_i, v_j).

In summary, the quivers R^α, R^β and R^γ are the quivers obtained from the quivers Q^α, Q^β and Q^γ in Section 5.2 by reversing the arrows. (See Figures 10–12.

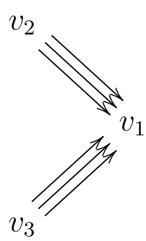

Figure 10.
R^α.

We define moduli spaces M^α, M^β and M^γ of representations of quivers as follows. M^α (resp. M^β) is the coarse moduli space of θ-semistable representations of the quiver R^α (resp. R^β) with dimension vector $(\delta(v_1), \delta(v_2), \delta(v_3)) = (m_2, m_3, m_3)$ (resp. $= (m_3, m_2, m_2)$), where

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

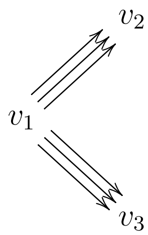

Figure 11.
R^β.

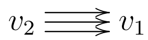

Figure 12.
R^γ.

the weight vector

$$(\theta(v_1), \theta(v_2), \theta(v_3)) = (2m_3, -m_2, -m_2)$$

(resp. $= (-2m_2, m_3, m_3)$). M^γ is the coarse moduli space of θ-semistable representations of the quiver R^γ with dimension vector $(\delta(v_1), \delta(v_2)) = (m_2, m_3)$, where the weight vector $(\theta(v_1), \theta(v_2)) = (m_3, -m_2)$. The nonemptyness of these moduli spaces will be verified later.

9.2 Rational map

We define a rational map from $M(\xi)$ to M^α, M^β or M^γ depending on which one of α, β, γ is even.

Case (1) or (2) A general sheaf $F \in M(\xi)$ has a resolution of the form

$$E_{-\alpha-2}^{m_3} \xrightarrow{(f,g)} E_{-\beta}^{m_2} \oplus E_{-\gamma}^{m_1}.$$

Let $M(\xi)^\circ$ be the open subscheme of $M(\xi)$ consisting of stable sheaves having the resolution of this form.

Subcase (1 or 2, α). If α is even, then $E_{-\alpha-2} = E'_{-\alpha-2} \oplus E''_{-\alpha-2}$. Giving the map

$$f = f' + f'' : {E'}_{-\alpha-2}^{m_3} \oplus {E''}_{-\alpha-2}^{m_3} \to E_{-\beta}^{m_2}$$

is equivalent to giving linear maps

(9.1)
$$\mathbb{C}^{m_3} \to \mathrm{Hom}(E'_{-\alpha-2}, E_{-\beta})^{m_2} \qquad \text{and} \qquad \mathbb{C}^{m_3} \to \mathrm{Hom}(E''_{-\alpha-2}, E_{-\beta})^{m_2},$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

which are obtained from f' and f'' by applying $\mathrm{Hom}(E'_{-\alpha-2}, -)$ and $\mathrm{Hom}(E''_{-\alpha-2}, -)$, respectively. The linear maps (9.1) determine a representation of the quiver R^{α}, denoted by $\pi(F)$, with dimension vector (m_2, m_3, m_3).

Subcase (1 or 2, β). If β is even, then $E_{-\beta} = E'_{-\beta} \oplus E''_{-\beta}$. As in the previous case, the map f determines a representation of the quiver R^{β}, denoted by $\pi(F)$, with dimension vector (m_3, m_2, m_2).

Subcase (1 or 2, γ). If γ is even, then both $E_{-\alpha-2}$ and $E_{-\beta}$ are simple bundles. Giving map f is equivalent to giving a linear map

$$\mathbb{C}^{m_3} \to \mathrm{Hom}(E_{-\alpha-2}, E_{-\beta})^{m_2}, \tag{9.2}$$

which is obtained from f by applying $\mathrm{Hom}(E_{-\alpha-2}, -)$. The map (9.2) determines a representation of the quiver R^{γ}, denoted by $\pi(F)$, with dimension vector (m_2, m_3).

Case (3). A general sheaf $F \in M(\xi)$ has a resolution of the form

$$E_{-\gamma-2}^{m_1} \oplus E_{-\alpha-2}^{m_3} \xrightarrow{g+f} E_{-\beta}^{m_2}. \tag{9.3}$$

Let $M(\xi)^{\circ}$ be the open subscheme of $M(\xi)$ consisting of stable sheaves having the resolution of this form. We consider the map $f : E_{-\alpha-2}^{m_3} \to E_{-\beta}^{m_2}$. As in the preceding cases, we see that f determines a representation of the quiver R^{α}, R^{β} or R^{γ}, denoted by $\pi(F)$, depending on whether α, β or γ is even.

For $\star \in \{\alpha, \beta, \gamma\}$, let $M^{\star s}$ be the open subscheme of $M^{\star}$ consisting of θ-stable representations, where the weight θ is the one defined before. Let $M(\xi)^{\circ\sigma}$ be the open subscheme of $M(\xi)^{\circ}$ consisting of such F that $\pi(F) \in M^{\star s}$. The rational map π from $M(\xi)$ to M^{α}, M^{β} or M^{γ} is defined to map $F \in M(\xi)^{\circ\sigma}$ to $\pi(F)$. The following proposition guarantees that the domain of definition of the rational map π is nonempty.

PROPOSITION 9.1. *$M(\xi)^{\circ\sigma}$ is nonempty. Moreover, in Case (2), we have $M(\xi)^{\circ} = M(\xi)^{\circ\sigma}$.*

Proof. In Case (2), $F \in M(\xi)$ fits in a short exact sequence

$$0 \to E_{-\alpha-2}^{m_3} \xrightarrow{f} E_{-\beta}^{m_2} \to F \to 0.$$

So the result follows from Proposition 5.7.

