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STABILITY OF GORENSTEIN FLAT CATEGORIES

GANG YANG
School of Mathematics, Physics and Software Engineering, Lanzhou Jiaotong University, Lanzhou 730070*, P.R. China*
e-mail: yanggang@mail.lzjtu.cn

and ZHONGKUI LIU
College of Mathematics and Information Science, Northwest Normal University, Lanzhou 730070*, P.R. China*
e-mail: liuzk@nwnu.edu.cn

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Abstract. A left R-module M is called two-degree Gorenstein flat if there exists an exact sequence of Gorenstein flat left R-modules $\cdots \to G_2 \to G_1 \to G_0 \to G_{-1} \to G_{-2} \to \cdots$ such that $M \cong \mathrm{Ker}(G_0 \to G_{-1})$ and it remains exact after applying $H \otimes_R -$ for any Gorenstein injective right R-module H. In this paper we first give some characterisations of Gorenstein flat objects in the category of complexes of modules and then use them to show that two notions of the two-degree Gorenstein flat and the Gorenstein flat left R-modules coincide when R is right coherent.

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1. Introduction. Let R be an associative ring. By Enochs and Jenda [**10**], an R-module M is called *Gorenstein injective* if there exists an exact sequence of injective R-modules

$$I = \cdots \xrightarrow{\delta_2^I} I_1 \xrightarrow{\delta_1^I} I_0 \xrightarrow{\delta_0^I} I_{-1} \xrightarrow{\delta_{-1}^I} \cdots$$

with $M = \mathrm{Ker}(\delta_0^I)$ such that the complex $\mathrm{Hom}_R(J, I)$ is exact for any injective R-module J. Dually, an R-module L is called *Gorenstein projective* if there exists an exact sequence of projective R-modules

$$P = \cdots \xrightarrow{\delta_2^P} P_1 \xrightarrow{\delta_1^P} P_0 \xrightarrow{\delta_0^P} P_{-1} \xrightarrow{\delta_{-1}^P} \cdots$$

with $L = \mathrm{Ker}(\delta_0^P)$ such that the complex $\mathrm{Hom}_R(P, Q)$ is exact for any projective R-module Q. Later, Enochs and co-authors [**9**] introduced the *Gorenstein flat left* R-modules, which are the modules of the form $\mathrm{Ker}(\delta_0^F)$ for some exact sequence of flat left R-modules

$$F = \cdots \xrightarrow{\delta_2^F} F_1 \xrightarrow{\delta_1^F} F_0 \xrightarrow{\delta_0^F} F_{-1} \xrightarrow{\delta_{-1}^F} \cdots$$

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such that the complex $J \otimes_R F$ is exact for any injective right R-module J, and so a relative homological theory of Gorenstein modules was initiated (see [**2**–**5**, **12**, **16**–**18**]). Recently, Sather-Wagstaff et al. [**21**] introduced modules that we call 'two-degree Gorenstein projective modules': An R-module N is *two-degree Gorenstein projective* if there exists an exact sequence of the Gorenstein projective R-modules

$$G = \cdots \xrightarrow{\delta_2^G} G_1 \xrightarrow{\delta_1^G} G_0 \xrightarrow{\delta_0^G} G_{-1} \xrightarrow{\delta_{-1}^G} \cdots$$

with $N \cong \mathrm{Ker}(\delta_0^G)$ such that the two complexes $\mathrm{Hom}_R(H, G)$ and $\mathrm{Hom}_R(G, H)$ are exact for any Gorenstein projective R-module H. They proved that any two-degree Gorenstein projective module is nothing but a Gorenstein projective module when R is commutative (Theorem A in [**21**]). Also, a similar notion was introduced and studied by Sather-Wagstaff et al. in [**22**]. Inspired by their work, in this paper, we introduce and investigate the notion of the so-called *two-degree Gorenstein flat* modules, which is different from the one given in [**22**], defined as being the modules isomorphic to the form $\mathrm{Ker}(\delta_0^G)$ for some exact sequence of the Gorenstein flat R-modules

$$G = \cdots \xrightarrow{\delta_2^G} G_1 \xrightarrow{\delta_1^G} G_0 \xrightarrow{\delta_0^G} G_{-1} \xrightarrow{\delta_{-1}^G} \cdots$$

such that the complex $H \otimes_R G$ is exact for any Gorenstein injective right R-module H. We show that over a left GF-closed ring R (see Definition 3.7), any two-degree Gorenstein flat left R-module is exactly a Gorenstein flat module. In fact the proof of this result is based on the characterisations of Gorenstein flatness of objects in the category of complexes of modules. The class of left GF-closed rings strictly contains the class of all right coherent rings (see [**2**]).

The current paper is organised as follows:

Section 2 provides some relevant definitions and notations that will be used throughout the paper. In Section 3 we give some characterisations of the Gorenstein flat complexes of modules. We show that if R is a left GF-closed ring, then a complex X is Gorenstein flat if and only if the modules X_i are Gorenstein flat for all $i \in \mathbb{Z}$. Hence, the Gorenstein flatness of complexes of modules is completely determined by the Gorenstein flatness of all its terms. As an immediate consequence of this result, we see that the Gorenstein flat objects in the category complexes possess many properties as the Gorenstein flat modules in the category of modules. In Section 4 we give an application of the characterisations of Gorenstein flat complexes to show that over a left GF-closed ring R (hence, over any right coherent ring), a two-degree Gorenstein flat left R-module is exactly a Gorenstein flat module.

2. Preliminaries. Throughout, let R be an associative ring with 1, R-Mod (respectively, Mod-R) be the category of left (respectively, right) R-modules and R-Comp (respectively, Comp-R) be the category of complexes of left (respectively, right) R-modules. Unless stated otherwise, an R-module (respectively, R-complex) will be understood to be a left R-module (respectively, a complex of left R-modules). For two R-modules M and N, we will let $\mathrm{Hom}_R(M, N)$ denote the group of morphisms from M to N in the category R-Mod.

