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Abstract

We prove a version of Kontsevich's formality theorem for two subspaces (branes) of a vector space X. The result implies, in particular, that the Kontsevich deformation quantizations of $\mathrm{S}(X^*)$ and $\wedge(X)$ associated with a quadratic Poisson structure are Koszul dual. This answers an open question in Shoikhet's recent paper on Koszul duality in deformation quantization.

1. Introduction

Kontsevich's proof of his formality theorem in [Kon03] is based on the Feynman diagram expansion of a topological quantum field theory. In [CF04], a program to extend Kontsevich's construction by including branes (i.e. submanifolds that define boundary conditions for the quantum fields) is sketched. The case of one brane leads to the relative formality theorem (see [CF07]) for the Hochschild cochains of the sections of the exterior algebra of the normal bundle of a submanifold, and is related to quantization of Hamiltonian reduction of coisotropic submanifolds in Poisson manifolds. Here we consider the case of two branes in the simplest situation where the branes are linear subspaces U and V of a real (or complex) vector space X. The new feature is that one should associate to the intersection $U \cap V$ an A_∞-bimodule over the algebras associated with U and V. The formality theorem that we prove holds for the Hochschild cochains of an A_∞-category corresponding to this bimodule. It is interesting that even when $U = \{0\}$ and $V = X = \mathbb{R}$, the A_∞-bimodule is one-dimensional but has infinitely many non-trivial structure maps.

Our discussion is inspired by the recent paper [Sho08] of Shoikhet, who proved a similar formality theorem in the framework of Tamarkin's approach based on Drinfeld associators. Our result implies that Shoikhet's theorem on Koszul duality in deformation quantization holds for the explicit Kontsevich quantization as well. In the next subsection, we review the question of Koszul duality in Kontsevich's deformation quantization, explain how it fits into the setting of formality theorems and state our results.

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1.1 Koszul duality

Let X be a real or complex finite-dimensional vector space. Then it is well known that the algebra $B = \mathrm{S}(X^*)$ of polynomial functions on X is a quadratic Koszul algebra and is Koszul dual to the exterior algebra $A = \wedge(X)$. In [Sho08], Shoikhet studied the question of quantization of Koszul duality. He asked whether the Kontsevich deformation quantization of A and B corresponding to a quadratic Poisson bracket leads to Koszul dual formal associative deformations of A and B. Recall that a quadratic Poisson structure on a finite-dimensional vector space X is, by definition, a Poisson bracket on $B = \mathrm{S}(X^*)$ with the property that the bracket of any two linear functions is a homogeneous quadratic polynomial. A quadratic Poisson structure on X also defines via duality a (graded) Poisson bracket on $A = \wedge(X)$. If $x^1, \ldots, x^n$ are linear coordinates on X and $\theta_1, \ldots, \theta_n$ is the dual basis of X, the brackets of generators have the form

$$\{x^i, x^j\}_B = \sum_{k,l} C_{k,l}^{i,j} x^k x^l, \quad \{\theta_k, \theta_l\}_A = \sum_{i,j} C_{k,l}^{i,j} \theta_i \wedge \theta_j.$$

Kontsevich gave a universal formula for an associative star-product, $f \star g = fg + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \cdots$ on $\mathrm{S}(X^*)[[\hbar]]$, where B_1 is any given Poisson bracket. Here, 'universal' means that $B_j(f, g)$ is a differential polynomial in f, g and the components of the Poisson bivector field with universal coefficients. Kontsevich's result also applies to super-manifolds such as the odd vector space $W = X^*[1]$, in which case $\mathrm{S}(W^*) = \mathrm{S}(X[-1]) = \wedge(X)$. Moreover, if the Poisson bracket is quadratic, then the deformed algebras $A[[\hbar]]$ and $B[[\hbar]]$ are quadratic, i.e. they are generated, respectively, by θ^i and x_i with quadratic defining relations. Shoikhet proved that Tamarkin's universal deformation quantization corresponding to any Drinfeld associator (see [Tam98]) leads to Koszul dual quantizations. Here we show that the same is true for the original Kontsevich deformation quantization.

1.2 Branes and bimodules

In Kontsevich's approach, the associative deformations of A and B are given by explicit formulæ that involve integrals over configuration spaces labelled by Feynman diagrams of a topological quantum field theory. We approach the question of Koszul duality from the quantum field theory point of view, following a variant of a suggestion of Shoikhet (see [Sho08, § 0.7]). The setting is the theory of quantization of coisotropic branes in a Poisson manifold [CF04]. In this setting, quantum field theory predicts the existence of an A_∞-category whose set of objects S is any given collection of submanifolds ('branes') of a Poisson manifold. If S consists of one object, then one obtains the A_∞-algebra related to Hamiltonian reduction [CF07]. Here we consider the next simplest case, that of two objects which are subspaces U and V of a finite-dimensional vector space X. In this case, the A_∞-category structure is given by two (possibly curved) A_∞-algebras A and B together with an A_∞-A-B-bimodule K over $\mathbb{R}[[\hbar]]$ or $\mathbb{C}[[\hbar]]$; the A_∞-algebras represent the spaces of endomorphisms of the two objects U and V, while the A_∞-A-B-bimodule represents the space of morphisms from U to V. More precisely, we have $A = \Gamma(U, \wedge(\mathrm{N}U))[[\hbar]] = \mathrm{S}(U^*) \otimes \wedge(X/U)[[\hbar]]$ and $B = \Gamma(V, \wedge(\mathrm{N}V))[[\hbar]] = \mathrm{S}(V^*) \otimes \wedge(X/V)[[\hbar]]$, the sections of the exterior algebras of the normal bundles, and

$$K = \Gamma(U \cap V, \wedge(\mathrm{T}X/(\mathrm{T}U + \mathrm{T}V)))[[\hbar]] = \mathrm{S}(U \cap V) \otimes \wedge(X/(U + V))[[\hbar]]. \quad (1)$$

Remark 1.1. We refer to § 2 for the definition of the category $\mathtt{GrMod}_{\mathbb{R}}$ we consider throughout the whole paper: in particular, A, B and K (without tensoring with respect to $\mathbb{R}[\hbar]$) are understood as objects of $\mathtt{GrMod}_{\mathbb{R}}$, thus, all morphisms have also to be understood as morphisms in $\mathtt{GrMod}_{\mathbb{R}}$.

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The structure maps of these algebras and bimodule are compositions of morphisms in the A_∞-category, and they can be described by sums over graphs with weights given by integrals of differential forms over configuration spaces on the upper half-plane. The differential forms are products of pull-backs of propagators which are 1-forms on the configuration space of two points in the upper half-plane. In addition to the Kontsevich propagator [Kon03], which vanishes when the first point approaches the real axis, there are three other propagators with brane boundary conditions; see [CF04, CT08]. The four propagators obey the four possible boundary conditions of vanishing as the first or second point approaches the positive or negative real axis. In the physical model, these are the Dirichlet boundary conditions for coordinate functions of maps from the upper half-plane to X such that the positive real axis is mapped to a coordinate plane U and the negative real axis to a coordinate plane V.

The new feature here is that, even for zero Poisson structure, the A_∞-bimodule has non-trivial structure maps. Let us describe the result first in the simplest case, where $U=\{0\}$ and $V=X$ so that $A=\wedge(X)$, $B=\mathrm{S}(X^*)$ and $K=\mathbb{R}$ (here it is not necessary to tensor by $\mathbb{R}[[\hbar]]$ since the structure maps are independent of $\hbar$).

Proposition 1.2. *Let A be the graded associative algebra $A=\wedge(X)=\mathrm{S}(X[-1])$ with generators of degree 1, and let $B=\mathrm{S}(X^*)$ be concentrated in degree 0. View A and B as A_∞-algebras whose Taylor-component products d^j are zero except when $j=2$. Then there exists an A_∞-A-B-bimodule K whose structure maps*

$$d_K^{j,k}: A[1]^{\otimes j}\otimes K[1]\otimes B[1]^{\otimes k}\to K[1]$$

obey $d_K^{1,1}(v\otimes k\otimes u)=\langle u,v\rangle k$ for $k\in K$, $v\in X\subset\wedge(X)$ and $u\in X^*\subset\mathrm{S}(X^*)$, where $\langle\ ,\ \rangle$ denotes the canonical pairing. In the general case of subspaces $U,V\subset X$, where A is generated by $W_A=U^*\oplus(X/U)[1]$ and V by $W_B=V^*\oplus(X/V)[1]$, we have $d_K^{1,1}(v\otimes k\otimes u)=\langle v,u\rangle k$ for $v\in(V/(U\cap V))^*\oplus U/(U\cap V)[1]\subset W_B$ and $u\in(U/(U\cap V))^*\oplus V/(U\cap V)[1]\subset W_A$.

The remaining $d_K^{i,j}$ are given by explicit finite-dimensional integrals corresponding to the graphs depicted in Figure 5; see § 6. There should exist a more direct description of such basic objects.

Example 1.3. If X is one-dimensional, $A=\mathbb{R}[\theta]$ with $\theta^2=0$ and $B=\mathbb{R}[x]$, then the non-trivial structure maps of K on monomials are

$$d_K^{j,1}(\underbrace{\theta\otimes\cdots\otimes\theta}_{j}\otimes 1\otimes x^j)=1.$$

In this case, they can be computed inductively from the A_∞-A-B-bimodule relations, using the fact that $d_K^{1,1}$ is simply the duality pairing between the generators $\theta=\partial_x$ and x.

Conjecture 1.4. The bimodule of Proposition 1.2 is A_∞-quasi-isomorphic to the Koszul free resolution $\wedge(X^*)\otimes\mathrm{S}(X^*)$ of the right $\mathrm{S}(X^*)$-module $\mathbb{R}$, where $\wedge(X)$ acts from the left by contraction.