Next we consider Case (3). We have only to show that $M^{\star s}$ is nonempty ($\star \in \{\alpha, \beta, \gamma\}$). Consider $f : E_{-\alpha-2}^{m_3} \to E_{-\beta}^{m_2}$. For general f, f is injective and the rank of $F := \mathrm{Coker}\, f$ is positive.

Claim 9.1.1. We have $\bar{\mu}(F) > -\gamma - x_\gamma$.

Proof of Claim. Let $(\bar{\mu}_1, \Delta_1)$ be the intersection of the parabolas

$$\Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}(\xi)) - \Delta(\xi) \qquad \text{and} \qquad \Delta = \mathrm{P}(\bar{\mu} - \gamma - 2) - \Delta_\gamma$$

in the $(\bar{\mu}, \Delta)$-plane. The point $(\bar{\mu}(F), \Delta(F))$ lies on the parabolas

$$\Delta = \mathrm{P}(\bar{\mu} + \gamma) - \Delta_\gamma \qquad \text{and} \qquad \Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}_1) - \Delta_1.$$

By calculation, we have

$$\begin{aligned}\bar{\mu}(F) &= \frac{-\gamma - \bar{\mu}_1 - 2}{2} + \frac{\Delta_1 - \Delta_\gamma}{2(\bar{\mu}_1 - \gamma)} \\ &> \frac{-\gamma - (\gamma + x_\gamma) - 2}{2} + \frac{(1/2) - \Delta_\gamma}{2x_\gamma} = -\gamma - x_\gamma.\end{aligned}$$

This is the end of proof of the claim. ∎

So for the K-class $\eta := [F]$, we can apply Proposition 5.7 and conclude that $M^{\star s} \neq \emptyset$.

Finally we consider Case (1).

Claim 9.1.2. We have $-\bar{\mu}(\xi) < \gamma$.

Proof. Recall that $(\bar{\mu}_0, 1/2)$ is the intersection point with larger $\bar{\mu}$-coordinate of the parabola $\Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}(\xi)) - \Delta(\xi)$ and the line $\Delta = 1/2$ in the $(\bar{\mu}, \Delta)$-plane. Since $\Delta(\xi) > 1/2$, we have $-\bar{\mu}(\xi) < \bar{\mu}_0$. Let $\nu \in \mathfrak{E}$ with $-\bar{\mu}(\xi) \in I_\nu$. We have $\nu \leqslant \gamma$. If $\nu < \gamma$, then clearly $-\bar{\mu}(\xi) < \gamma$ and we are done.

Consider the case $\nu = \gamma$. We claim that $\gamma - x_\gamma < -\bar{\mu}(\xi) < \gamma$. Indeed, if $\gamma \leqslant -\bar{\mu}(\xi) < \gamma + x_\gamma$, then $-\gamma - x_\gamma < \bar{\mu}(\xi) \leqslant -\gamma$, and since the point $(\bar{\mu}(\xi), \Delta(\xi))$ in the $(\bar{\mu}, \Delta)$-plane is above the graph of the function $\delta(\bar{\mu})$, we have $\Delta(\xi) \geqslant \mathrm{P}(\bar{\mu}(\xi) + \gamma) - \Delta_\gamma$. On the other hand, in Case (1), we have $\Delta_\gamma < \mathrm{P}(\bar{\mu}(\xi) + \gamma) - \Delta(\xi)$. This is a contradiction. This is the end of proof of the claim. ∎

Let $(\bar{\mu}_1, \Delta_1)$ be the intersection of the parabolas

$$\Delta = \mathrm{P}(\bar{\mu} - \gamma) - \Delta_\gamma \qquad \text{and} \qquad \Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}(\xi)) - \Delta(\xi).$$

Using Claim 9.1.2, we can easily check $\gamma - x_\gamma < \bar{\mu}_1$. We consider three cases:
Case (1-i) $\gamma - x_\gamma < \bar{\mu}_1 < \gamma$,
Case (1-ii). $\bar{\mu}_1 = \gamma$,

Case (1-iii). $\gamma < \bar{\mu}_1$.

By calculation we can obtain

$$\bar{\mu}_1 = \gamma + \frac{1 - 2\Delta_\gamma - (\mathrm{P}(\bar{\mu}(\xi) + \gamma) - \Delta(\xi) - \Delta_\gamma)}{2(\gamma + \bar{\mu}(\xi))}$$

and

$$\mathrm{r}(\xi) - \mathrm{r}(E_{-\gamma}^{m_1}) = \frac{\mathrm{r}(\xi)\,\{1 - 2\Delta_\gamma - (\mathrm{P}(\bar{\mu}(\xi) + \gamma) - \Delta(\xi) - \Delta_\gamma)\}}{1 - 2\Delta_\gamma}.$$

From these equations, we see that the cases (1-i), (1-ii) and (1-iii) correspond to $\mathrm{r}(E_{-\alpha-2}^{m_3}) > \mathrm{r}(E_{-\beta}^{m_2})$, $\mathrm{r}(E_{-\alpha-2}^{m_3}) = \mathrm{r}(E_{-\beta}^{m_2})$ and $\mathrm{r}(E_{-\alpha-2}^{m_3}) < \mathrm{r}(E_{-\beta}^{m_2})$, respectively.

Let η be the K-class $[E_{-\beta}^{m_2}] - [E_{-\alpha-2}^{m_3}]$.

Case (1-ii). In this case, we can apply Proposition 5.9 for the K-class η and conclude $M^{\star s} \neq \emptyset$.