To every complex

$$C =: \cdots \longrightarrow C_{m+1} \xrightarrow{\delta_{m+1}^C} C_m \xrightarrow{\delta_m^C} C_{m-1} \xrightarrow{\delta_{m-1}^C} C_{m-2} \xrightarrow{\delta_{m-2}^C} \cdots$$

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we associate the numbers

$$\sup C = \sup\{\, l \mid C_l \neq 0\} \text{ and } \inf C = \inf\{\, l \mid C_l \neq 0\}.$$

The complex C is called *bounded left* when $\sup C < \infty$, *bounded right* when $\inf C > -\infty$ and *bounded* when it is bounded left and right. The mth *cycle module* is defined as $\mathrm{Ker}(\partial_m^C)$ and is denoted $Z_m C$. The mth *cycle module* of C is defined as $\mathrm{Ker}(\delta_m^C)$ and is denoted $Z_m C$, the mth *boundary module* is defined as $\mathrm{Im}(\delta_{m+1}^C)$ and is denoted $B_m C$.

In this paper, we will use both subscripts and superscripts to distinguish complexes. So if C_α and C^β are two complexes, then C_α will be $\cdots \to (C_\alpha)_{m+1} \to (C_\alpha)_m \to (C_\alpha)_{m-1} \to (C_\alpha)_{m-2} \to \cdots$, and C^β will be $\cdots \to C^\beta_{m+1} \to C^\beta_m \to C^\beta_{m-1} \to C^\beta_{m-2} \to \cdots$. If K is an R-module, then we will denote by $D^i(K)$ the complex

$$\cdots \longrightarrow 0 \longrightarrow K \xrightarrow{id} K \longrightarrow 0 \longrightarrow \cdots$$

with K in the i and i-1st degrees and $S^i(K)$ the complex

$$\cdots \longrightarrow 0 \longrightarrow K \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

with K in the ith degree. Given a complex C, for each $i \in \mathbb{Z}$, let $\Sigma^i C$ denote the complex with $(\Sigma^i C)_m = C_{m-i}$ and $\delta_m^{\Sigma^i C} = (-1)^i \delta_{m-i}^C$.

Given two complexes M and N, a homomorphism $\varphi : M \longrightarrow N$ of degree m is a family $(\varphi_i)_{i\in\mathbb{Z}}$ of homomorphisms of R-modules $\varphi_i : M_i \longrightarrow N_{i+m}$. All such homomorphisms form an abelian group, denoted as $\mathcal{H}om(M, N)_m$; it is clearly isomorphic to $\prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+m})$. We will let $\mathcal{H}om(M, N)$ denote the complex of $\mathbb{Z}$-modules with mth entry $\mathcal{H}om(M, N)_m$ and boundary map

$$(\delta_m(\varphi))_i = \delta^N_{i+m}\varphi_i - (-1)^m \varphi_{i-1}\delta^M_i.$$

A homomorphism $\varphi \in \mathcal{H}om(M, N)_m$ is called a *chain map* if $\delta_m(\varphi) = 0$, i.e. if

$$\delta^N_{i+m}\varphi_i = (-1)^m \varphi_{i-1}\delta^M_i \text{ for all } i \in \mathbb{Z}.$$

A chain map of degree 0 is called a *morphism*. We would let $\mathrm{Hom}(M, N)$ denote the set of all morphisms from M to N. Note that the category of complexes of modules has enough projectives and injectives. This can be seen from the fact that any complex of the form $\cdots \to 0 \to K \to K \to 0 \to \cdots$ with K projective (injective) is projective (injective). This, in turn, follows from the fact that a complex C is projective (respectively, injective) in the category of complexes of modules if and only if it is exact, and the modules $Z_i C$ are projective (respectively, injective) for all $i \in \mathbb{Z}$ (see **[13]**). We will let the $\mathrm{Ext}^i(M, N)$ for all $i \geq 1$ denote the groups we get from the right-derived functors of $\mathrm{Hom}(-, -)$. Recall that a complex P is called *semi-projective* (i.e. *dg-projective*) if the modules P_m are projective for all $m \in \mathbb{Z}$, and $\mathcal{H}om(P, E)$ is exact for any exact complex E (see **[11]**). In fact if P is a bounded right complex of projective modules, then P is semi-projective. It is also shown in **[13]** that a complex P is semi-projective if and only if $\mathrm{Ext}^1(P, E) = 0$ for any exact complex E.

If M is a complex of right R-modules and N is a complex of left R-modules, then their tensor product $M \otimes^{\cdot} N$ is defined by $(M \otimes^{\cdot} N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j$ in degree n, the boundary map δ_n is defined on the generators by $\delta^M(x) \otimes y + (-1)^{|x|} x \otimes \delta^N(y)$, where $|x|$ is the degree of the element x. One can easily check that $\delta_{n-1}\delta_n = 0$ for all $n \in \mathbb{Z}$ (and this would not be true if we did not introduce the sign $(-1)^{|x|}$). Let $M \overline{\otimes} N$