1.3 Formality theorem

Our main result is a formality theorem for the differential graded Lie algebra of Hochschild cochains of the A_∞-category associated with the A_∞-A-B-bimodule K (for zero Poisson structure). As above, let U and V be vector subspaces of X, the objects of the category, and let $A=\Gamma(U,\wedge(\mathrm{N}U))=\mathtt{Hom}(U,U)$, $B=\Gamma(V,\wedge(\mathrm{N}V))=\mathtt{Hom}(V,V)$, $K=\Gamma(U\cap V,\wedge(\mathrm{T}X/(\mathrm{T}U+\mathrm{T}V)))=\mathtt{Hom}(V,U)$ and $\mathtt{Hom}(U,V)=0$. The non-zero composition maps in this A_∞-category are

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the products on A and B and the A_∞-bimodule maps $d_K^{k,l} : A[1]^{\otimes k} \otimes K \otimes B[1]^{\otimes l} \to K[1]$ for $k, l \geqslant 0,\ i + j \geqslant 1$. Let us call this category $\mathtt{Cat}_\infty(A, B, K)$. As for any A_∞-category, its shifted Hochschild cochain complex $\mathrm{C}^{\bullet+1}(\mathtt{Cat}_\infty(A, B, K))$ is a graded Lie algebra with respect to the (obvious extension of the) Gerstenhaber bracket. Moreover, there are natural projections to the differential graded Lie algebras $\mathrm{C}^{\bullet+1}(A, A)$ and $\mathrm{C}^{\bullet+1}(B, B)$ of Hochschild cochains of A and B. By Kontsevich's formality theorem, these differential graded Lie algebras are L_∞-quasi-isomorphic to their cohomologies, which are both isomorphic to the Schouten Lie algebra $T_{\mathrm{poly}}^{\bullet+1}(X) = \mathrm{S}(X^*) \otimes \wedge^{\bullet+1} X$ of poly-vector fields on X.

Thus, we have the following diagram of L_∞-quasi-isomorphisms.

$$\begin{array}{ccccc} & & \mathrm{C}^{\bullet+1}(A, A) & & \\ & \overset{\mathcal{U}_A}{\nearrow} & & \overset{\mathrm{p}_A}{\nwarrow} & \\ T_{\mathrm{poly}}^{\bullet+1}(X) & & & & \mathrm{C}^{\bullet+1}(\mathtt{Cat}_\infty(A, B, K)) \\ & \underset{\mathcal{U}_B}{\searrow} & & \underset{\mathrm{p}_B}{\swarrow} & \\ & & \mathrm{C}^{\bullet+1}(B, B) & & \end{array} \tag{2}$$

Theorem 1.5. *There is an L_∞-quasi-isomorphism $T_{\mathrm{poly}}^{\bullet+1}(X) \to \mathrm{C}^{\bullet+1}(\mathtt{Cat}_\infty(A, B, K))$ that completes (2) to a commutative diagram of L_∞-morphisms.*

The coefficients of the L_∞-morphisms are given by integrals over configuration spaces of points in the upper half-plane of differential forms that are similar to Kontsevich's but have different (brane) boundary conditions. This 'formality theorem for pairs of branes' is an A_∞ analogue of Shoikhet's formality theorem [Sho08], which dealt with the case where $U = \{0\}$, $V = X$ and K is replaced by the Koszul complex, and used Tamarkin's L_∞-morphism instead of Kontsevich's. Theorem 1.5 follows from Theorem 7.2, which is formulated and proved in § 7.

1.4 Maurer–Cartan elements

An L_∞-quasi-isomorphism $\mathfrak{g}_1^\bullet \to \mathfrak{g}_2^\bullet$ induces an isomorphism between the sets $MC(\mathfrak{g}_i) = \{x \in \hbar\mathfrak{g}_i^1[\![\hbar]\!] : dx + \frac{1}{2}[x, x] = 0\}/\exp(\hbar\mathfrak{g}_i^0[\![\hbar]\!])$, $i = 1, 2$ of equivalence classes of Maurer–Cartan elements (MCEs for short); see [Kon03]. MCEs in $T_{\mathrm{poly}}(X)$ are formal Poisson structures on X. They are mapped to MCEs in $\mathrm{C}^\bullet(A, A)$ and $\mathrm{C}^\bullet(B, B)$, which are A_∞-deformations of the product in A and B. The previous theorem implies that the image of a Poisson structure in X in $\mathrm{C}^\bullet(A, B, K)$ is an A_∞-bimodule structure on $K[\![\hbar]\!]$ over the A_∞-algebras $A[\![\hbar]\!]$ and $B[\![\hbar]\!]$.

1.5 Keller's condition

The key property of the bimodule K, which is preserved under deformation and implies the Koszul duality as well as the fact that the projections p_A and p_B are quasi-isomorphisms, is that it obeys an A_∞ version of *Keller's condition* [Kel03]. Before formulating this condition, we introduce some necessary notions; see § 4 for further details.

Recall that an A_∞-algebra over a commutative unital ring R is a $\mathbb{Z}$-graded free R-module A with a codifferential d_A on the counital tensor R-coalgebra $\mathrm{T}(A[1])$. The *differential graded (DG) category of right A_∞-modules* over an A_∞-algebra A has as objects pairs (M, d_M) where M is a $\mathbb{Z}$-graded free R-module and d_M is a codifferential on the cofree right $\mathrm{T}(A[1])$-comodule $FM = M[1] \otimes_R \mathrm{T}(A[1])$. The complex of morphisms $\underline{\mathrm{Hom}}_{-A}(M, N)$ is the graded R-module whose degree-j subspace consists of homomorphisms $FM \to FN$ of comodules of degree j with

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differential $\phi \mapsto d_N \circ \phi - \phi \circ d_M$. In particular, for any module M, $\underline{\mathrm{End}}_{-A}(M) = \underline{\mathrm{Hom}}_{-A}(M, M)$ is a differential graded algebra. If A is an ordinary associative algebra and M and N are ordinary modules, the cohomology of $\underline{\mathrm{Hom}}_{-A}(M, N)$ is the direct sum of the Ext-groups $\mathrm{Ext}^i_{-A}(M, N)$. The DG category of left A-modules is defined analogously; its morphism spaces are denoted by $\underline{\mathrm{Hom}}_{A-}(M, N)$. If A and B are A_∞-algebras, an A_∞-A-B-bimodule structure on K is the same as a codifferential on the cofree $(\mathrm{T}(A[1]), \mathrm{T}(B[1]))$-comodule $T(A[1]) \otimes K[1] \otimes T(B[1])$, that is, a codifferential compatible with coproducts and codifferentials d_A and d_B.

The *curvature* of an A_∞-algebra (A, d_A) is the component $\mathrm{F}_A \in A^2$ in $A[1] = T^1(A[1])$ of $d_A(1)$ where $1 \in R = T^0(A[1])$. If F_A vanishes, then $d_A(1) = 0$ and A is said to be *flat*. If A and B are flat, then an A_∞-A-B-bimodule is, in particular, an A_∞ left A-module and an A_∞ right B-module. The left-action of A then induces a *derived left-action*

$$\mathrm{L}_A : A \to \underline{\mathrm{End}}_{-B}(K),$$

which is a morphism of A_∞-algebras (the differential graded algebra $\underline{\mathrm{End}}_{-B}(K)$ is considered as an A_∞-algebra with two non-trivial structure maps, namely the differential and the product). Similarly, we have a morphism of A_∞-algebras

$$\mathrm{R}_B : B \to \underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}.$$

We say that an A_∞-A-B-bimodule K, for flat A_∞-algebras A and B, obeys the Keller condition if L_A and R_B are both quasi-isomorphisms.

Lemma 1.6. *The bimodule K of Proposition 1.2 obeys the Keller condition.*

The A_∞ version of Keller's theorem [Kel03] that we prove in § 4 (see Theorem 4.9) states that if K obeys the Keller condition, then p_A and p_B in (2) are quasi-isomorphisms. Moreover, the Keller condition is an A_∞ version of the Koszul duality between A and B, and reduces to it in the case where $U = \{0\}$ and $V = X$ with quadratic Poisson brackets, in which both A and B are ordinary associative algebras. Indeed, in this case, L_A and R_B induce algebra isomorphisms $B \cong \mathrm{Ext}^\bullet_{A-}(K, K)^{\mathrm{op}}$ and $A \cong \mathrm{Ext}^\bullet_{-B}(K, K)^{\mathrm{op}}$.

1.6 The trouble with the curvature

Let us again consider the simplest case where $U = \{0\}$ and $V = X$, and suppose that π is a Poisson bivector field on X. Then Kontsevich's deformation quantization gives rise to an associative algebra $(B_\hbar = \mathrm{S}(X^*)[[\hbar]], \star_B)$ and a possibly curved A_∞-algebra $(A_\hbar = \wedge(X)[[\hbar]], d_{A_\hbar})$, both over $\mathbb{R}[[\hbar]]$. The one-dimensional A-B-bimodule K deforms to an A_∞-$A_\hbar$-$B_\hbar$-bimodule $K_\hbar$. If we restrict the structure maps of this bimodule to $K_\hbar \otimes \mathrm{T}(B_\hbar)$, we get a deformation $\circ : K_\hbar \otimes B_\hbar \to B_\hbar$ of the right-action of B as the only non-trivial map. However, this map is not an action; instead, we get

$$(k \circ b_1) \circ b_2 - k \circ (b_1 \star b_2) = \langle \mathrm{F}_{A_\hbar}, db_1 \wedge db_2 \rangle k.$$

The curvature $\mathrm{F}_{A_\hbar}$ is a formal power series in $\hbar$ whose coefficients are differential polynomials in the components of the Poisson bivector field evaluated at zero. Its leading term vanishes if $\pi(0) = 0$ (i.e. if V is coisotropic). The next term is proportional to $\hbar^3$; it represents an obstruction to the quantization of the augmentation module over $\mathrm{S}(X^*)$. Willwacher constructed in [Wil07] an example of a zero of a Poisson bivector field on a five-dimensional space, whose module over the Kontsevich deformation of the algebra of functions cannot be deformed. On the other hand, there are several interesting examples of Poisson structures such that $\mathrm{F}_{A_\hbar} = 0$. Apart from quadratic Poisson structures, there are many examples related to Lie theory, which we will study elsewhere.

1.7 Organization of the paper

After fixing our notation and conventions in § 2, we recall in § 3 the basic notions of A_∞-categories and their Hochschild cochain complex. In § 4 we formulate an A_∞ version of Keller's condition and extend Keller's theorem to this case. In § 5, integrals over configuration spaces of differential forms with brane boundary conditions are described. The differential graded Lie algebra of Hochschild cochains of an A_∞-category is discussed in § 6. Our main result and its consequences are presented and proved in § 7.