In cases (1-i) and (1-iii), put $\bar{\mu}_2 = \bar{\mu}(\eta)$ and $\Delta_2 = \Delta(\eta)$. Since the point $(\bar{\mu}_2, \Delta_2)$ in the $(\bar{\mu}, \Delta)$-plane lies on the parabolas

$$\Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}_1) - \Delta_1 \qquad \text{and} \qquad \Delta = \mathrm{P}(\bar{\mu} + \gamma) - \Delta_\gamma,$$

we have, by calculation,

$$\bar{\mu}_2 = -\gamma + \frac{1 - 2\Delta_\gamma}{2(\bar{\mu}_1 - \gamma)}.$$

Case (1-iii). In this case, we have $-\gamma < \bar{\mu}_2$. So we can apply Proposition 5.7 for the K-class η and conclude that $M^{\star s} \neq \emptyset$.

Case (1-i). In this case, we consider the dual map

$$f^* : E_{\beta}^{m_2} \to E_{\alpha+2}^{m_3}.$$

Let $\eta^\dagger$ be the K-class $[E_{\alpha+2}^{m_3}] - [E_{\beta}^{m_2}]$. We have

$$\bar{\mu}(\eta^\dagger) = -\bar{\mu}_2 = \gamma + \frac{1 - 2\Delta_\gamma}{2(\gamma - \bar{\mu}_1)} > \gamma + \frac{1 - 2\Delta_\gamma}{2x_\gamma} = \gamma + 2 - x_\gamma.$$

Then we can apply Proposition 5.7 for the K-class $\eta^\dagger$ and conclude that $M^{\star s} \neq \emptyset$. ∎

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

9.3 Moving curves

Let us see that there are moving curves in the domain of definition of the rational map π.

PROPOSITION 9.2. *In Case (1) or Case (3), for a general point x of $M(\xi)$, there exists a complete curve C passing through x in $M(\xi)^{\circ\sigma}$ such that $\pi(C)$ is a point.*

PROPOSITION 9.3. *In Case (2), the map π is birational. More precisely, it gives rise to an open immersion of $M(\xi)^{\circ}$ to $M^{\star s}$. For a general point x of $M(\xi)$, there exists a complete curve in $M(\xi)^{\circ}$ passing through x.*

Proof of Proposition 9.2. We give a proof for Case (1). We first check that the dimension of the fiber of the rational map π is positive. By calculation, the dimension of the fiber of the rational map π is $-\chi(\xi, E_{-\gamma})$. We need to show $\chi(\xi, E_{-\gamma}) < 0$.

When $-\bar{\mu}(\xi) \in I_{\gamma}$, we have

$$\Delta(\xi) > \mathrm{P}(\gamma - 2 + \bar{\mu}(\xi)) - \Delta_{\gamma}$$

since the height of the moduli space $M(\xi)$ is positive. From this, we have $\chi(\xi, E_{-\gamma}) = \chi(E_{-\gamma+2}, \xi) < 0$.

When $-\bar{\mu}(\xi) \notin I_{\gamma}$, we have $-\bar{\mu}(\xi) < \gamma - x_{\gamma}$. Moving upward the graph $\Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}(\xi)) - \Delta(\xi)$ in the $(\bar{\mu}, \Delta)$-plane, we can find a positive number ρ such that the graph $\Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}(\xi)) - \Delta(\xi) + \rho$ passes through the point $(\gamma - x_{\gamma}, 1/2)$. Then by calculation,

$$\begin{aligned} \mathrm{P}(-\bar{\mu}(\xi) - \gamma) - \Delta(\xi) - \Delta_{\gamma} + \rho &= (x_{\gamma} - 2)\big(2(\gamma + \bar{\mu}(\xi)) - x_{\gamma}\big) - \Delta_{\gamma} + 1/2 \\ &< x_{\gamma}(x_{\gamma} - 2) - \Delta_{\gamma} + 1/2 = 0. \end{aligned}$$

By this, we have $\chi(\xi, E_{-\gamma}) < 0$.

We turn to the existence of a complete curve in $M(\xi)^{\circ\sigma}$ passing through a general point. Recall that we put $U := E_{-\alpha-2}^{m_3}$ and $V := E_{-\beta}^{m_2} \oplus E_{-\gamma}^{m_1}$. Put

$$\mathrm{H}_1 := \mathrm{Hom}(U, E_{-\gamma}^{m_1}) \qquad \text{and} \qquad \mathrm{H}_2 := \mathrm{Hom}(U, E_{-\beta}^{m_2})$$

so that $\mathrm{Hom}(U, V) \simeq \mathrm{H}_2 \oplus \mathrm{H}_1$. Take a general $f \in \mathrm{H}_2$. Then the representation of the quiver corresponding to f is θ-stable. Consider the open subset $Y_f \subset \mathrm{H}_1$ consisting of $g \in \mathrm{H}_1$ such that the morphism $(f, g) : U \to V$ is injective with stable cokernel. Since the open subset Y_f is invariant under the action of $\mathbb{C}^{\times}$ by multiplication, we can consider the projectivization

$\mathbb{P}(Y_f) \subset \mathbb{P}(\mathrm{H}_1)$. Since $\operatorname{codim}(\mathrm{H}_1 \setminus Y_f, \mathrm{H}_1) \geqslant 2$ by the explanation before Remark 8.2, a general line in $\mathrm{P}(\mathrm{H}_1)$ lies inside $\mathrm{P}(Y_f)$ and determines a complete curve in $M(\xi)^{\circ\sigma}$. This curve is mapped to a point by π because f is fixed.