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$= \frac{(M\otimes^{\cdot} N)}{B(M\otimes^{\cdot} N)}$, that is, $M\overline{\otimes}N$ is the complex of abelian groups with nth entry $(M\overline{\otimes}N)_n = \frac{(M\otimes^{\cdot} N)_n}{B_n(M\otimes^{\cdot} N)}$ and boundary map

$$\frac{(M\otimes^{\cdot} N)_n}{B_n(M\otimes^{\cdot} N)} \longrightarrow \frac{(M\otimes^{\cdot} N)_{n-1}}{B_{n-1}(M\otimes^{\cdot} N)}$$

given by $\overline{x\otimes y} \to \overline{\delta^M(x)\otimes y}$, where $\overline{x\otimes y}$ is used to denote the coset in $\frac{(M\otimes^{\cdot} N)_n}{B_n(M\otimes^{\cdot} N)}$. This gives us a bifunctor

$$-\overline{\otimes}- : \text{Comp-}R \times R\text{-Comp} \longrightarrow \text{Comp-}\mathbb{Z},$$

where Comp-$\mathbb{Z}$ denotes the category of complexes of abelian groups. For a complex M (respectively, N) of right (respectively, left) R-modules, the functor $M\overline{\otimes}-$ (respectively, $-\overline{\otimes}N$) is right exact. We can construct left-derived functors, which are denoted by $\overline{\text{Tor}}_i(M,-)$ (respectively, $\overline{\text{Tor}}_i(-,N)$). These functors in the category of complexes of modules possess many nice properties analogous to $\text{Tor}_i^R(-,-)$ in the category of R-modules, where $\text{Tor}_i^R(-,-)$ denote left-derived functors of $-\otimes_R-$ in the category of R-modules. Consult references [**7**, **14**] for more details.

3. Gorenstein flat complexes. In this section we investigate the Gorenstein flatness of complexes of left R-modules. The notion of the Gorenstein flat complex was introduced and studied by Enochs and García Rozas [**7**]. They proved that if R is an *n-Gorenstein ring* (i.e. R is a two-side Noetherian ring with both left and right self-injective dimensions at most n for some non-negative integer n), then any complex X is Gorenstein flat in the category of complexes if and only if each component X_m is Gorenstein flat in the category of modules (Theorem 4.3 in [**7**]).

Recall from Definition 4.1.2 in [**14**] that a complex F is called *flat* if F is exact and each Z_iF is flat for all $i\in\mathbb{Z}$. In fact, a complex F is flat if and only if $\overline{\text{Tor}}_1(X,F)=0$ for any complex X (Lemma 5.4.1 in [**14**]).

We continue with the following definition from [**7**].

Definition 3.1. We call a complex X Gorenstein flat if there exists an exact sequence of flat complexes

$$\mathscr{F} =: \ \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $X = \text{Ker}(F^0 \longrightarrow F^1)$ and which remains exact after applying $I\overline{\otimes}-$ for any injective complex I of right R-modules. In this case, $\mathscr{F}$ is said to be a complete flat resolution of X.

Remark 3.2. Let M be a right R-module, and N a left R-module. Then, naturally, M and N can be taken as the complexes $S^0(M)$ and $S^0(N)$, respectively. Clearly, $S^0(M)\overline{\otimes}S^0(N) = S^0(M)\otimes^{\cdot} S^0(N) = S^0(M\otimes_R N)$, and thus if we substitute the complexes in Definition 3.1 with corresponding modules, then we get the definition of Gorenstein flat modules.

Remark 3.3. (1) Note that if F is a flat complex, then the sequence, $\cdots\to 0\to F\to F\to 0\to\cdots$, is a complete flat resolution of F, and so F is Gorenstein flat.

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(2) Since the class of flat complexes is closed under direct sums, and the functor $\overline{\otimes}$ commutes with sums, we get that the class of Gorenstein flat complexes is closed under direct sums.

(3) If $\mathscr{F} =: \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ is a complete flat resolution of a complex $X = \mathrm{Ker}(F^0 \to F^1)$, then by symmetry, all the images, the kernels and the cokernels of $\mathscr{F}$ are Gorenstein flat in R-Comp.

(4) If X is a Gorenstein flat complex, then by definition, there is an exact sequence of flat complexes $\cdots \to F_2 \to F_1 \to F_0 \to X \to 0$, which remains exact after applying $I\overline{\otimes}-$ for any injective complex I of right R-modules, and so $\overline{\mathrm{Tor}}_j(I, X) = 0$ for all injective complexes I of right R-modules and all $j \geq 1$.

Lemma 3.4. *Let X be a complex. If X_m is Gorenstein flat for all $m \in \mathbb{Z}$, then $\overline{\mathrm{Tor}}_1(I, X) = 0$ for all injective complexes I of right R-modules.*

Proof. Since a complex I of right R-modules is injective if and only if I is exact and all Z_mI, $m \in \mathbb{Z}$ are injective right R-modules, we get that any injective complex has the form $\bigoplus_{i\in\mathbb{Z}} D^i(J_i)$ with each J_i injective in Mod-R. On the other hand, for any injective right R-module J, we get from Example 4.1 in [7] that $\overline{\mathrm{Tor}}_1(D^i(J), X) = 0$ for all $i \in \mathbb{Z}$, since each X_m is Gorenstein flat for all $m \in \mathbb{Z}$. Thus, if $I = \bigoplus_{i\in\mathbb{Z}} D^i(J_i)$ is an injective complex with each J_i injective in Mod-R, then $\overline{\mathrm{Tor}}_1(I, X) = \overline{\mathrm{Tor}}_1(\bigoplus_{i\in\mathbb{Z}} D^i(J_i), X) \cong \bigoplus_{i\in\mathbb{Z}} \overline{\mathrm{Tor}}_1(D^i(J_i), X) = 0$, as desired. □

Recall that a complex X is called *semi-flat* if X_m is flat in R-Mod for all $m \in \mathbb{Z}$ and $E \otimes^{\cdot} X$ is exact for any exact complex E of right modules. It is shown in Lemma 5.4.1(c) in [**14**] that a complex X is semi-flat if and only if $\overline{\mathrm{Tor}}_i(E, X) = 0$ for any exact complex E of right modules and all $i \geq 1$. A complex X is called *graded flat* if X_m is flat in R-Mod for all $m \in \mathbb{Z}$. The semi-flat and graded flat objects are very important in characterising homological dimensions of complexes and modules, see [**1**] and [**20**] (the semi-flat and graded flat complexes, respectively, is called dg-flat and #-flat in [**1**]). Since the class of flat modules is closed under extensions, every flat complex is graded flat, but in general a graded flat complex may not be flat, for instance, the complex, $\cdots \to 0 \to F \to 0 \to \cdots$, is graded flat complex but not flat when F is some flat module. The following proposition extends Proposition 3.10 in [**15**] and implies that there are abundant Gorenstein flat objects that are not flat in the category of complexes.