2. Notation and conventions

We consider a ground field k of characteristic zero; for instance, $k=\mathbb{R}$ or $k=\mathbb{C}$.

Further, we consider the category $\mathtt{GrMod}_k$ of topological, complete (with respect to the given topology), $\mathbb{Z}$-graded vector spaces over k, i.e. an object V of $\mathtt{GrMod}_k$ decomposes into a direct sum of topological, complete vector spaces,

$$V=\bigoplus_{n\in\mathbb{Z}} V_n,$$

where V_n is a topological, complete vector space over k, the so-called homogeneous component of degree n. Morphisms in the category $\mathtt{GrMod}_k$ are, by definition, k-linear, continuous maps between objects of $\mathtt{GrMod}_k$ of degree 0, and we use the notation $\hom(V,W)$ for the space of morphisms between two objects V, W of $\mathtt{GrMod}_k$. We denote by $\mathtt{Mod}_k$ the full subcategory of $\mathtt{GrMod}_k$ with objects being the ones concentrated in degree 0. We denote by $[\bullet]$ the degree-shifting functor on $\mathtt{GrMod}_k$.

The category $\mathtt{GrMod}_k$ is a symmetric tensor category: for two general objects of $\mathtt{GrMod}_k$, the tensor product $V\otimes W$ (where, with an abuse of notation, we have not written out the explicit dependence on the ground field k) is the tensor product of V and W as k-vector spaces, with the grading induced by

$$(V\otimes W)_p=\prod_{m+n=p} V_m\otimes W_n \quad \text{for } p\in\mathbb{Z}.$$

The symmetry isomorphism σ is given by 'signed transposition',

$$\sigma_{V,W}: V\otimes W\longrightarrow W\otimes V, \quad v\otimes w\longmapsto (-1)^{|v||w|} w\otimes v.$$

Finally, observe that the category $\mathtt{GrMod}_k$ has inner Hom-spaces: given two graded vector spaces V and W, one can consider the graded vector space $\operatorname{Hom}(V,W)$ defined by

$$\operatorname{Hom}^i(V,W)=\hom(V,W[-i])=\prod_{k\in\mathbb{Z}} \hom_{\mathtt{Mod}_k}(V_k,W_{k+i}) \quad \text{for } i\in\mathbb{Z}.$$

Concretely, this means that we will always tacitly assume Koszul's sign rule when dealing with linear maps between graded vector spaces; for example,

$$(\phi\otimes\psi)(v\otimes w)=(-1)^{|\psi||v|}\phi(v)\otimes\psi(w).$$

The identity morphism of a general object V of the category $\mathtt{GrMod}_k$ induces an isomorphism $s: M\to M[1]$ of degree -1, called the *suspension*; its inverse $s^{-1}: M[1]\to M$, which obviously has degree 1, is called the *desuspension*. It is standard to denote by $|\cdot|$ the degree of homogeneous elements of objects of $\mathtt{GrMod}_k$; thus, by the definition of suspension and desuspension, we have $|s(\bullet)|=|\bullet|-1$.

For a general object V of $\texttt{GrMod}_k$, we denote by the graded counital tensor coalgebra cogenerated by V. The counit is the canonical projection onto $V^{\otimes 0} = k$, and the coproduct is given by

$$\Delta(v_1|\cdots|v_n) = 1 \otimes (v_1|\cdots|v_n) + \sum_{j=1}^{n-1}(v_1|\cdots|v_j) \otimes (v_{j+1}|\cdots|v_n) + (v_1|\cdots|v_n) \otimes 1$$

where, for the sake of simplicity, we write $(v_1|\cdots|v_n)$ for the tensor product $v_1 \otimes \cdots \otimes v_n$ in $V^{\otimes n}$.

The tensor coalgebra $\mathrm{T}(V)$ is graded via

$$\mathrm{T}(V)^n = \prod_{\substack{r\geqslant 0,\ m_1,\dots,m_r\geqslant 0,\ n_1,\dots,n_r\in\mathbb{Z}\\ m_1n_1+\cdots+m_rn_r=n}} \mathrm{T}^{m_1}(V_{n_1}) \otimes \cdots \otimes \mathrm{T}^{m_r}(V_{n_r}),$$

where $\mathrm{T}^m(V_n)$ denotes the component of tensor degree m of the tensor algebra of V_n (as a topological, complete vector space).

Further, the symmetric algebra $\mathrm{S}(V)$ is defined to be $\mathrm{S}(V) = \mathrm{T}(V)/\langle (v_1|v_2) - (-1)^{|v_1||v_2|}(v_2|v_1) : v_1, v_2 \in V\rangle$. A general, homogeneous element of $\mathrm{S}(V)$ will be denoted by $v_1\cdots v_n$, with v_i in V for $i = 1,\dots,n$. The symmetric algebra is endowed with a coalgebra structure, with coproduct given by

$$\Delta_{\mathrm{sh}}(v_1\cdots v_n) = \sum_{p+q=n}\sum_{\sigma\in\mathfrak{S}_{p,q}} \epsilon(\sigma, v_1,\dots,v_n)(v_{\sigma(1)}\cdots v_{\sigma(p)}) \otimes (v_{\sigma(p+1)}\cdots v_{\sigma(n)})$$

where $\mathfrak{S}_{p,q}$ is the set of (p,q)-shuffles, i.e. permutations $\sigma \in \mathfrak{S}_{p+q}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$, with corresponding sign

$$\epsilon(\sigma, v_1,\dots,v_n) = (-1)^{\sum_{i<j,\sigma(i)>\sigma(j)} |\gamma_i||\gamma_j|}, \tag{3}$$

and counit specified by the canonical projection onto k. Again, the symmetric algebra of V is graded in a way similar to the tensor algebra of V, for V a general object of $\texttt{GrMod}_k$: namely,

$$\mathrm{S}(V)^n = \prod_{\substack{r\geqslant 0,\ m_1,\dots,m_r\geqslant 0,\ n_1,\dots,n_r\in\mathbb{Z}\\ m_1n_1+\cdots+m_rn_r=n}} \mathrm{S}^{m_1}(V_{n_1}) \otimes \cdots \otimes \mathrm{S}^{m_r}(V_{n_r}),$$

with the same notations as before.

We observe that the symmetric algebra in the framework of $\texttt{GrMod}_k$ is already completed: if V is a finite-dimensional, graded vector space concentrated in degree 0 (endowed with the discrete topology), then $\mathrm{S}(V)$ coincides with the completed symmetric algebra of V with respect to the adic topology, and $\mathrm{S}(V)$ is concentrated in (internal) degree 0. On the other hand, if V is finite-dimensional and concentrated in degree 1, $\mathrm{S}(V)$ coincides with the exterior algebra of V (which is already complete by construction).

From now on, if not otherwise explicitly stated, symmetric algebras of $\mathbb{Z}$-graded, finite-dimensional vector spaces (endowed with the discrete topology) are always meant to be objects of $\texttt{GrMod}_k$.

We define the cocommutative coalgebra of invariants on V as $\mathrm{C}(V) = \bigoplus_{n\geqslant 0} \mathrm{I}_n(V)$, where $\mathrm{I}_n(V) = \{x \in V^{\otimes n} : x = \sigma x\ \forall \sigma \in \mathfrak{S}_n\}$; it is a sub-coalgebra of $\mathrm{T}(V)$, with coproduct given by the restriction of the natural coproduct onto a standard counit. We define also the cocommutative coalgebra without counit to be $\mathrm{C}^+(V) = \mathrm{C}(V)/k$. We have an obvious isomorphism of coalgebras,

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Sym : S(V) → C(V), which is explicitly given by

$$\mathrm{S}(V) \ni v_1 \cdots v_n \overset{\mathrm{Sym}}{\mapsto} \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma, v_1, \ldots, v_n)(v_{\sigma(1)}| \cdots |v_{\sigma(n)}) \in \mathrm{C}(V).$$

Finally, we need to consider the category $\texttt{GrMod}_k^{I\times I}$ of $(I \times I)$-graded objects in $\texttt{GrMod}_k$, where I is a finite set. In this category, the tensor product is defined by

$$(V \otimes_I W)_{i,j} = \bigoplus_{k \in I} V_{i,k} \otimes W_{k,j},$$

and Hom-spaces are given by

$$\mathrm{Hom}_{I\times I}(V, W)_{i,j} = \mathrm{Hom}(V_{i,j}, W_{i,j}).$$

This monoidal category is of course *not* symmetric at all; but we will often allow ourselves to use the symmetry isomorphism σ of $\texttt{GrMod}_k$ in explicit computations, since $V \otimes_I W \subset V \otimes W$ and $\mathrm{Hom}_{I\times I}(V, W) \subset \mathrm{Hom}(V, W)$ for any $(I \times I)$-graded objects V and W in $\texttt{GrMod}_k$.

For example, we have the graded counital tensor coalgebra $T_I(V) := \bigoplus_{n\in\mathbb{N}} V^{\otimes_I n}$ cogenerated by V as above, but we do not have the symmetric algebra in $\texttt{GrMod}_k^{I\times I}$.

3. A_∞-categories

In this section, we introduce the concept of (small) A_∞-categories and related A_∞-functors.

Definition 3.1. A (small and finite) A_∞-*category* is a triple $\mathcal{A} = (I, A, d_A)$ where:

- I is a finite set (whose elements are called objects);
- $A = (A_{\mathbf{a},\mathbf{b}})_{(\mathbf{a},\mathbf{b})\in I\times I}$ is an element in $\texttt{GrMod}_k^{I\times I}$ ($A_{\mathbf{a},\mathbf{b}}$ is called the space of morphisms from $\mathbf{b}$ to $\mathbf{a}$);
- d_A is a codifferential on $\mathrm{T}_I(A[1])$, i.e. a degree-one endomorphism (in $\texttt{GrMod}_k^{I\times I}$) of $\mathrm{T}_I(A[1])$, which satisfies $\Delta \circ d_A = (d_A \otimes_I 1 + 1 \otimes_I d_A) \circ \Delta$, $\varepsilon_A \circ d_A = 0$ and $(d_A)^2 = 0$.

The above conditions are equivalent to requiring that $(I, \mathrm{T}(A[1]), d_A)$ be a (small) differential graded cocategory.