The argument for Case (3) is similar (and easier). ∎

Proof of Proposition 9.3. The argument goes as in the proof of Proposition 9.2. We employ the same notation used there. In Case (2), we have $m_1 = 0$. In the last paragraph of Section 8, we considered the open subset $Y' \subset \operatorname{Hom}(U, V)$ consisting of injective morphisms with stable cokernel. By the action of $\mathbb{C}^\times$ by multiplication, Y' is invariant. Consider the projectivization $\mathbb{P}(Y') \subset \mathbb{P}\operatorname{Hom}(U, V)$. Since the complement of $\mathbb{P}(Y')$ in $\mathbb{P}\operatorname{Hom}(U, V)$ has codimension at least 2, a general line in $\mathbb{P}\operatorname{Hom}(U, V)$ lies in $\mathbb{P}(Y')$, which determines a complete curve in $M(\xi)^\circ$. ∎

Remark 9.4. Let $M(\xi)^{\mu s}$ be the open subscheme of $M(\xi)$ consisting of μ-stable bundles. When $\mathrm{r}(\xi) \geqslant 3$, substituting Y'' in Remark 8.2 for Y' in the above proof, we can find a complete curve passing through a general point in Propositions 9.2 and 9.3 inside $M(\xi)^{\circ\sigma} \cap M(\xi)^{\mu s}$.

§10. Effective cone of the moduli space

10.1 Line bundles on moduli spaces

We fix notation for line bundles on a moduli space of sheaves.

For a flat family $\mathcal{E}$ of coherent sheaves on S parametrized by a scheme T, we let $\lambda_{\mathcal{E}} : K(S) \to \operatorname{Pic}(T)$ be the composition of homomorphisms

$$K(S) \xrightarrow{q^*} K^0(S \times T) \xrightarrow{\cdot[\mathcal{E}]} K^0(S \times T) \xrightarrow{p_!} K^0(T) \xrightarrow{\det} \operatorname{Pic}(T),$$

where $p : S \times T \to T$ and $q : S \times T \to S$ are projections, and $K^0(?)$ is the Grothendieck group of locally free sheaves on ?

Fix a semistable class $\xi \in K(S)$. Let H be a smooth divisor in the very ample linear system $|\mathcal{O}_S(1, 1)|$ and let $h = [\mathcal{O}_H]$.

As in [HL, Definition 8.1.4], we put

$$K_\xi := \xi^\perp \qquad \text{and} \qquad K_{\xi,H} := \xi^\perp \cap \{1, h, h^2\}^{\perp\perp}.$$

Since a class $\eta \in K(S)$ is in $\{1, h, h^2\}^{\perp\perp}$ if and only if η has symmetric c_1, we can also describe $K_{\xi,H}$ as

$$K_{\xi,H} = \{\eta \in K_\xi \mid \eta \text{ has symmetric } c_1\}.$$

So, from now on, we write K_ξ^{sym} for $K_{\xi,H}$.

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

By [HL, Theorem 8.1.5], we have a homomorphism

$$\lambda : K_\xi^{\mathrm{sym}} \to \mathrm{Pic}\,(M(\xi)) \tag{10.1}$$

such that if $\mathcal{E}$ is a flat family of semistable sheaves with K-class ξ on S parametrized by a scheme T, then $\phi_{\mathcal{E}}^* \circ \lambda = \lambda_{\mathcal{E}}$, where $\phi_{\mathcal{E}} : T \to M(\xi)$ is the classifying morphism.

As in [HL, Definition 8.1.9], we define a line bundle $\mathcal{L}_i$ by

$$\mathcal{L}_i := \lambda\,(u_i(\xi)),$$

where $u_i(\xi) = -\mathrm{r}(\xi) \cdot h^i + \chi(\xi \otimes h^i)[\mathcal{O}_x]$ with $x \in S$ a closed point.

For $m \gg 0$, $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ is an ample line bundle on $M(\xi)$ (cf. [HL, Remark 8.1.12]).

10.2 Edges of the cone

In the rest of this section, we assume that the semistable class $\xi \in K(S)$ has symmetric c_1 such that the height of the moduli space $M(\xi)$ is positive. Let γ be the symmetric exceptional slope determined by Lemma 8.1.

We denote also by λ the map

$$\lambda : K_\xi^{\mathrm{sym}} \otimes \mathbb{R} \to \mathrm{NS}\,(M(\xi))_{\mathbb{R}} \tag{10.2}$$

induced by (10.1).

LEMMA 10.1. *The map λ in (10.2) is injective.*

Proof. When $\mathrm{r}(\xi) = 1$, the exceptional divisor of the Hilbert–Chow morphism is effective but not ample, so rank $\lambda = 2$.

When $\mathrm{r}(\xi) = 2$, the divisor of $M(\xi)$ consisting of nonlocally free sheaves is effective but not ample, so rank $\lambda = 2$.

The proof for the case $\mathrm{r}(\xi) \geqslant 3$ is postponed after the proof of Theorem 10.2. ∎

Put $V := \lambda(K_\xi^{\mathrm{sym}} \otimes \mathbb{R})$. In this section, we determine the cone $\mathrm{Eff}\,(M(\xi)) \cap V$.

We define $(\bar{\mu}^+, \Delta^+)$ as follows (see the paragraph after Lemma 8.1 for Cases (1), (2) and (3)):

In Case (1), we define $(\bar{\mu}^+, \Delta^+)$ to be the intersection of the parabolas $\Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}(\xi)) - \Delta(\xi)$ and $\Delta = \mathrm{P}(\bar{\mu} - \gamma) - \Delta_\gamma$.

In Case (2), we define $(\bar{\mu}^+, \Delta^+) := (\gamma, \Delta_\gamma)$.

In Case (3), we define $(\bar{\mu}^+, \Delta^+)$ to be the intersection of the parabolas $\Delta = \mathrm{P}(\bar{\mu} + \bar{\mu}(\xi)) - \Delta(\xi)$ and $\Delta = \mathrm{P}(\bar{\mu} - \gamma - 2) - \Delta_\gamma$.