Proposition 3.5. *Every graded flat complex G is Gorenstein flat. In particular, every semi-flat complex is Gorenstein flat.*

Proof. If we define the morphism $\alpha : G \longrightarrow \oplus_{i\in\mathbb{Z}} D^{i+1}(G_i) = F^0$ of complexes as the following:

$$\begin{array}{ccccccccccc}
G =: \cdots & \longrightarrow & G_{i+2} & \xrightarrow{\delta^G_{i+2}} & G_{i+1} & \xrightarrow{\delta^G_{i+1}} & G_i & \xrightarrow{\delta^G_i} & G_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle (1_{G_{i+2}}, \delta^G_{i+2})} & & \downarrow{\scriptstyle (1_{G_{i+1}}, \delta^G_{i+1})} & & \downarrow{\scriptstyle (1_{G_i}, \delta^G_i)} & & \downarrow{\scriptstyle (1_{G_{i-1}}, \delta^G_{i-1})} & & \\
F^0 =: \cdots & \longrightarrow & G_{i+2} \oplus G_{i+1} & \longrightarrow & G_{i+1} \oplus G_i & \longrightarrow & G_i \oplus G_{i-1} & \longrightarrow & G_{i-1} \oplus G_{i-2} & \longrightarrow & \cdots
\end{array}$$

that is, $\alpha_i = (1_{G_i}, \delta^G_i)$ for each $i \in \mathbb{Z}$, then we get a short exact sequence of complexes

$$0 \longrightarrow G \xrightarrow{\alpha} F^0 \longrightarrow K^1 \longrightarrow 0$$

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with F^0 flat. In fact, $K^1 = \mathrm{Coker}(\alpha)$ is graded flat. To see this, for any $i \in \mathbb{Z}$, consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_i & \longrightarrow & G_i \oplus G_{i-1} & \longrightarrow & (K^1)_i & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \pi_i} & & \downarrow & & \\ 0 & \longrightarrow & G_i & = & G_i & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

where $\pi_i : G_i \oplus G_{i-1} \longrightarrow G_i$ is the canonical projection. By the snake lemma, we get that $(K^1)_i \cong G_{i-1}$ is flat. Note that the short exact sequence

$$0 \longrightarrow G \xrightarrow{\alpha} F^0 \longrightarrow K^1 \longrightarrow 0$$

is still exact after applying $I \overline{\otimes} -$ for any injective complex I of right R-modules, since $\overline{\mathrm{Tor}}_1(I, K^1) = 0$ by Lemma 3.4, and K^1 has the same properties as G. Then, we can use the same procedure to construct an exact sequence of flat complexes

$$0 \longrightarrow G \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots, \qquad (\dagger)$$

which remains exact when the functor $I \overline{\otimes} -$ is applied for it for any injective complex I of right R-modules. Again take

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0 \qquad (\ddagger)$$

as a flat resolution of G, which is still exact when $I \overline{\otimes} -$ is applied for it for any injective complex I of right R-modules, since it is easy to check that each $\mathrm{Ker}(F_i \longrightarrow F_{i-1})$ is again graded flat for all $i \geq 0$, where $F_{-1} = G$. Assembling the sequences (†) and (‡), we get a complete flat resolution of G, and so G is Gorenstein flat.

Clearly, every semi-flat complex is graded flat, and so it is Gorenstein flat. □

The following result provides the characterisation of Gorenstein flat complexes by using graded flat complexes.

THEOREM 3.6. *Let X be a complex. Then X is Gorenstein flat if and only if there exists an exact sequence of graded flat complexes*

$$\mathscr{H} =: \cdots \longrightarrow H_1 \longrightarrow H_0 \longrightarrow H^0 \longrightarrow H^1 \longrightarrow \cdots$$

with $X = \mathrm{Ker}(H^0 \longrightarrow H^1)$ and which remains exact after applying $I \overline{\otimes} -$ for any injective complex I of right R-modules.

Proof. The necessity follows from the fact that every flat complex is graded flat. For sufficiency, split the sequence $\mathscr{H}$ as follows:

$$0 \longrightarrow X \longrightarrow H^0 \longrightarrow X^1 \longrightarrow 0 \qquad (\dagger^1)$$

$$0 \longrightarrow X^1 \longrightarrow H^1 \longrightarrow X^2 \longrightarrow 0 \qquad (\dagger^2)$$

$$\cdots\cdots$$

$$0 \longrightarrow X^i \longrightarrow H^i \longrightarrow X^{i+1} \longrightarrow 0 \qquad (\dagger^{i+1})$$

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$$\cdots\cdots$$

and

$$0 \longrightarrow X_1 \longrightarrow H_0 \longrightarrow X \longrightarrow 0 \qquad (\dagger_1)$$

$$0 \longrightarrow X_2 \longrightarrow H_1 \longrightarrow X_1 \longrightarrow 0 \qquad (\dagger_2)$$

$$\cdots\cdots$$

$$0 \longrightarrow X_{i+1} \longrightarrow H_i \longrightarrow X_i \longrightarrow 0 \qquad (\dagger_{i+1})$$

$$\cdots\cdots$$

Since H^0 is graded flat, by the proof of Proposition 3.5, there is an exact sequence

$$0 \longrightarrow H^0 \longrightarrow F^0 \longrightarrow G^1 \longrightarrow 0$$