The fact that d_A is a coderivation on $\mathrm{T}_I(A[1])$ lying in the kernel of the counit implies that d_A is uniquely determined by its Taylor components $d_A^n : A[1]^{\otimes_I n} \to A[1]$, $n \geqslant 0$, via

$$d_A|_{\mathrm{T}_I^n(A[1])} = \sum_{m=0}^{n} \sum_{l=0}^{n-m} 1^{\otimes_I l} \otimes_I d_A^m \otimes_I 1^{\otimes_I (n-m-l)},$$

where $1^{\otimes_I l}$ denotes the identity on $A[1]^{\otimes_I l}$. Then, the condition $(d_A)^2 = 0$ is equivalent to the following infinite set of quadratic equations with respect to the Taylor components of d_A:

$$\sum_{i=0}^{k} \sum_{j=1}^{k-i+1} d_A^{k-i+1} \circ \left(1^{\otimes_I (j-1)} \otimes_I d_A^i \otimes_I 1^{\otimes_I (k+1-j-i)}\right) = 0, \quad k \geqslant 0. \tag{4}$$

Equivalently, if we consider the maps $\mu_A^n : A^{\otimes_I n} \to A[2-n]$ obtained by appropriately twisting d_A with respect to suspension and desuspension, the quadratic relations (4) become

$$\sum_{i=0}^{k} \sum_{j=1}^{k-i+1} (-1)^{i \sum_{l=1}^{j-1} |a_l| + j(i+1)} \mu_A^{k-i+1}(a_1, \ldots, a_{j-1}, \mu_A^i(a_j, \ldots, a_{i+j-1}), a_{i+j}, \ldots, a_k) = 0 \tag{5}$$

for $k \geqslant 0$, where $a_i \in A_{\mathbf{a}_{i-1},\mathbf{a}_i}$ with $\mathbf{a_0}, \ldots, \mathbf{a}_k \in I$.

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An A_∞-category $\mathcal{A} = (I, A, d_A)$ is said to be *flat* if $d_A^0 = 0$; in this case, d_A^1 is a differential on A, d_A^2 is an associative product up to homotopy, and so on. Otherwise, $\mathcal{A}$ is said to be *curved*. If a flat A_∞-category is such that it has $d_A^k = 0$ for $k \geqslant 3$, then it is called a *differential graded* (DG) category.

We now assume that $\mathcal{A} = (I, A, d_A)$ and $\mathcal{B} = (J, B, d_B)$ are two (possibly curved) A_∞-categories in the sense of Definition 3.1; then an A_∞-functor from $\mathcal{A}$ to $\mathcal{B}$ is the *datum* of a functor $\mathcal{F}$ between the corresponding DG cocategories. More precisely, $\mathcal{F}$ is given by:

- a map $f : I \to J$;
- an $(I \times I)$-graded coalgebra morphism $F : \mathrm{T}_I(A[1]) \to \mathrm{T}_I(B[1])$ of degree 0 which intertwines the codifferentials d_A and d_B, i.e. $F \circ d_A = d_B \circ F$.

It follows immediately from the coalgebra (or, better, cocategory) structure on $\mathrm{T}_I(A[1])$ and $\mathrm{T}_I(B[1])$ (and from the fact that F is compatible with the corresponding counits, whence $F_0(1) = 1$) that an A_∞-functor from A to B is uniquely specified by its Taylor components $F_n : A[1]^{\otimes_I n} \to B[1]$ via

$$F|_{A[1]^{\otimes_I n}} = \sum_{k=0}^{n} \sum_{\substack{\mu_1,\dots,\mu_k \geqslant 0 \\ \sum_{i=1}^k \mu_i = n}} F_{\mu_1} \otimes_I \cdots \otimes_I F_{\mu_k}.$$

As a consequence, the condition that F intertwine the codifferentials d_A and d_B can be rewritten as the following infinite series of equations with respect to the Taylor components of d_A, d_B and F:

$$\sum_{m=0}^{n} \sum_{l=0}^{n-m} F_{n-m+1} \circ (1^{\otimes_I l} \otimes_I d_A^m \otimes_I 1^{\otimes_I (n-m-l)}) = \sum_{k=0}^{n} d_B^k \circ \Bigg(\sum_{\substack{\mu_1,\dots,\mu_k \geqslant 0 \\ \sum_{i=1}^k \mu_i = n}} F_{\mu_1} \otimes_I \cdots \otimes_I F_{\mu_k} \Bigg).$$

We finally observe that, by twisting the Taylor components F_n of an A_∞-morphism F from A to B, we get a semi-infinite series of morphisms $\phi_n : A^{\otimes_I n} \to B[1-n]$ of degree $1-n$ for $n \geqslant 0$. The natural signs in the previous relations can be computed immediately by using suspension and desuspension.

Example 3.2. An A_∞-category with only one object is an A_∞-algebra; a DG algebra is a DG category with only one object.

Given an A_∞-category $\mathcal{A} = (I, A, d_A)$ and a subset J of objects, there is an obvious notion of full A_∞-subcategory with respect to J. In particular, the space of endomorphisms $A_{\mathbf{a},\mathbf{a}}$ of a given object $\mathbf{a}$ is naturally an A_∞-algebra.

Example 3.3. We consider an A_∞-category $\mathcal{C} = (I, C, d_C)$ with two objects, $I = \{\mathbf{a}, \mathbf{b}\}$. We further assume $C_{\mathbf{b},\mathbf{a}} = 0$. Let us define

$$A = C_{\mathbf{a},\mathbf{a}}, \quad B = C_{\mathbf{b},\mathbf{b}}, \quad K = C_{\mathbf{a},\mathbf{b}}.$$

Here, A and B are A_∞-algebras, and we say that K is an A_∞-A-B-bimodule. Observe that we can alternatively define an A_∞-A-B-bimodule structure on K as a codifferential d_K on the cofree $(T(A[1]), T(B[1]))$-bicomodule cogenerated by $K[1]$: we write $d_K^{m,n}$ for the restriction of the Taylor component d_C^{m+n+1} onto the subspace $A[1]^{\otimes m} \otimes K[1] \otimes B[1]^{\otimes m} \subset (C[1]^{\otimes_I m+n+1})_{\mathbf{a},\mathbf{b}}$ (which takes values in $K[1] = C_{\mathbf{a},\mathbf{b}}[1]$). We will often denote by $\mathtt{Cat}_\infty(A, B, K)$ the corresponding A_∞-category.

Remark 3.4. Observe that an A_∞-algebra structure d_A on A determines an A_∞-A-A-bimodule structure on A via the Taylor components

$$d_A^{m,n} := d_A^{m+n+1}. \tag{6}$$

3.1 The Hochschild cochain complex of an A_∞-category

Consider an object $A = (A_{\mathbf{a},\mathbf{b}})_{\mathbf{a},\mathbf{b} \in I \times I}$ of $\mathtt{GrMod}_k^{I\times I}$. We associate to it another element $\mathrm{C}^\bullet(A, A)$ of $\mathtt{GrMod}_k^{I\times I}$, defined as follows:

$$\begin{aligned}\mathrm{C}^\bullet(A, A) &= \bigoplus_{p\geqslant 0} \mathrm{Hom}_{I\times I}(A^{\otimes_I p+1}, A)\\ &= \bigoplus_{p\geqslant 0} \bigoplus_{\mathbf{a_0},\dots,\mathbf{a_{p+1}}\in I} \mathrm{Hom}(A_{\mathbf{a_0},\mathbf{a_1}} \otimes \cdots \otimes A_{\mathbf{a_p},\mathbf{a_{p+1}}}, A_{\mathbf{a_0},\mathbf{a_{p+1}}}).\end{aligned}$$

The $\mathbb{Z}$-grading on $\mathrm{C}^\bullet(A, A)$ is given as the total grading of the following $\mathbb{Z}^2$-grading:

$$\mathrm{C}^{(p,q)}(A, A) = \mathrm{Hom}^q_{I\times I}(A^{\otimes_I p+1}, A).$$

We have the standard brace operations on $\mathrm{C}^\bullet(A, A)$. Specifically, the brace operations are defined via the usual higher compositions (whenever they make sense, of course); that is,

$$\begin{aligned}&P\{Q_1, \dots, Q_q\}(a_1, \dots, a_n)\\ &\quad= \sum_{i_1,\dots,i_q} (-1)^{\sum_{k=1}^q \|Q_k\|(i_k-1+\sum_{j=1}^{i_k-1}|a_j|)} P(a_1, \dots, Q_1(a_{i_1}, \dots), \dots, Q_q(a_{i_q}, \dots), \dots, a_n).\end{aligned}$$

Here $n = p + \sum_{a=1}^q (q_a - 1)$, $i_1 \geqslant 1$, $i_k + q_k \leqslant i_{k+1}$ for $k = 1, \dots, q-1$, $i_q + q_q - 1 \leqslant n$ and, for $i = 1, \dots, n$, a_i is a general element of A; $|Q_k|$ denotes the degree of Q_k, while q_k is the number of entries. We use the standard notation and sign rules; see, for instance, [CR08b, GJ90, TT01, VG95] for more details. In particular, $\| \bullet \|$ denotes the total degree with respect to the previous bigrading. Finally, let us recall that the graded commutator of the (non-associative) pairing defined by the brace operations on two elements satisfies the requirements for being a graded Lie bracket (with respect to the total degree), namely the so-called *Gerstenhaber bracket*.

Remark 3.5. Another (more intrinsic) definition of the Hochschild complex is as the space of $(I \times I)$-graded coderivations of $\mathrm{T}_I(A[1])$:

$$\mathrm{CC}(A) := \mathrm{Coder}_{I\times I}(\mathrm{T}_I(A[1])) = \mathrm{Hom}_{I\times I}(\mathrm{T}_I(A[1]), A[1]).$$

In this description, the Gerstenhaber bracket becomes more transparent: it is simply the natural Lie bracket of coderivations. The identification between $\mathrm{CC}(A)$ and $\mathrm{C}(A, A)$ is again given by an appropriate twisting with respect to suspension and desuspension.

According to the previous remark, the structure of an A_∞-category with I as the set of objects and A as the $(I \times I)$-graded space of morphisms translates into the existence of a Maurer–Cartan element (MCE) γ in $\mathrm{C}^\bullet(A, A)$, i.e. an element γ of $\mathrm{C}^\bullet(A, A)$ which is of (total) degree 1 and satisfies $[\gamma, \gamma]/2 = \gamma\{\gamma\} = 0$. Finally, the MCE γ specifies a degree-one differential $d_\gamma = [\gamma, \bullet]$, where $[\bullet, \bullet]$ denotes the Gerstenhaber bracket on $\mathrm{C}^\bullet(A, A)$. We obtain in this way a DG Lie algebra.