Let r^+ be the smallest positive integer such that $r^+\bar{\mu} \in \mathbb{Z}$ and $r^+(\mathrm{P}(\bar{\mu}^+) - \Delta^+) \in \mathbb{Z}$. We define ξ^+ to be an element of K_ξ^{sym} such that $\mathrm{r}(\xi^+) = r^+$, $\bar{\mu}(\xi^+) = \bar{\mu}^+$ and $\Delta(\xi^+) = \Delta^+$.

THEOREM 10.2.

(a) *The ray spanned by* $\lambda(-\xi^+)$ *in* $\mathrm{NS}(M(\xi))_{\mathbb{R}}$ *is an edge of* $\mathrm{Eff}(M(\xi)) \cap V$.
(b) *The other edge of* $\mathrm{Eff}(M(\xi)) \cap V$ *is spanned by* $\lambda(\eta)$. *Here* η *is the element of* $K_\xi^{\mathrm{sym}} \otimes \mathbb{R}$ *with* $\bar{\mu}(\eta) = -\bar{\mu}(\xi)$ *and* $\mathrm{r}(\eta) > 0$ *when* $\mathrm{r}(\xi) = 1$; $\eta \in K_\xi^{\mathrm{sym}} \otimes \mathbb{R}$ *is such that* $\bar{\mu}(\eta) = -1 - \bar{\mu}(\xi)$ *and* $\mathrm{r}(\eta) > 0$ *when* $\mathrm{r}(\xi) = 2$; *and* $\eta = ((\xi^* \otimes K_S)^+)^*$ *when* $\mathrm{r}(\xi) \geqslant 3$.

Sketch of Proof. The argument is the same as in [CHW].

We let $\star = \alpha, \beta$ or γ depending on whether α, β or γ is even.

(a) *Case* (1). To show $\mathbb{R}_{\geqslant 0}\lambda(-\xi^+) \subset \mathrm{Eff}(M(\xi))$, it suffices to find $E \in M(\xi)$ and $F \in M(N\xi^+)$ for some $N > 0$ such that $\mathrm{H}^i(E \otimes F) = 0$ for all i. In fact, then, $\lambda(-N\xi^+)$ is the class of the effective divisor $\Theta_F := \{E \in M(\xi) \mid \mathrm{H}^0(E \otimes F) \neq 0\}$.

Consider $E \in M(\xi)^{\circ\sigma}$ fitting in a short exact sequence

$$0 \to E_{-\alpha-2}^{m_3} \xrightarrow{(f,g)} E_{-\beta}^{m_2} \oplus E_{-\gamma}^{m_1} \to E \to 0,$$

and $F \in M(N\xi^+)$ fitting in a short exact sequence

$$0 \to E_{\beta-2}^{m} \xrightarrow{h} E_{\alpha}^{n} \to F \to 0,$$

where $m = -\tilde{\chi}(E_\beta, F)$ and $n = -\tilde{\chi}(E_{\gamma.\beta}, F)$. Define two-term complexes $C_1^\bullet$, $C_2^\bullet$ and $D^\bullet$ with terms in degree -1, 0 by

$$C_1^\bullet = [E_{-\alpha-2}^{m_3} \xrightarrow{(f,g)} E_{-\beta}^{m_2} \oplus E_{-\gamma}^{m_1}], \quad C_2^\bullet = [E_{-\alpha-2}^{m_3} \xrightarrow{f} E_{-\beta}^{m_2}],$$
$$D^\bullet = [E_{\beta-2}^{m} \xrightarrow{h} E_{\alpha}^{n}].$$

We have $\chi(E \otimes F) = 0$ and $\mathrm{H}^2(E \otimes F) = 0$. Moreover, we have

$$\begin{aligned} \mathrm{H}^1(E \otimes F) &\simeq \mathrm{H}^1(C_1^\bullet \otimes F) \\ &\simeq \mathrm{H}^1(C_2^\bullet \otimes F) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(S)}(D^{\bullet\vee}, C_2^\bullet[1]) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(S)}(C_2^\bullet, D^{\bullet\vee} \otimes K_S[1])^*, \end{aligned} \tag{10.3}$$

https://doi.org/10.1017/nmj.2016.50 Published online by Cambridge University Press

where the second isomorphism follows from $\mathrm{H}^i(E_{-\gamma}\otimes F)=0$, for all i, and the last isomorphism from the duality. $D^{\bullet\vee}\otimes K_S[1]$ is the complex

$$E^n_{-\alpha-2}\xrightarrow{h'}E^m_{-\beta} \tag{10.4}$$

with terms in degree $-1,0$, where h' is the morphism induced by h. The complex $C_2^\bullet$ corresponds to the representation $\pi(E)$ of the quiver $R^\star$, we let $\tau(F)$ be the representation of $R^\star$ corresponding to the complex (10.4). Using [DW, Theorem 1], we can see that for some $N>0$ and general $F\in M(N\xi^+)$, we have $\mathrm{Hom}(\pi(E),\tau(F))=0$, hence $\mathrm{H}^1(E\otimes F)=0$.

By Proposition 9.2, we have a moving complete curve $C\subset M(\xi)^{\circ\sigma}$. From (10.3), it follows that $C\subset\Theta_F$ or $C\cap\Theta_F=\emptyset$ because $\pi(C)$ is a point. Then the intersection number $C\cdot\Theta_F=0$. This implies that the ray $\mathbb{R}_{\geqslant 0}\lambda(-\xi^+)$ is an edge of $\mathrm{Eff}(M(\xi))\cap V$.