with F^0 flat and G^1 graded flat. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & H^0 & \longrightarrow & X^1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & F^0 & \longrightarrow & U^1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G^1 & = & G^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By assumption, the sequence $(\dagger^1)$ remains exact after applying $I\overline{\otimes}-$ for any injective complex I of right R-modules. Hence, it is easily seen that $\overline{\mathrm{Tor}}_1(I, X^1) = 0$ for any injective complex I of right R-modules, and so the long exact sequence lemma yields that $\overline{\mathrm{Tor}}_1(I, U^1) = 0$ for such I. Thus, the sequence

$$0 \longrightarrow X \longrightarrow F^0 \longrightarrow U^1 \longrightarrow 0$$

is still exact when $I\overline{\otimes}-$ is applied for it for any injective complex I of right R-modules. Now consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X^1 & \longrightarrow & H^1 & \longrightarrow & X^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U^1 & \longrightarrow & V^1 & \longrightarrow & X^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G^1 & = & G^1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Visibly, V^1 is graded flat since G^1 and H^1 are so. Thus, by the proof of Proposition 3.5 there is an exact sequence

$$0 \longrightarrow V^1 \longrightarrow F^1 \longrightarrow G^2 \longrightarrow 0$$

with F^1 flat and G^2 graded flat. Again, consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U^1 & \longrightarrow & V^1 & \longrightarrow & X^2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U^1 & \longrightarrow & F^1 & \longrightarrow & U^2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G^2 & = & G^2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It is easy to see that the sequence

$$0 \longrightarrow U^1 \longrightarrow F^1 \longrightarrow U^2 \longrightarrow 0$$

is still exact when $I\overline{\otimes}-$ is applied for it for any injective complex I of right R-modules. Assemble the sequences $0 \longrightarrow X \longrightarrow F^0 \longrightarrow U^1 \longrightarrow 0$ and

$$0 \longrightarrow U^1 \longrightarrow F^1 \longrightarrow U^2 \longrightarrow 0 .$$

Then we have an exact sequence

$$0 \longrightarrow X \longrightarrow F^0 \longrightarrow F^1 \longrightarrow U^1 \longrightarrow 0,$$

which remains exact when $I\overline{\otimes}-$ is applied for it for any injective complex I of right R-modules. Inductively, we can get an exact sequence

$$0 \longrightarrow X \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots, \qquad (*)$$

which remains exact when $I\overline{\otimes}-$ is applied for it for any injective complex I of right R-modules.

Using the method dual to the above, we can get an exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow X \longrightarrow 0, \qquad (**)$$

which remains exact when $I\overline{\otimes}-$ is applied for it for any injective complex I of right R-modules. Assembling the sequences $(*)$ and $(**)$, we get a complete flat resolution of X, this proves that X is Gorenstein flat. Thus, the result follows. □

In [**2**], Bennis introduced the notion of a left GF-closed ring and studied Gorenstein flat modules over such rings.

DEFINITION 3.7. ([**2**]) A ring R is called left *GF-closed* if the class of the Gorenstein flat left R-modules is closed under extensions, i.e. if $0 \to X \to Y \to Z \to 0$ is an exact sequence with X and Z Gorenstein flat modules, then Y is also Gorenstein flat. A right GF-closed ring can be defined similarly.

Bennis showed in [**2**] that all right coherent rings and all rings with finite weak dimension are left GF-closed. Also, the class of left GF-closed rings includes strictly the one of the right coherent rings and the one of the rings with finite weak dimension.

Let $\mathcal{X}$ be a class of modules. Following [**17**], the class $\mathcal{X}$ is called *projectively resolving* if all projective modules are contained in $\mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X \in \mathcal{X}$ and $X' \in \mathcal{X}$ are equivalent. Bennis proved (Theorem 2.3 in [**2**]) that a ring R is left GF-closed if and only if the class of the Gorenstein flat left R-modules is projectively resolving.

LEMMA 3.8. *Let R be a left GF-closed ring,* $0 \longrightarrow M \stackrel{f}{\longrightarrow} F \stackrel{g}{\longrightarrow} N \longrightarrow 0$ *be a short exact sequence of R-modules. If N is Gorenstein flat and F is flat, then* $\mathrm{Coker}(\alpha)$ *is Gorenstein flat for any homomorphism $f' : M \longrightarrow F'$ with F' flat, where $\alpha = (f, f') : M \longrightarrow F \oplus F'$.*

Proof. Suppose $f' : M \longrightarrow F'$ is any homomorphism with F' flat. Then the sequence $0 \longrightarrow M \stackrel{\alpha}{\longrightarrow} F \oplus F' \longrightarrow \mathrm{Coker}(\alpha) \longrightarrow 0$ is exact. By the factor lemma, there is a homomorphism $\mu : \mathrm{Coker}(\alpha) \longrightarrow N$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \stackrel{\alpha}{\longrightarrow} & F \oplus F' & \longrightarrow & \mathrm{Coker}(\alpha) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mu} & & \\ 0 & \longrightarrow & M & \stackrel{f}{\longrightarrow} & F & \stackrel{g}{\longrightarrow} & N & \longrightarrow & 0 \end{array}$$

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where $\pi : F \oplus F' \longrightarrow F$ is the canonical projection. By the five lemma, we get that μ is an epimorphism, and $\mathrm{Ker}(\mu) \cong \mathrm{Ker}(\pi) = F'$ is flat by the snake lemma. Since R is left GF-closed, the short exact sequence $0 \longrightarrow \mathrm{Ker}(\mu) \longrightarrow \mathrm{Coker}(\alpha) \xrightarrow{\mu} N \longrightarrow 0$ yields that $\mathrm{Coker}(\alpha)$ is Gorenstein flat. □

LEMMA 3.9. *Let R be a left GF-closed ring, and X be a complex. If X_i is Gorenstein flat in R-Mod for all $i \in \mathbb{Z}$, then X is Gorenstein flat.*