Example 3.6. We now make more explicit the case of the A_∞-category $\mathtt{Cat}_\infty(A, B, K)$ from Example 3.3. First of all, the bigrading on $C = \mathtt{Cat}_\infty(A, B, K)$ can be read immediately from the above conventions; that is,

$$\mathrm{C}^n(C,C)=\bigoplus_{p+q=n}\mathrm{Hom}^q(A^{\otimes(p+1)},A)\oplus\bigoplus_{p+q+r=n}\mathrm{Hom}^r(A^{\otimes p}\otimes K\otimes B^{\otimes q},K)$$
$$\oplus\bigoplus_{p+q=n}\mathrm{Hom}^q(B^{\otimes(p+1)},B).$$

The A_∞-structure on $\mathtt{Cat}_\infty(A,B,K)$ specifies an MCE γ, which splits into three pieces as $\gamma=d_A+d_K+d_B$. By the very construction of the Hochschild differential, d_γ splits into five components since, for a general element $\varphi=\varphi_A+\varphi_K+\varphi_B$ of $\mathrm{C}^\bullet(C,C)$, we have

$$d_\gamma\varphi=[d_A,\varphi_A]+d_K\{\varphi_A\}+[\gamma,\varphi_K]+d_K\{\varphi_A\}+[d_B,\varphi_B].$$

We observe that $[\gamma,\varphi_K]=[d_K,\varphi_K]-(-1)^{\|\varphi_K\|}\varphi_K\{d_A+d_K+d_B\}$ and denote by $\mathrm{C}^\bullet(A,K,B)$ the subcomplex consisting of elements φ_K in the middle term of the above splitting.

We want to explain the meaning of the five components in the alternative description of the Hochschild complex. An element ϕ in $\mathrm{CC}^n(C)$ consists of a triple (ϕ_A,ϕ_K,ϕ_B) where ϕ_A and ϕ_B are coderivations of $\mathrm{T}(A[1])$ and $\mathrm{T}(B[1])$, respectively, and ϕ_K is a coderivation of the bicomodule $\mathrm{T}(A[1])\otimes K[1]\otimes\mathrm{T}(B[1])$ with respect to ϕ_A and ϕ_B. Now, the MCE γ gives such an element (d_A,d_K,d_B), which moreover squares to zero. The five components can then be interpreted as

$$d_\gamma\phi=[d_A,\phi_A]+\mathrm{L}_A\circ\phi_A+[d_K,\phi_K]+\mathrm{R}_B\circ\phi_B+[d_B,\phi_B].$$

The meaning of the morphisms L_A and R_B, the derived left- and right-actions, is explained in full detail in § 4.

Sign considerations. We now discuss the signs appearing in the brace operations, which correspond to the natural Koszul signs that occur when one considers all possible higher compositions between different elements of $\mathrm{Hom}(\mathrm{T}(A[1]),A[1])$.

Before going into details, we need to make the grading conventions more precise: if ϕ is a general element of $\mathrm{Hom}^m(B[1]^{\otimes n},B[1])$, then we write $|\phi|=m$, and similar notation will be used when ϕ is an element of $\mathrm{Hom}^r(B[1]^{\otimes p}\otimes K[1]\otimes A[1]^{\otimes q},K[1])$. On the other hand, we shall write $\|\phi\|=m+n-1$; similarly, if ϕ is in $\mathrm{Hom}^r(B[1]^{\otimes p}\otimes K[1]\otimes A[1]^{\otimes q},K[1])$, we write $\|\phi\|=p+q+r$.

Consider, for instance, the Gerstenhaber bracket on B. For ϕ_i in $\mathrm{Hom}^{m_i}(B[1]^{\otimes n_i},B[1])$, $i=1$ or 2, we have

$$[\phi_1,\phi_2]:=\sum_{j=1}^{n_1}\phi_1\circ(1^{\otimes(j-1)}\otimes\phi_2\otimes 1^{\otimes(n_1-j)})-(-1)^{|\phi_1||\phi_2|}(\phi_2\leftrightarrow\phi_1).$$

Upon twisting with respect to suspension and desuspension (recall that the suspension $s:B\to B[1]$ has degree -1 and the desuspension s^{-1} has degree 1), we introduce the desuspended maps $\widetilde{\phi}_i\in\mathrm{Hom}^{1+m_i-n_i}(B^{\otimes n_i},B)$ and then set $|\widetilde{\phi}_i|:=1+m_i-n_i$ and $\|\widetilde{\phi}_i\|=m_1$; in other words, $\phi_i=s\circ\widetilde{\phi}_i\circ(s^{-1})^{\otimes n_i}$ for $i=1,2$.

Observe that

$$\|\widetilde{\phi}_i\|=|\phi_i|=m_i\quad\text{and}\quad|\tilde{\phi}_i|=\|\phi_i\|=m_i+n_i-1\quad\text{modulo 2}.$$

We then get, via explicit computations, that

$$[\widetilde{\phi}_1,\widetilde{\phi}_2]=\widetilde{\phi}_1\bullet\widetilde{\phi}_2-(-1)^{\|\widetilde{\phi}_1\|\|\widetilde{\phi}_2\|}\widetilde{\phi}_2\bullet\widetilde{\phi}_2,$$

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where the new desuspended signs for the higher composition $\bullet$ are given by

$$\widetilde{\phi}_1 \bullet \widetilde{\phi}_2 = \sum_{j=1}^{n_1} (-1)^{(|\widetilde{\phi}_2|+n_2-1)(n_1-1)+(j-1)(n_2-1)} \widetilde{\phi}_1 \circ (1^{\otimes(j-1)} \otimes \widetilde{\phi}_2 \otimes 1^{\otimes(n_1-j)}). \tag{7}$$

Note that these signs appear also in [CF07]. Obviously, by replacing B with A, we can repeat all the previous arguments to arrive at the signs for the Gerstenhaber bracket on $\mathrm{C}^\bullet(A, A)$.

Further, assuming that ϕ_i ($i = 1$ or 2) is a general element of $\mathrm{Hom}^{r_i}(B[1]^{\otimes p_i} \otimes K[1] \otimes A[1]^{\otimes q_i}, K[1])$, we introduce the desuspended map via $\phi_i = s \circ \widetilde{\phi}_i \circ (s^{-1})^{\otimes p_i+q_i+1}$; this is an element of $\mathrm{Hom}^{r_i-p_i-q_i}(B^{\otimes p_i} \otimes K \otimes A^{\otimes q_i}, K)$.

Setting $|\widetilde{\phi}_i| = r_i - p_i - q_i$ and $\|\widetilde{\phi}_i\| = r_i$, we have

$$\|\widetilde{\phi}_i\| = |\phi_i| = r_i \quad \text{and} \quad |\widetilde{\phi}_i| = \|\phi_i\| = r_i + p_i + r_i \quad \text{modulo 2}.$$

We further get the higher composition $\bullet$ between $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$, coming from the natural brace operations, with corresponding signs

$$\widetilde{\phi}_1 \bullet \widetilde{\phi}_2 = (-1)^{(|\widetilde{\phi}_2|+p_2+q_2)(p_1+q_1)+p_1(p_2+q_2)} \widetilde{\phi}_1 \circ (1^{\otimes p_1} \otimes \widetilde{\phi}_2 \otimes 1^{\otimes q_1}).$$

Now, suppose ϕ_1 is in $\mathrm{Hom}^{r_1}(B[1]^{\otimes p_1} \otimes K[1] \otimes A[1]^{\otimes q_1}, K[1])$ and ϕ_2 is in $\mathrm{Hom}^{m_2}(B[1]^{\otimes n_2}, B[1])$; then, by introducing the desuspended maps $\widetilde{\phi}_i$, $i = 1, 2$, whose (total) degrees satisfy the same relations as above, we get the following higher composition with corresponding signs between $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$, coming from the previously described brace operations:

$$\widetilde{\phi}_1 \bullet \widetilde{\phi}_2 = \sum_{j=1}^{p_1} (-1)^{(|\widetilde{\phi}_2|+n_2-1)(p_1+q_1)+(j-1)(n_2-1)} \widetilde{\phi}_2 \circ (1^{\otimes(j-1)} \otimes \widetilde{\phi}_2 \otimes 1^{\otimes(p_1+q_1+1-j)}).$$

Finally, if ϕ_1 and ϕ_2 lie in $\mathrm{Hom}^{r_1}(B[1]^{\otimes p_1} \otimes K[1] \otimes A[1]^{\otimes q_1}, K[1])$ and $\mathrm{Hom}^{m_2}(A[1]^{\otimes n_1}, A[1])$, respectively, then the higher composition between the desuspended maps $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ with corresponding signs, coming from the brace operations, has the explicit form

$$\widetilde{\phi}_1 \bullet \widetilde{\phi}_2 = \sum_{j=1}^{q_1} (-1)^{(|\widetilde{\phi}_2|+n_2-1)(p_1+q_1)+(p_1+j)(n_2-1)} \widetilde{\phi}_2 \circ (1^{\otimes(p_1+j)} \otimes \widetilde{\phi}_2 \otimes 1^{\otimes(q_1-j)}).$$

4. Keller's condition in the A_∞ framework

We now discuss some cohomological features of the Hochschild cochain complex of the A_∞-category $\mathtt{Cat}_\infty(A, B, K)$ from Example 3.3; in particular, we will extend to this framework the classical result of Keller for DG categories [Kel03], which is a central piece in the proof of the main result of [Sho08].

4.1 The derived left- and right-actions

Let A, B and K be as in Example 3.3, using the same notation.

We consider the restriction $d_{K,B}$ of d_K to $K[1] \otimes \mathrm{T}(B[1])$, i.e. the map

$$d_{K,B} = \mathrm{P}_{K,B} \circ d_K,$$

where $\mathrm{P}_{K,B}$ denotes the natural projection from $\mathrm{T}(A[1]) \otimes K[1] \otimes \mathrm{T}(B[1])$ onto $K[1] \otimes \mathrm{T}(B[1])$.