Case (2). In this case we have $\mathrm{H}^i(E\otimes E_\gamma)=0$, for all i, for any $E\in M(\xi)^\circ$. The effective divisor $\Theta_{E_\gamma}=\{E\in M(\xi)\mid \mathrm{H}^0(E\otimes E_\gamma)\neq 0\}$ has the class $\lambda(-\xi^+)$. Using Proposition 9.3 instead of Proposition 9.2, arguing as in Case (1), we can see that the ray $\mathbb{R}_{\geqslant 0}\lambda(-\xi^+)$ is an edge of $\mathrm{Eff}(M(\xi))\cap V$.

Case (3). We can argue as in Case (1).

(b) In the case $\mathrm{r}(\xi)=1$ (resp. $\mathrm{r}(\xi)=2$), we can see, by calculation, that the ray $\mathbb{R}_{\geqslant 0}\lambda(\eta)$ is spanned by the exceptional divisor of the Hilbert–Chow morphism (resp. the morphism to the Donaldson–Uhlenbeck compactification). So it is an edge of $\mathrm{Eff}(M(\xi))\cap V$.

Consider the case $\mathrm{r}(\xi)\geqslant 3$. By (a), for general $E'\in M(\xi^*\otimes K_S)$ and $F'\in M(N(\xi^*\otimes K_S)^+)$ for some $N>0$, we have $\mathrm{H}^i(E'\otimes F')=0$ for all i. This implies, by considering the dual, that for general $E\in M(\xi)$ and $F\in M(N\eta)$ we have $\mathrm{H}^i(E\otimes F)=0$ for all i. Then $\Xi_F:=\{E\in M(\xi)\mid \mathrm{H}^2(E\otimes F)\neq 0\}$ is an effective divisor with the class $\lambda(N\eta)$. Let us show that the ray $\mathbb{R}_{\geqslant 0}\lambda(\eta)$ is an edge of $\mathrm{Eff}(M(\xi))\cap V$. Let $M(\xi)^{\mu s}$ and $M(\xi^*\otimes K_S)^{\mu s}$ be the open subschemes of $M(\xi)$ and $M(\xi^*\otimes K_S)$, respectively consisting of μ-stable bundles. We have an isomorphism

$$f:M(\xi)^{\mu s}\to M(\xi^*\otimes K_S)^{\mu s}$$

by $f(E)=E^*\otimes K_S$. By Remark 9.4, we can find a moving complete curve C in $M(\xi^*\otimes K_S)^{\mu s}$ that is orthogonal to the class $\lambda((\xi^*\otimes K_S)^+)$. Then the moving curve $f^{-1}(C)$ is orthogonal to $\lambda(\eta)$. So the ray $\mathbb{R}_{\geqslant 0}\lambda(\eta)$ is extremal. $\square$

Proof of Lemma 10.1 in the case $\mathrm{r}(\xi) \geqslant 3$. We first claim that ξ^+ and $\left((\xi^* \otimes K_S)^+\right)^*$ are linearly independent in $K_\xi^{\mathrm{sym}} \otimes \mathbb{R}$. Indeed, by definition, we have

$$\bar{\mu}(\xi^+) > -1 - \bar{\mu}(\xi) \qquad \text{and} \qquad \bar{\mu}\left((\xi^* \otimes K_S)^+\right) > -1 - \bar{\mu}(\xi^* \otimes K_S).$$

From this, we have

$$\bar{\mu}(((\xi^* \otimes K_S)^+)^*) < -1 - \bar{\mu}(\xi) < \bar{\mu}(\xi^+).$$

So ξ^+ and $\left((\xi^* \otimes K_S)^+\right)^*$ are linearly independent.

Now suppose that λ is not injective. Then $\operatorname{rank} \lambda = 1$ because $\operatorname{Im} \lambda$ contains an ample class. For $\tau \in K_\xi^{\mathrm{sym}} \otimes \mathbb{R}$, $\lambda(\tau)$ is proportional to an ample class. So, if there exists a complete curve $C \subset M(\xi)$ such that the intersection number $\lambda(\tau) \cdot C$ is zero, then $\lambda(\tau) = 0$. In the proof of Theorem 10.2, we showed that for ξ^+ and $\left((\xi^* \otimes K_S)^+\right)^*$, such a curve exists. So $\lambda\left(\xi^+\right) = \lambda\left(\left((\xi^* \otimes K_S)^+\right)^*\right) = 0$. But then, we have $\operatorname{rank} \lambda = 0$ because ξ^+ and $\left((\xi^* \otimes K_S)^+\right)^*$ span K_ξ^{sym}. This is a contradiction. ∎

REMARK 10.3. I mention related works. The effective and ample cones of the Hilbert scheme of Del Pezzo surfaces are discussed in [BC]. Ryan [Ry] studies the effective cone of the moduli space of sheaves on a quadric surface without the assumption c_1 symmetric. The ample and movable cone of the moduli space of sheaves on a surface is also studied by several authors [BM], [CH1], [CH2], [LZ], [Y].

§11. Strange duality

Let $\xi, \xi' \in \mathrm{K}(S)$ be semistable class with symmetric c_1. Assume that $\mu(\xi \otimes \xi') \geqslant 0$ and $\chi(\xi \otimes \xi') = 0$. Define a subscheme Θ of $M(\xi) \times M(\xi')$ by

$$\Theta = \left\{(F, F') \in M(\xi) \times M(\xi') \,\middle|\, \mathrm{H}^0(S, F \otimes F') \neq 0\right\}.$$

Assume that $\Theta \neq M(\xi) \times M(\xi')$. Then Θ is a Cartier divisor and we have

$$\mathcal{O}(\Theta) \simeq \mathcal{D} \boxtimes \mathcal{D}',$$

where $\mathcal{D} = \lambda(\xi')^*$ and $\mathcal{D}' = \lambda(\xi)^*$. The section defining Θ induces a linear map

$$\mathrm{H}^0(M(\xi), \mathcal{D})^* \to \mathrm{H}^0(M(\xi'), \mathcal{D}'). \tag{11.1}$$

THEOREM 11.1. *Assume that either $M(\xi)$ or $M(\xi')$ is of height zero. Then the map (11.1) is an isomorphism.*

Once we have Proposition 6.1, and a rational map to moduli spaces of quiver representations constructed in Section 9, the proof of the above theorem is quite parallel to that of [A15, Theorem 1.1]. So we omit the proof.