Proof. Since each X_i is Gorenstein flat, there exists a short exact sequence of modules $0 \longrightarrow X_i \xrightarrow{f_i} H_i \longrightarrow Y_i \longrightarrow 0$ such that H_i is flat and Y_i is Gorenstein flat for each $i \in \mathbb{Z}$. If we define the morphism $\alpha : X \longrightarrow \oplus_{i \in \mathbb{Z}} D^{i+1}(H_i) = F^0$ of complexes as the following:

$$\begin{array}{ccccccccc}
X =: \cdots \longrightarrow & X_{i+2} & \xrightarrow{\delta^X_{i+2}} & X_{i+1} & \xrightarrow{\delta^X_{i+1}} & X_i & \xrightarrow{\delta^X_i} & X_{i-1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle (f_{i+2}, f_{i+1}\delta^X_{i+2})} & & \downarrow{\scriptstyle (f_{i+1}, f_i\delta^X_{i+1})} & & \downarrow{\scriptstyle (f_i, f_{i-1}\delta^X_i)} & & \downarrow{\scriptstyle (f_{i-1}, f_{i-2}\delta^X_{i-1})} & \\
F^0 =: \cdots \longrightarrow & H_{i+2} \oplus H_{i+1} & \longrightarrow & H_{i+1} \oplus H_i & \longrightarrow & H_i \oplus H_{i-1} & \longrightarrow & H_{i-1} \oplus H_{i-2} & \longrightarrow \cdots
\end{array}$$

that is, $\alpha_i = (f_i, f_{i-1}\delta^X_i)$ for each $i \in \mathbb{Z}$, then we get that an exact sequence

$$0 \longrightarrow X \xrightarrow{\alpha} F^0 \longrightarrow K^1 \longrightarrow 0$$

with F^0 flat and $K^1 = \mathrm{Coker}(\alpha)$. It follows from Lemma 3.8 that each $(K^1)_i = \mathrm{Coker}(\alpha_i)$ is Gorenstein flat. Thus, the above short exact sequence is still exact after applying $I \overline{\otimes} -$ for any injective complex I of right R-modules since $\overline{\mathrm{Tor}}_1(I, K^1) = 0$ by Lemma 3.4. Note that K^1 has the same properties as X. Then, we can use the same procedure to construct an exact sequence of flat complexes

$$0 \longrightarrow X \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots, \qquad (\flat)$$

which remains exact when the functor $I \overline{\otimes} -$ is applied for it for any injective complex I of right R-modules.

Suppose that the sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow X \longrightarrow 0 \qquad (\flat\flat)$$

is a flat resolution of X. Since R is left GF-closed, the class of all Gorenstein flat modules is projectively resolving. Then it is easy to see that each $K_i = \mathrm{Ker}(F_i \longrightarrow F_{i-1})$ has the same properties as X for all $i \geq 0$, where $F_{-1} = X$. Thus, we get from Lemma 3.4 that the sequence ($\flat\flat$) is still exact when $I \overline{\otimes} -$ is applied for it for any injective complex I of right R-modules. Assembling the sequences ($\flat$) and ($\flat\flat$), we get a complete flat resolution of X, so X is Gorenstein flat. □

LEMMA 3.10. *If X is a Gorenstein flat complex, then each X_m is a Gorenstein flat module for $m \in \mathbb{Z}$.*

Proof. See the proof of Theorem 4.3 in [7]. □

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The following result generalises Theorem 4.3 in [7], which was proved for Gorenstein rings.

THEOREM 3.11. *Let R be a left GF-closed ring, and X be a complex. Then X is Gorenstein flat if and only if X_i is Gorenstein flat for all $i \in \mathbb{Z}$.*

Proof. Use Lemma 3.9 and Lemma 3.10. □

COROLLARY 3.12. *Let R be a right coherent ring, and X be a complex. Then X is Gorenstein flat if and only if X_i is Gorenstein flat for all $i \in \mathbb{Z}$.*

Proof. Use Theorem 3.11 and Proposition 2.2 in [2]. □

It is easily seen in Theorem 3.11 that when R is left GF-closed then the Gorenstein flat objects in the category complexes possess many properties as the Gorenstein flat modules in the category of modules. For example, the class of all Gorenstein flat complexes of modules is closed under direct limits by Lemma 3.1 in [23] and under extensions, any direct summand of a Gorenstein flat complex is Gorenstein flat by Corollary 2.6 in [2].

Let C be a complex. We define the Gorenstein flat dimension, $\mathrm{Gfd}(C)$ of C as $\mathrm{Gfd}(C) = \inf\{n|$ there exists an exact sequence $0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to C \to 0$ with each X_i Gorenstein flat$\}$. If no such n exists, set $\mathrm{Gfd}(C) = \infty$. Details and results on Gorenstein flat dimension of modules or of complexes appeared in [**2**, **5**, **6**, **11**, **17**–**19**].

THEOREM 3.13. *Let R be a left GF-closed ring and C be a complex. Then $Gfd(C) = sup\{Gfd(C_i)|i \in \mathbb{Z}\}$.*

Proof. We begin by showing that $\mathrm{Gfd}(C) \leq \sup\{\mathrm{Gfd}(C_i)|i \in \mathbb{Z}\}$. If $\sup\{\mathrm{Gfd}(C_i)|i \in \mathbb{Z}\} = \infty$, then $\mathrm{Gfd}(C) \leq \sup\{\mathrm{Gfd}(C_i)|i \in \mathbb{Z}\}$. So naturally we may assume that $\sup\{\mathrm{Gfd}(C_i)|i \in \mathbb{Z}\} = n$ is finite. Consider a partial flat resolution

$$0 \longrightarrow K_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

of C, where each F_j is flat. Then $(K_n)_i$ is Gorenstein flat for all $i \in \mathbb{Z}$ by Theorem 2.8 in [2]. Now, by Theorem 3.11, K_n is a Gorenstein flat complex. This shows that $\mathrm{Gfd}(C) \leq \sup\{\mathrm{Gfd}(C_i)|i \in \mathbb{Z}\}$.