A direct check confirms that $\mathrm{P}_{K,B}$ is a morphism of right $\mathrm{T}(B[1])$-comodules, from which it follows directly that $d_{K,B}$ is a coderivation on $K[1] \otimes \mathrm{T}(B[1])$.

Remark 4.1. Similarly, the restriction of d_K to $\mathrm{T}(A[1]) \otimes K[1]$ defines a left coderivation $d_{A,K}$ on $\mathrm{T}(A[1]) \otimes K[1]$.

For A, B and K as above, we set

$$\underline{\mathrm{End}}_{-B}(K) = \mathrm{End}_{\mathrm{comod}-\mathrm{T}(B[1])}(K[1] \otimes \mathrm{T}(B[1])) = \mathrm{Hom}(K[1] \otimes \mathrm{T}(B[1]), K[1]),$$

where End and Hom have to be understood as the inner space of endomorphisms of the given object of $\mathtt{GrMod}_k$. Obviously, $\underline{\mathrm{End}}_{-B}(K)$ becomes, with respect to the composition, a graded algebra (GA for short).

The derived left-action of A on K, denoted by L_A, is defined as a coalgebra morphism from $\mathrm{T}(A[1])$ to $\mathrm{T}(\underline{\mathrm{End}}_{-B}(K)[1])$ (both endowed with the obvious coalgebra structures), whose mth Taylor component, viewed as an element of $\underline{\mathrm{End}}_{-B}(K)[1]$, decomposes as

$$\mathrm{L}_A^m(a_1|\cdots|a_m)^n(k|b_1|\cdots|b_n) = d_K^{m,n}(a_1|\cdots|a_m|k|b_1|\cdots|b_n) \quad \text{for } m \geqslant 1, n \geqslant 0. \tag{8}$$

In a more formal way, the Taylor component L_A^m can be defined as

$$\mathrm{L}_A^m(a_1|\cdots|a_m) = (\mathrm{P}_{K,B} \circ d_K)(a_1|\cdots|a_m|\cdots).$$

It is not difficult to check that $\mathrm{L}_A^m(a_1|\cdots|a_m)$ is an element of $\underline{\mathrm{End}}_{-B}(K)$.

The grading conditions on d_K imply, via direct computations, that L_A^m is a morphism from $A[1]^{\otimes n}$ to $\underline{\mathrm{End}}_{-B}(K)[1]$ of degree 0.

For later computations, we write down explicitly the Taylor series of the derived left-action up to order two, namely,

$$\mathrm{L}_A(a_1|\cdots|a_n) = \mathrm{L}_A^n(a_1|\cdots|a_n) + \sum_{\substack{n_1+n_2=n \\ n_i \geqslant 1, i=1,2}} (\mathrm{L}_A^{n_1}(a_1|\cdots|a_{n_1})|\mathrm{L}_A^{n_2}(a_{n_1+1}|\cdots|a_n)) + \cdots .$$

Next, we wish to define an A_∞-algebra structure on $\underline{\mathrm{End}}_{-B}(K)$. For this purpose, we first consider $d_{K,B}^2$: since $d_{K,B}$ is a right coderivation on $K[1] \otimes \mathrm{T}(B[1])$, its square is easily verified to be an element of $\underline{\mathrm{End}}_{-B}(K)$.

Lemma 4.2. *The operator $d_{K,B}^2$ satisfies*

$$d_{K,B}^2 = -\mathrm{L}_A^1(d_A^0(1)).$$

Proof. By its very definition, $d_{K,B}$ obeys

$$d_{K,B}^2 = \mathrm{P}_{K,B} \circ d_K \circ \mathrm{P}_{K,B} \circ d_K|_{K[1]\otimes \mathrm{T}(B[1])}.$$

Since d_K is a bicomodule morphism, upon taking into account the definition of the left and right coactions Δ_L and Δ_R on $\mathrm{T}(A[1]) \otimes K[1] \otimes \mathrm{T}(B[1])$, we get

$$\mathrm{P}_{K,B} \circ d_K\big|_{K[1]\otimes \mathrm{T}(B[1])} = d_K\big|_{K[1]\otimes \mathrm{T}(B[1])} - (d_A^0(1)|\bullet).$$

Since $d_K^2 = 0$, the claim follows directly. □

Therefore, $\underline{\mathrm{End}}_{-B}(K)$ inherits the structure of an A_∞-algebra, i.e. there is a degree-one codifferential Q whose only non-trivial Taylor components are

$$Q^0(1) = \mathrm{L}_A^1(d_A^0(1)), \quad Q^1(\varphi) = -[d_{K,B}, \varphi], \quad Q^2(\varphi_1|\varphi_2) = (-1)^{|\varphi_1|}\varphi_1 \circ \varphi_2.$$

Remark 4.3. In a similar way, we can introduce the A_∞-algebra $\underline{\mathrm{End}}_{A-}(K) = \mathrm{End}_{\mathrm{T}(A[1])\text{-comod}}(\mathrm{T}(A[1]) \otimes K[1])$ and the derived right-action R_B. Accordingly, $\underline{\mathrm{End}}_{A-}(K)$ is an A_∞-algebra with A_∞-structure given by the curvature $Q^0(1) = \mathrm{R}_B(d_B^0(1))$, degree-one derivation $[d_{A,K}, \bullet]$ and composition as product.

It is clear that if A and B are flat A_∞-algebras, then $\underline{\mathrm{End}}_{-B}(K)$ and $\underline{\mathrm{End}}_{A-}(K)$ are DG algebras.

Remark 4.4. The DG algebras $\underline{\mathrm{End}}_{-B}(K)$ and $\underline{\mathrm{End}}_{A-}(K)$ were introduced by Keller in [Kel01].

Lemma 4.5. *The derived left-action* L_A *is an* A_∞*-morphism from* A *to* $\underline{\mathrm{End}}_{-B}(K)$*.*

Proof. The condition for L_A to be an A_∞-morphism can be checked by means of its Taylor components. Specifically, recalling that the A_∞-structure on $\underline{\mathrm{End}}_{-B}(K)$ has only three non-trivial components, we need to check the two identities

$$\begin{aligned} &(\mathrm{L}_A \circ d_A)(1) = (Q \circ \mathrm{L}_A)(1),\\ &\sum_{k=0}^{m}\sum_{i=1}^{m-k+1}(-1)^{\sum_{j=1}^{i-1}(|a_j|-1)}\mathrm{L}_A^{m-k+1}\big(a_1|\cdots|d_A^k(a_i|\cdots|a_{i+k-1})|a_{i+k}|\cdots|a_m\big)\\ &\quad= -[d_{K,B}, \mathrm{L}_A^m(a_1|\cdots|a_m)]\\ &\qquad+\sum_{\substack{m_1+m_2=m\\ m_i\geqslant 1, i=1,2}}(-1)^{\sum_{k=1}^{m_1}(|a_k|-1)}\mathrm{L}_A^{m_1}(a_1|\cdots|a_{m_1})\circ \mathrm{L}_A^{m_2}(a_{m_1+1}|\cdots|a_m). \end{aligned} \tag{9}$$

The first identity in (9) follows immediately from the construction of the A_∞-structure on $\underline{\mathrm{End}}_{-B}(K)$. In order to prove the second one, we evaluate both sides of the equality explicitly on a general element of $K[1]\otimes \mathrm{T}(B[1])$, projecting down to $K[1]$: upon writing down the natural signs arising from Koszul's sign rule and the differential $[d_{K,B}, \bullet]$, we see immediately that this identity is equivalent to the condition that K be an A_∞-A-B-bimodule. □

Of course, similar arguments imply that there is an A_∞-morphism R_B from B to $\underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}$, where the suffix 'op' refers to the fact that we consider the opposite product on $\underline{\mathrm{End}}_{A-}(K)$: again, R_B being an A_∞-morphism is equivalent to K being an A_∞-A-B-bimodule.

Furthermore, L_A endows $\underline{\mathrm{End}}_{-B}(K)$ with the structure of a A_∞-A-A-bimodule, and R_B endows $\underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}$ with the structure of a A_∞-B-B-bimodule.

In a more conceptual manner, given two A_∞-algebras A and B along with an A_∞-morphism F from A to B, we first view both A and B as A_∞-bimodules in the sense of Remark 3.4. Then we define an A_∞-A-A-bimodule structure on B simply via the codifferential $d_B\circ(F\otimes 1\otimes F)$, where d_B here denotes the codifferential inducing the A_∞-B-B-bimodule structure on B.

To be explicit, we write down the Taylor components of the A_∞-A-A-bimodule structure on $\underline{\mathrm{End}}_{-B}(K)$. Since the A_∞-structure on $\underline{\mathrm{End}}_{-B}(K)$ has only three non-trivial Taylor components, a direct computation shows that

$$\begin{aligned} \mathrm{Q}^{0,0}(\varphi) &= -[d_{K,B},\varphi], \quad \mathrm{Q}^{m,n}=0 \quad \text{for } n,m\geqslant 1,\\ \mathrm{Q}^{m,0}(a_1|\cdots|a_m|\varphi) &= (-1)^{\sum_{k=1}^{m}(|a_k|-1)}\mathrm{L}_A^m(a_1|\cdots|a_m)\circ\varphi \quad \text{for } m\geqslant 1,\\ \mathrm{Q}^{0,n}(\varphi|a_1|\cdots|a_n) &= (-1)^{\varphi}\varphi\circ \mathrm{L}_A^m(a_1|\cdots|a_n) \quad \text{for } n\geqslant 1. \end{aligned} \tag{10}$$

Similar formulæ hold for the derived right-action.

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4.2 The Hochschild cochain complex of an A_∞-algebra

For the A_∞-algebra A, we consider its Hochschild cochain complex with values in itself. As we have already seen in § 3.1, this is defined as

$$\mathrm{C}^\bullet(A, A) = \mathrm{Coder}(\mathrm{T}(A[1])) = \mathrm{Hom}(\mathrm{T}(A[1]), A[1]),$$

the vector space of coderivations of the coalgebra $\mathrm{T}(A[1])$ (with the obvious coalgebra structure), with differential $[d_A, \bullet]$.