Appendix

From a full exceptional collection, one obtains a Beilinson-type spectral sequence (cf. [GK, Section 4.5]). For readers' convenience, we present a spectral sequence obtained from a full d-block exceptional collection.

We let $\mathcal{D} = \mathrm{D}(Y)$ for a smooth projective variety Y. Let

$$\mathbb{E} = (\mathbb{E}^{(1)}, \ldots, \mathbb{E}^{(d)})$$

be a full d-block exceptional collection, where $\mathbb{E}^{(k)} = (E_1^{(k)}, \ldots, E_{N_k}^{(k)})$. We define a block $\mathbb{G}^{(k)} = (G_1^{(k)}, \ldots, G_{N_k}^{(k)})$ of exceptional objects by

$$\mathbb{G}^{(k)} = R_{\mathbb{E}^{(d)}} \ldots R_{\mathbb{E}^{(k+1)}}(\mathbb{E}^{(k)}).$$

Then we have

$$\text{(A1)} \qquad \mathrm{Hom}^m(G_i^{(k)}, E_j^{(l)}) = \begin{cases} \mathbb{C} & (k,i) = (l,j) \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F \in \mathcal{D}$ and put $F_1 := F$. Since the pair $(\langle \mathbb{E}^{(k)} \rangle, \langle \mathbb{E}^{(k+1)}, \ldots, \mathbb{E}^{(d)} \rangle)$ is a semi-orthogonal decomposition of $\langle \mathbb{E}^{(k)}, \ldots, \mathbb{E}^{(d)} \rangle$, we can define F_k $(2 \leqslant k \leqslant d)$, and C_k $(1 \leqslant k \leqslant d-1)$ inductively by a triangle

$$\text{(A2)} \qquad F_{k+1} \to F_k \to C_k \to F_{k+1}[1],$$

where $F_{k+1} \in \langle \mathbb{E}^{(k+1)}, \ldots, \mathbb{E}^{(d)} \rangle$ and $C_k \in \langle \mathbb{E}^{(k)} \rangle$. Put $C_d := F_d$. Using the triangle (A2), we can see that

$$C_k = \bigoplus_{1 \leqslant \alpha \leqslant N_k,\, j \in \mathbb{Z}} \mathrm{Ext}^j(G_\alpha^{(k)}, F) \otimes E_\alpha^{(k)}[-j].$$

From the triangle (A2), we see that F_k is quasi-isomorphic to the mapping cone of $C_k[-1] \to F_{k+1}$. By considering inductively, we can see that F is quasi-isomorphic to a complex $A^\bullet$ with filtration

$$A^\bullet = A_1^\bullet \supset \cdots \supset A_d^\bullet \supset A_{d+1}^\bullet = 0$$

such that the graded complex $A_k^\bullet/A_{k+1}^\bullet$ is quasi-isomorphic to C_k. So we have a spectral sequence

$$E_1^{p,q} = \bigoplus_{1\leqslant\alpha\leqslant N_p,\, j\in\mathbb{Z}} \mathrm{Ext}^j(G_\alpha^{(p)}, F)\otimes \mathrm{H}^{p+q-j}(E_\alpha^{(p)}) \tag{A3}$$

converging to $\mathrm{H}^{p+q}(F)$.

Now suppose in addition that $d = \dim Y + 1$ and all $E_i^{(k)}$ are exceptional sheaves. Put $\bar{G}_i^{(k)} := G_i^{(k)}[d-k]$. Then $\bar{G}_i^{(k)}$ are sheaves by [BS, Theorem 4.5 and Lemma 5.2]. We have a spectral sequence

$$E_1^{p,q} = \bigoplus_{1\leqslant\alpha\leqslant N_p} \mathrm{Ext}^q(\bar{G}_\alpha^{(p+d)}, F)\otimes E_\alpha^{(p+d)} \tag{A4}$$

converging to $\mathrm{H}^{p+q}(F)$.

We apply the above spectral sequence to symmetric exceptional triples. Let $\mathcal{E} = (E^{(1)}, E^{(2)}, E^{(3)})$ be a symmetric exceptional triple on S. Put $(G^{(3)}, G^{(2)}, G^{(1)}) := \tau_1^R\tau_2^R\tau_1^R(\mathcal{E})$. Note the parity of $E^{(k)}$ and $G^{(k)}$ is the same. Then for $F \in \mathcal{D}$, we have a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q(G^{(p+3)}, F)\hat{\otimes} E^{(p+3)} \tag{A5}$$

converging to $\mathrm{H}^{p+q}(F)$, where

$$\mathrm{Ext}^q(G^{(k)}, F)\hat{\otimes}E^{(k)} = \begin{cases} \mathrm{Ext}^q(G^{(k)}, F)\otimes E^{(k)} & \text{if } G^{(k)} \text{ and } E^{(k)} \text{ are odd,} \\ \mathrm{Ext}^q(G^{(k)\prime}, F)\otimes E^{(k)\prime} & \text{if } G^{(k)} = G^{(k)\prime}\oplus G^{(k)\prime\prime} \\ \qquad\qquad\oplus & \text{and } E^{(k)} = E^{(k)\prime}\oplus E^{(k)\prime\prime} \\ \mathrm{Ext}^q(G^{(k)\prime\prime}, F)\otimes E^{(k)\prime\prime} & \text{are even.} \end{cases}$$

References

[A15] T. Abe, *Strange duality for height zero moduli spaces of sheaves on* $\mathbb{P}^2$, Michigan Math. J. **64**(3) (2015), 569–586.