Next, we will show that $\sup\{\mathrm{Gfd}(C_i)|i \in \mathbb{Z}\} \leq \mathrm{Gfd}(C)$. Naturally, we may assume that $\mathrm{Gfd}(C) = n$ is finite. Then there exists an exact sequence of complexes

$$0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow C \longrightarrow 0$$

with each X_j Gorenstein flat. By Theorem 3.11, we get that $(X_j)_i$ is Gorenstein flat for all $i \in \mathbb{Z}$ and all $j = 0, 1, \cdots, n$. Thus, $\mathrm{Gfd}(C_i) \leq n$ for all $i \in \mathbb{Z}$, and so $\sup\{\mathrm{Gfd}(C_i)|i \in \mathbb{Z}\} \leq n = \mathrm{Gfd}(C)$. □

4. Stability of Gorenstein flat categories of modules. In this section, we give an application of the characterisations of Gorenstein flat complexes in the former section and show that an iteration of the procedure used to define the Gorenstein flat modules over a left GF-closed ring R yields exactly the Gorenstein flat modules. We are inspired by Sather-Wagstaff et al. [**21, 22**] to introduce the following definition.

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Definition 4.1. An R-module M is said to be two-degree Gorenstein flat if there exists an exact sequence of Gorenstein flat R-modules

$$G = \cdots \xrightarrow{\delta_2^G} G_1 \xrightarrow{\delta_1^G} G_0 \xrightarrow{\delta_0^G} G_{-1} \xrightarrow{\delta_{-1}^G} \cdots$$

such that the complex $H \otimes_R G$ is exact for each Gorenstein injective right R-module H and $M \cong \mathrm{Ker}(\delta_0^G)$.

We denote the class of all two-degree Gorenstein flat R-modules by $\mathcal{G}^2(\mathcal{F}(R))$, and denote the class of all Gorenstein flat R-modules by $\mathcal{G}(\mathcal{F}(R))$. Let $X \in \mathcal{G}(\mathcal{F}(R))$. Then using a resolution of the form $\cdots \to 0 \to X \to X \to 0 \to \cdots$, one sees that $\mathcal{G}(\mathcal{F}(R)) \subseteq \mathcal{G}^2(\mathcal{F}(R))$. Our aim is to prove that the containment $\mathcal{G}(\mathcal{F}(R)) \subseteq \mathcal{G}^2(\mathcal{F}(R))$ is always an equality when R is a left GF-closed ring. To achieve this, we need the following lemma.

Lemma 4.2. *Let X be an exact Gorenstein flat complex. If $J \otimes_R X$ is exact for any injective right R-module J, then $Z_m X$ are Gorenstein flat modules for all $m \in \mathbb{Z}$.*

Proof. Since X is Gorenstein flat, there exists an exact sequence of flat complexes

$$\cdots \longrightarrow F_1 \xrightarrow{u_1} F_0 \xrightarrow{u_0} F^0 \xrightarrow{u^0} F^1 \xrightarrow{u^1} \cdots \qquad (\natural)$$

with $X = \mathrm{Ker}(u^0) = \mathrm{Im}(u_0)$ and remains exact when $I\overline{\otimes}-$ is applied to it for any injective complex I of right R-modules. Split this sequence as follows:

$$0 \longrightarrow \mathrm{Im}(u_1) \longrightarrow F_0 \longrightarrow X \longrightarrow 0 \qquad (\natural_1)$$

$$0 \longrightarrow \mathrm{Im}(u_2) \longrightarrow F_1 \longrightarrow \mathrm{Im}(u_1) \longrightarrow 0 \qquad (\natural_2)$$

$$\cdots\cdots$$

$$0 \longrightarrow \mathrm{Im}(u_{i+1}) \longrightarrow F_i \longrightarrow \mathrm{Im}(u_i) \longrightarrow 0 \qquad (\natural_{i+1})$$

$$\cdots\cdots$$

and

$$0 \longrightarrow X \longrightarrow F^0 \longrightarrow \mathrm{Ker}(u^1) \longrightarrow 0 \qquad (\natural^1)$$

$$0 \longrightarrow \mathrm{Ker}(u^1) \longrightarrow F^1 \longrightarrow \mathrm{Ker}(u^2) \longrightarrow 0 \qquad (\natural^2)$$

$$\cdots\cdots$$

$$0 \longrightarrow \mathrm{Ker}(u^i) \longrightarrow F^i \longrightarrow \mathrm{Ker}(u^{i+1}) \longrightarrow 0 \qquad (\natural^{i+1})$$

$$\cdots\cdots$$

Then, it is easy to see that all complexes $\mathrm{Im}(u_i)$ and $\mathrm{Ker}(u^i)$ are exact Gorenstein flat for $i \geq 1$. An argument using Lemma 3.10 yields that sequences $\mathrm{Im}(u_i)$ and $\mathrm{Ker}(u^i)$ remain exact when $J \otimes_R -$ is applied to them for any injective right R-module J and for any $i \geq 1$. Hence, each $\mathrm{Im}(u_i)$ and $\mathrm{Ker}(u^i)$ have the same properties as X.