Now, given a general A_∞-A-A-bimodule M, we define the Hochschild cochain complex of A with values in M, denoted by $\mathrm{C}^\bullet(A, M)$, as the vector space of morphisms φ from $\mathrm{T}(A[1])$ to the bicomodule $\mathrm{T}(A[1]) \otimes M[1] \otimes \mathrm{T}(A[1])$ such that

$$\Delta_L \circ \varphi = (1 \otimes \varphi) \circ \Delta_A, \quad \Delta_R \circ \varphi = (\varphi \otimes 1) \circ \Delta_A.$$

The differential is then simply given by $d_M \varphi = d_M \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_A$. It is clear that

$$\mathrm{C}^\bullet(A, M) = \mathrm{Hom}(\mathrm{T}(A[1]), M[1]).$$

Remark 4.6. The previous definition of the Hochschild cochain complex $\mathrm{C}^\bullet(A, M)$ in the case where $M = A$ agrees with the definition of $\mathrm{C}^\bullet(A, A)$. This is because, in both cases, $C^\bullet(A, A) = \mathrm{Hom}(\mathrm{T}(A[1]), A[1])$ and A becomes an A_∞-A-A-bimodule in the sense of Remark 3.4, which implies that the differentials on the two complexes coincide.

We further consider the complex $\mathrm{C}^\bullet(A, B, K)$ with differential $[d_K, \bullet]$ as in § 3.1.

Finally, for A, B and K as above, we consider the A_∞-A-A-bimodule $\underline{\mathrm{End}}_{-B}(K)$; similar arguments work for the A_∞-B-B-bimodule $\underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}$.

Lemma 4.7. *The complexes* $(\mathrm{C}^\bullet(A, B, K), [d_K, \bullet])$ *and* $(\mathrm{C}^\bullet(A, \underline{\mathrm{End}}_{-B}(K)), d_{\underline{\mathrm{End}}_{-B}(K)})$ *are isomorphic.*

Proof. It suffices to give an explicit formula for the isomorphism: a general element φ of $\mathrm{C}^\bullet(A, B, K)$ is uniquely determined by its Taylor components $\varphi^{m,n}$ from $A[1]^{\otimes m} \otimes K[1] \otimes B[1]^{\otimes n}$ to $K[1]$.

On the other hand, a general element ψ of $\mathrm{C}^\bullet(A, \underline{\mathrm{End}}_{-B}(K))$ is also uniquely determined by its Taylor components ψ^m from $A[1]^{\otimes m}$ to $\underline{\mathrm{End}}_{-B}(K)$; in turn, any Taylor component $\psi^m(a_1|\cdots|a_m)$ is, by definition, completely determined by its Taylor components $(\psi^m(a_1|\cdots|a_m))^n$ from $K[1] \otimes B[1]^{\otimes n}$ to $K[1]$.

Thus, the isomorphism from $\mathrm{C}^\bullet(A, B, K)$ to $\mathrm{C}^\bullet(A, \underline{\mathrm{End}}_{-B}(K))$ is explicitly described via

$$(\widetilde{\varphi}^m(a_1|\cdots|a_m))^n(k|b_1|\cdots|b_n) = \varphi^{m,n}(a_1|\cdots|a_m|k|b_1|\cdots|b_n) \quad \text{for } m, n \geqslant 0.$$

It remains to prove that the previous isomorphism is a chain map. For simplicity, we omit the signs here, since they can all be deduced quite easily from our previous conventions and Koszul's sign rule; we shall write down only the formulæ from which we deduce the claim immediately. It also suffices, by construction, to prove the claim on the corresponding Taylor components.

Therefore, we consider

$$\begin{aligned}\left(\widetilde{[d_K, \varphi]}^m(a_1|\cdots|a_m)\right)^n(k|b_1|\cdots|b_n) &= ([d_K, \varphi])^{m,n}(a_1|\cdots|a_m|k|b_1|\cdots|b_n)\\ &= (d_K \circ \varphi)^{m,n}(a_1|\cdots|a_m|k|b_1|\cdots|b_n)\\ &\quad - (-1)^{|\varphi|}(\varphi \circ d_K)^{m,n}(a_1|\cdots|a_m|k|b_1|\cdots|b_n).\end{aligned}$$

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The first term in the last expression can be rewritten as a sum of terms of the form

$$d_K^{i-1,n-j}\big(a_1|\cdots|a_{i-1}|\varphi^{(m-i+1,j)}(a_i|\cdots|a_m|k|b_1|\cdots|b_j)|b_{j+1}|\cdots|b_n\big) \quad \text{with } 1\leqslant i\leqslant m+1,\ 0\leqslant j\leqslant n. \tag{11}$$

On the other hand, the second term in the last expression above is the sum of the following three types of terms:

$$\varphi^{i-1,n-j}\big(a_1|\cdots|a_{i-1}|d_K^{(m-i+1,j)}(a_i|\cdots|a_m|k|b_1|\cdots|b_j)|b_{j+1}|\cdots|b_n\big) \quad \text{with } 1\leqslant i\leqslant m+1,\ 0\leqslant j\leqslant n, \tag{12}$$

$$\varphi^{m-j+1,n}\big(a_1|\cdots|a_{i-1}|d_B^j(a_i|\cdots|a_{i+j-1})|a_{i+j}|\cdots|a_m|k|b_1|\cdots|b_n\big) \quad \text{with } 0\leqslant i\leqslant m+1,\ 0\leqslant j\leqslant m, \tag{13}$$

$$\varphi^{m,n-j+1}\big(a_1|\cdots|a_m|k|b_1|\cdots|b_{i-1}|d_A^j(b_i|\cdots|b_{i+j-1})|b_{i+j}|\cdots|b_n\big) \quad \text{with } 0\leqslant i\leqslant n+1,\ 0\leqslant j\leqslant n. \tag{14}$$

We now consider the expression

$$(d_{\underline{\mathrm{End}}_{-B}(K)}\widetilde{\varphi})^m(a_1|\cdots|a_m) = (d_{\underline{\mathrm{End}}_{-B}(K)}\circ\widetilde{\varphi})^m(a_1|\cdots|a_m) - (-1)^{|\widetilde{\varphi}|}(\widetilde{\varphi}\circ d_A)^m(a_1|\cdots|a_m).$$

If we apply the previous identity to an element $(k|b_1|\cdots|b_n)$ as above, then the second term on the right-hand side is, by definition, a sum of terms of the type (13).

Looking at the first term on the right-hand side of the previous expression, if we recall the Taylor components (10) of the A_∞-A-A-bimodule structure on $\underline{\mathrm{End}}_{-B}(K)$, we obtain

$$\begin{aligned}(d_{\underline{\mathrm{End}}_{-B}(K)}&\circ\widetilde{\varphi})^m(a_1|\cdots|a_m)\\ &= -[d_{K,B},\widetilde{\varphi}^m(a_1|\cdots|a_m)]\\ &\quad+\sum_{\substack{m_1+m_2=m\\ m_i\geqslant 1, i=1,2}}(-1)^{|\widetilde{\varphi}|+\sum_{k=1}^{m_1}(|a_k|-1)}\widetilde{\varphi}^{m_1}(a_1|\cdots|a_{m_1})\circ \mathrm{L}_A^{m_2}(a_{m_1+1}|\cdots|a_m)\\ &\quad+\sum_{\substack{m_1+m_2=m\\ m_i\geqslant 1, i=1,2}}(-1)^{(|\widetilde{\varphi}|+1)(\sum_{k=1}^{m_1}(|a_k|-1))}\mathrm{L}_A^{m_1}(a_1|\cdots|a_{m_1})\circ\widetilde{\varphi}^{m_2}(a_{m_1+1}|\cdots|a_m).\end{aligned} \tag{15}$$

The sum of expressions of type (12) with $i=m+1$ and expressions of type (14) equals, by definition, the first term on the right-hand side of (15); expressions of type (12) with $i\leqslant m$ sum up to the second term on the right-hand side of (15), while expressions of type (11) sum up to the third term on the right-hand side. □

The same arguments, with obvious adjustments, imply that the complex $(\mathrm{C}^\bullet(A,B,K), [d_K,\bullet])$ is isomorphic to the Hochschild chain complex $(\mathrm{C}^\bullet(B,\underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}), d_{\underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}})$, upon replacing L_A by R_B.

Finally, composition with L_A and composition with R_B define morphisms of complexes

$$\mathrm{L}_A:\mathrm{C}^\bullet(A,A)\to\mathrm{C}^\bullet(A,B,K)\cong\mathrm{C}^\bullet(A,\underline{\mathrm{End}}_{-B}(K)),$$
$$\mathrm{R}_B:\mathrm{C}^\bullet(B,B)\to\mathrm{C}^\bullet(A,B,K)\cong\mathrm{C}^\bullet(B,\underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}).$$

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More precisely, composition with L_A on $\mathrm{C}^\bullet(A,A)$ is defined via the assignment

$$(\mathrm{L}_A\circ\varphi)^{m,n}(a_1|\cdots|a_m|k|b_1|\cdots|b_n) = \sum_{i=0}^{m}\sum_{j=0}^{m+1}(-1)^{|\varphi|(\sum_{k=1}^{j-1}(|a_k|-1))}d_K^{m-i+1,n}(a_1|\cdots|\varphi^i(a_j|\cdots|a_{j+i-1})|\cdots|a_m|k|b_1|\cdots|b_n),$$

and a similar formula defines composition with R_B. The fact that composition with L_A or with R_A is a map of complexes is a direct consequence of the computations in the proofs of Lemmata 4.5 and 4.7.

Remark 4.8. Observe that, in the notation of § 3.1, the previous formula coincides with $d_K\{\varphi\}$.

4.3 Keller's condition

From the arguments in § 3.1, it is easy to verify that the natural projections p_A and p_B from $\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A,B,K))$ onto $\mathrm{C}^\bullet(A,A)$ and $\mathrm{C}^\bullet(B,B)$, respectively, are well-defined morphisms of complexes.

A natural question that arises in our context is the following: under what conditions are the projections p_A and p_B quasi-isomorphisms? This question generalizes, in the framework of A_∞-algebras and modules, a similar problem for DG algebras and DG modules, which was solved by Keller in [Kel03] and recently treated in the framework of deformation quantization by Shoikhet [Sho08].

In fact, when A and B are DG algebras and K is a DG A-B-bimodule, we may consider the DG category $\mathtt{Cat}(A,B,K)$ as in § 3. Analogously, we can consider the Hochschild cochain complex of $\mathtt{Cat}(A,B,K)$ with values in itself; again, this splits into three pieces, and the Hochschild differential d_γ, uniquely determined by the DG structures on A, B and K, splits into five pieces.