[AB] D. Arcara and A. Bertram, *Bridgeland-stable moduli spaces for K-trivial surfaces*, J. Eur. Math. Soc. (JEMS) **15**(1) (2013), 1–38. With an appendix by Max Lieblich.

[ABCH] D. Arcara, A. Bertram, I. Coskun and J. Huizenga, *The minimal model program for the Hilbert scheme of points on* $\mathbb{P}$ *and Bridgeland stability*, Adv. Math. **235** (2013), 580–626.

[AM] D. Arcara and E. Miles, *Bridgeland stability of line bundles on surfaces*, J. Pure Appl. Algebra **220**(4) (2016), 1655–1677.

[BM] A. Bayer and E. Macr, *Projectivity and birational geometry of Bridgeland moduli spaces*, J. Amer. Math. Soc. **27**(3) (2014), 707–752.

[Bo] A. I. Bondal, *Representations of associative algebras and coherent sheaves*, Math. USSR-Izv. **34**(1) (1990), 23–42.

[BC] A. Bertram and I. Coskun, "*The birational geometry of the Hilbert scheme of points on surfaces*", in *Birational Geometry, Rational Curves, and Arithmetic*, Springer, New York, 2013, 15–55.

[BS] T. Bridgeland and D. Stern, *Helices on del Pezzo surfaces and tilting Calabi–Yau algebras*, Adv. Math. **224**(4) (2010), 1672–1716.

[CH1] I. Coskun and J. Huizenga, *The ample cone of moduli spaces of sheaves on the plane*, Algebr. Geom. **3**(1) (2016), 106–136.

[CH2] I. Coskun and J. Huizenga, *The nef cone of the moduli space of sheaves and strong Bogomolov inequalities*, preprint, arXiv:1512.02661.

[CHW] I. Coskun, J. Huizenga and M. Woolf, *The effective cone of the moduli space of sheaves on the plane*, J. Eur. Math. Soc. (JEMS), to appear.

[DW] H. Derksen and J. Weyman, *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients*, J. Amer. Math. Soc. **13**(3) (2000), 467–479.

[D] J. M. Drezet, *Fibrés exceptionnels et variétés de modules de faisceaux semi-stables sur* $\mathbb{P}_2(\mathbb{C})$, J. Reine Angew. Math. **380** (1987), 14–58.

[DL] J. M. Drezet and J. Le Potier, *Fibrés stables et fibrés exceptionnels sur* $\mathbb{P}_2$, Ann. Sci. Éc. Norm. Supér. (4) **18**(2) (1985), 193–243.

[GK] A. L. Gorodentsev and S. A. Kuleshov, *Helix theory*, Mosc. Math. J. **4**(2) (2004), 377–440.

[H] J. Huizenga, *Effective divisors on the Hilbert scheme of points in the plane and interpolation for stable bundles*, J. Algebraic Geom. **25**(1) (2016), 19–75.

[HL] D. Huybrechts and M. Lehn, "*The geometry of moduli spaces of sheaves*", in *Aspects of Mathematics*, E31. Friedr., Vieweg & Sohn, Braunschweig, 1997, xiv+269.

[Ka] B. V. Karpov, *Semistable sheaves on a two-dimensional quadric and Kronecker modules (in Russian)*, Izv. Ross. Akad. Nauk Ser. Mat. **56**(1) (1992), 38–74; translation in Russian Acad. Sci. Izv. Math. **40**(1) (1993), 33–66.

[Ku] S. A. Kuleshov, *On the moduli variety of stable bundles on a quadric (in Russian)*, Mat. Zametki **62**(6) (1997), 843–864; English translation in Math. Notes **62**(5–6) (1997), 707–725.

[KO] S. A. Kuleshov and D. O. Orlov, *Exceptional sheaves on Del Pezzo surfaces*, Izv. Math. **44**(3) (1995), 479–513.

[Le] J. Le Potier, "*Lectures on vector bundles*", Translated by A. Maciocia. Cambridge Studies in Advanced Mathematics, **54**, Cambridge University Press, Cambridge, 1997.

[LZ] C. Li and X. Zhao, *Birational models of moduli spaces of coherent sheaves on the projective plane*, preprint, arXiv:1603.05035.

[Oh] R. Ohkawa, *Moduli of Bridgeland semistable objects on* $\mathbb{P}^2$, Kodai Math. J. **33**(2) (2010), 329–366.

[R89] A. N. Rudakov, *Exceptional vector bundles on a quadric*, Math. USSR-Izv. **33**(1) (1989), 115–138.

[R94] A. N. Rudakov, *A description of Chern classes of semistable sheaves on a quadric surface*, J. Reine Angew. Math. **453** (1994), 113–135.

[Ry] T. Ryan, *The effective cone of moduli spaces of moduli spaces of sheaves on* $\mathbb{P}^1 \times \mathbb{P}^1$, arXiv:1607.07114.

[Y] K. Yoshioka, "*Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface*", in *Developement of Moduli Theory (Kyoto 2013)*, Adv. Stud. Pure Math. **69**, Math. Soc. Japan, Tokyo, 2016, 473–537.

Graduate School of Science and Technology
Kumamoto University
2-39-1 Kurokami
Kumamoto 860-8555
Japan
abeken@sci.kumamoto-u.ac.jp