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Notice that $\mathrm{Hom}(S^m(R), C) \cong \mathrm{Ker}(\delta_m^C) = Z_mC$ for any complex C and for any $m \in \mathbb{Z}$, thus if F is a flat complex, then $\mathrm{Hom}(S^m(R), F) = Z_mF$ is a flat R-module. On the other hand, since $S^m(R)$ is semi-projective, we get that $\mathrm{Ext}^1(S^m(R), E) = 0$ for any exact complex E ([**13**]). Then, we apply the functor $\mathrm{Hom}(S^m(R), -)$ to $(\natural_i)$ and $(\natural^i)$ and get exact sequences:

$$0 \longrightarrow Z_m(\mathrm{Im}(u_1)) \longrightarrow Z_mF_0 \longrightarrow Z_mX \longrightarrow 0 \qquad (z\natural_1)$$

$$0 \longrightarrow Z_m(\mathrm{Im}(u_2)) \longrightarrow Z_mF_1 \longrightarrow Z_m(\mathrm{Im}(u_1)) \longrightarrow 0 \qquad (z\natural_2)$$

$$\cdots\cdots$$

$$0 \longrightarrow Z_m(\mathrm{Im}(u_{i+1})) \longrightarrow Z_mF_i \longrightarrow Z_m(\mathrm{Im}(u_i)) \longrightarrow 0 \qquad (z\natural_{i+1})$$

$$\cdots\cdots$$

and

$$0 \longrightarrow Z_mX \longrightarrow Z_mF^0 \longrightarrow Z_m(\mathrm{Ker}(u^1)) \longrightarrow 0 \qquad (z\natural^1)$$

$$0 \longrightarrow Z_m(\mathrm{Ker}(u^1)) \longrightarrow Z_mF^1 \longrightarrow Z_m(\mathrm{Ker}(u^2)) \longrightarrow 0 \qquad (z\natural^2)$$

$$\cdots\cdots$$

$$0 \longrightarrow Z_m(\mathrm{Ker}(u^i)) \longrightarrow Z_mF^i \longrightarrow Z_m(\mathrm{Ker}(u^{i+1})) \longrightarrow 0 \qquad (z\natural^{i+1})$$

$$\cdots\cdots$$

Thus, we assemble these sequences $(z\natural_i)$ and $(z\natural^i)$ and get an exact sequence of flat modules

$$\cdots \longrightarrow Z_mF_1 \longrightarrow Z_mF_0 \longrightarrow Z_mF^0 \longrightarrow Z_mF^1 \longrightarrow \cdots \qquad (\sharp)$$

with $Z_mX \cong \mathrm{Im}(Z_mF_0 \to Z_mF^0) \cong \mathrm{Ker}(Z_mF^0 \to Z_mF^1)$. Now we claim that the sequence $(\sharp)$ remains exact when the functor $J \otimes_R -$ is applied to it for any injective right R-module J.

Let J be an injective right R-module. By assumption, the sequence $J \otimes_R X$ is exact, so we get an exact sequence

$$0 \longrightarrow Z_{m+1}(J \otimes_R X) \longrightarrow J \otimes_R X_{m+1} \longrightarrow Z_m(J \otimes_R X) \longrightarrow 0\,.$$

On the other hand, by Lemma 3.10, X_{m+1} is Gorenstein flat in R-Mod, and so $\mathrm{Tor}_1^R(J, X_{m+1}) = 0$. Hence, applying the functor $J \otimes_R -$ to the exact sequence $0 \to Z_{m+1}X \to X_{m+1} \to Z_mX \to 0$ yields that the sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(J, Z_mX) \longrightarrow J \otimes_R Z_{m+1}X \longrightarrow J \otimes_R X_{m+1} \longrightarrow J \otimes_R Z_mX \longrightarrow 0$$

is exact. Thus, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & Z_{m+1}(J \otimes_R X) & \longrightarrow & J \otimes_R X_{m+1} & \longrightarrow & Z_m(J \otimes_R X) & \longrightarrow & 0 \\
 & & \downarrow & & \simeq\downarrow & & \| & & \simeq\downarrow & & \\
0 & \longrightarrow & \mathrm{Tor}_1^R(J, Z_m X) & \longrightarrow & J \otimes_R Z_{m+1} X & \longrightarrow & J \otimes_R X_{m+1} & \longrightarrow & J \otimes_R Z_m X & \longrightarrow & 0
\end{array}$$

Now it follows by the five lemma that $\mathrm{Tor}_1^R(J, Z_m X) = 0$. Since each $\mathrm{Im}(u_i)$ and $\mathrm{Ker}(u^i)$ have the same properties as X, we can use the same way to show that $\mathrm{Tor}_1^R(J, Z_m(\mathrm{Im}(u_i))) = \mathrm{Tor}_1^R(J, Z_m(\mathrm{Ker}(u^i))) = 0$ for all $i \geq 1$. Then, it is easily seen that the sequence ($\sharp$) remains exact when the functor $J \otimes_R -$ is applied to it, this proves that $Z_m X$ is a Gorenstein flat module, as desired. $\square$

We are now in the position to give the main result in this section.

THEOREM 4.3. *If R is a left GF-closed ring, then $\mathcal{G}(\mathcal{F}(R)) = \mathcal{G}^2(\mathcal{F}(R))$.*

Proof. We need only to show any R-module $M \in \mathcal{G}^2(\mathcal{F}(R))$ is contained in $\mathcal{G}(\mathcal{F}(R))$. Let $M \in \mathcal{G}^2(\mathcal{F}(R))$. Then there is an exact sequence of the Gorenstein flat R-modules

$$G = \cdots \xrightarrow{\delta_2^G} G_1 \xrightarrow{\delta_1^G} G_0 \xrightarrow{\delta_0^G} G_{-1} \xrightarrow{\delta_{-1}^G} \cdots$$

such that the complex $H \otimes_R G$ is exact for each Gorenstein injective right R-module H and $M \cong \mathrm{Ker}(\delta_0^G)$. By Theorem 3.11, G is Gorenstein flat in R-Comp. On the other hand, the exact complex G remains exact after applying $I \otimes_R -$ for any injective right R-module, since it is so after applying $H \otimes_R -$ for any Gorenstein injective right R-module. Thus, it follows from Lemma 4.3 that M is Gorenstein flat. $\square$

Since right coherent rings are left GF-closed (Proposition 2.2 in [**2**], the following result is easily seen by Theorem 4.3.

COROLLARY 4.4. *If R is a right coherent ring, then $\mathcal{G}(\mathcal{F}(R)) = \mathcal{G}^2(\mathcal{F}(R))$.*

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