Again, the two natural projections p_A and p_B from $\mathrm{C}^\bullet(\mathtt{Cat}(A,B,K))$ onto $\mathrm{C}^\bullet(A,A)$ and $\mathrm{C}^\bullet(B,B)$ are morphisms of complexes. Keller proved in [Kel03] that both projections are quasi-isomorphisms if the derived left- and right-actions L_A and R_B from A and B to $\mathrm{RHom}^\bullet_{-B}(K,K)$ and $\mathrm{RHom}^\bullet_{A-}(K,K)^{\mathrm{op}}$, respectively, are quasi-isomorphisms. Here $\mathrm{RHom}_{-B}(K,K)$, for example, denotes the right derived functor of $\mathrm{Hom}_{-B}(\bullet,K)$ in the derived category $\mathcal{D}(\mathtt{Mod}_B)$ of the category $\mathtt{Mod}_B$ of graded right B-modules, whose spaces of morphisms are specified by

$$\mathrm{Hom}_{-B}(V,W)=\bigoplus_{p\in\mathbb{Z}}\mathrm{Hom}^p_{-B}(V,W)=\bigoplus_{p\in\mathbb{Z}}\mathrm{hom}_{-B}(V,W[p]).$$

The cohomology of the complex $\mathrm{RHom}^\bullet_{-B}(K,K)$ computes the derived functor $\mathrm{Ext}^\bullet_{-B}(K,K)$; accordingly, L_A denotes the derived right-action of A on K in the framework of derived categories.

We observe that the DG algebras $\underline{\mathrm{End}}_{-B}(K)$ and $\underline{\mathrm{End}}_{A-}(K)$ represent, respectively, $\mathrm{RHom}_{-B}(K,K)$ and $\mathrm{RHom}_{A-}(K,K)^{\mathrm{op}}$, where we take the bar resolution of K in $\mathtt{Mod}_B$ and ${}_A\mathtt{Mod}$, respectively (of course, the product structure on $\mathrm{RHom}_{A-}(K,K)^{\mathrm{op}}$ is induced by the opposite of the Yoneda product). Thus, the derived left- and right-actions in the A_∞ framework truly generalize the corresponding derived left- and right-actions in the case of a DG category, with the obvious advantage of providing explicit formulæ involving homotopies. Furthermore, in the framework of derived categories, the derived left- and right-actions L_A and R_B induce DG bimodule structures on $\mathrm{RHom}_{-B}(K,K)$ and $\mathrm{RHom}^\bullet_{A-}(K,K)^{\mathrm{op}}$ in a natural way; moreover, two components of the Hochschild differential d_γ are determined by composition with L_A and R_B.

Theorem 4.9. *Suppose that A, B and K are as above, with A and B assumed to be flat.*

If L_A is a quasi-isomorphism, then the canonical projection

$$\mathrm{p}_B : \mathrm{C}^\bullet(\mathtt{Cat}_\infty(A, B, K)) \twoheadrightarrow \mathrm{C}^\bullet(B, B)$$

is a quasi-isomorphism.

If R_B is a quasi-isomorphism, then the canonical projection

$$\mathrm{p}_A : \mathrm{C}^\bullet(\mathtt{Cat}_\infty(A, B, K)) \twoheadrightarrow \mathrm{C}^\bullet(A, A)$$

is a quasi-isomorphism.

Proof. We prove the claim for the derived left-action; the proof of the claim for the derived right-action is almost the same, with obvious modifications.

Since p_B is a chain map, that it is a quasi-isomorphism is tantamount to the acyclicity of $\mathrm{Cone}^\bullet(\mathrm{p}_B)$, the cone of p_B. First of all, $\mathrm{Cone}^\bullet(\mathrm{p}_B)$ is quasi-isomorphic to the subcomplex $\mathrm{Ker}(\mathrm{p}_B)$.[1]

Observe that $\mathrm{Ker}(\mathrm{p}_B) = \mathrm{C}^\bullet(A, B, K) \oplus \mathrm{C}^\bullet(A, A)$; by the arguments of § 3.1, $\mathrm{C}^\bullet(A, B, K)$ is a subcomplex thereof. Lemma 4.7 from § 4.2 then yields the isomorphism of complexes

$$\mathrm{C}^\bullet(A, B, K) \cong \mathrm{C}^\bullet(A, \underline{\mathrm{End}}_{-B}(K)).$$

As has already been observed, composition with the derived left-action L_A defines a morphism of complexes from $\mathrm{C}^\bullet(A, A)$ to $\mathrm{C}^\bullet(A, \underline{\mathrm{End}}_{-A}(K))$. From this and the arguments of § 3.1, it is easy to see that $\mathrm{C}^\bullet(A, B, K) \oplus \mathrm{C}^\bullet(A, A)$ is precisely the cone of the morphism induced by composition with L_A, which we shall denote by $\mathrm{Cone}(\mathrm{L}_A)$.

It is now a standard fact that for any A_∞-quasi-isomorphism of A_∞-algebras $A \to B$, the induced cochain map $\mathrm{C}^\bullet(A, A) \to \mathrm{C}^\bullet(A, B)$ is a quasi-isomorphism, where B is viewed as an A_∞-A-A-bimodule as explained at the end of § 4.1.

Therefore, $\mathrm{Cone}(\mathrm{L}_A)$ is quasi-isomorphic to the cone of the identity map on $\mathrm{C}^\bullet(A, A)$, which is obviously acyclic. □

5. Configuration spaces, their compactifications and colored propagators

In this section we discuss in some detail compactifications of configuration spaces of points in the complex upper half-plane $\mathbb{H}$ and on the real axis $\mathbb{R}$.

We will focus our attention on Kontsevich's eye $\mathcal{C}_{2,0}$ and the I-cube $\mathcal{C}_{2,1}$, in order to better formulate the properties of the two-colored and four-colored propagators, which will play a central role in the proof of the main result.

5.1 Configuration spaces and their compactifications

In this subsection, we recall compactifications of configuration spaces of points in the complex upper half-plane $\mathcal{H}$ and on the real axis $\mathbb{R}$.

[1] To be specific, as in [Sho08] we regard $\mathrm{Cone}(\mathrm{p}_B)$ as a bicomplex whose vertical differential is the sum of the corresponding Hochschild differentials of the two complexes involved and whose horizontal differential is $\mathrm{p}_B[1]$. It has only two columns, hence the associated spectral sequence stabilizes at E_2 and, moreover, E_1 coincides with $\mathrm{Ker}(\mathrm{p}_B)$.

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Consider a finite set A and a finite (totally) ordered set B. The open configuration space $C^+_{A,B}$ is defined by

$$C^+_{A,B} := \mathrm{Conf}^+_{A,B}/G_2 = \{(p,q) \in \mathbb{H}^A \times \mathbb{R}^B \mid p(a) \neq p(a') \text{ if } a \neq a', q(b) < q(b') \text{ if } b < b'\}/G_2,$$

where G_2 is the semidirect product $\mathbb{R}^+ \ltimes \mathbb{R}$, which acts diagonally on $\mathbb{H}^A \times \mathbb{R}^B$ via

$$(\lambda,\mu)(p,q) = (\lambda p + \mu, \lambda q + \mu) \quad \text{for } \lambda \in \mathbb{R}^+,\ \mu \in \mathbb{R}.$$

The action of the two-dimensional Lie group G_2 on such $(n+m)$-tuples is free precisely when $2|A| + |B| - 2 \geqslant 0$; in this case, $C^+_{A,B}$ is a smooth real manifold of dimension $2|A| + |B| - 2$. (Of course, when $|B|$ is either 0 or 1, we may simply drop the suffix $+$, as no ordering of B is involved.)

The configuration space C_A is defined as

$$C_A := \{p \in \mathbb{C}^A \mid p(a) \neq p(a') \text{ if } a \neq a'\}/G_3,$$

where G_3 is the semidirect product $\mathbb{R}^+ \ltimes \mathbb{C}$, which acts diagonally on $\mathbb{C}^A$ via

$$(\lambda,\mu)p = \lambda p + \mu \quad \text{for } \lambda \in \mathbb{R}^+,\ \mu \in \mathbb{C}.$$

The action of G_3, which is a real Lie group of dimension three, is free precisely when $2|A| - 3 \geqslant 0$, in which case C_A is a smooth real manifold of dimension $2|A| - 3$.

The configuration spaces $C^+_{A,B}$ and C_A admit compactifications à la Fulton and MacPherson, obtained by successive real blow-ups. We will not discuss here the construction of their compactifications $\mathcal{C}^+_{A,B}$ and $\mathcal{C}_A$, which are smooth manifolds with corners, and refer the reader to [CR08a, Kon03] for more details. We shall focus mainly on their stratification, in particular on the boundary strata of codimension one of $\mathcal{C}^+_{A,B}$.

To be specific, the compactified configuration space $\mathcal{C}^+_{A,B}$ is a stratified space, and its boundary strata of codimension one can be described as follows.

(i) There exist a subset A_1 of A and an ordered subset B_1 of successive elements of B such that

$$\partial_{A_1,B_1} \mathcal{C}^+_{A,B} \cong \mathcal{C}^+_{A_1,B_1} \times \mathcal{C}^+_{A \smallsetminus A_1, B \smallsetminus B_1 \sqcup \{*\}}. \tag{16}$$

Intuitively, this corresponds to the situation where points in $\mathbb{H}$, labelled by A_1, and successive points in $\mathbb{R}$, labelled by B_1, collapse to a single point labelled by $*$ in $\mathbb{R}$. Obviously, we must have $2|A_1| + |B_1| - 2 \geqslant 0$ and $2(|A| - |A_1|) + (|B| - |B_1| + 1) - 2 \geqslant 0$.

(ii) There is a subset A_1 of A such that

$$\partial_{A_1} \mathcal{C}^+_{A,B} \cong \mathcal{C}_{A_1} \times \mathcal{C}^+_{A \smallsetminus A_1 \sqcup \{*\}, B}. \tag{17}$$

This corresponds to the situation where points in $\mathbb{H}$, labelled by A_1, all collapse to a single point in $\mathbb{H}$, labelled by $*$. Again, we must have $2|A_1| - 3 \geqslant 0$ and $2(|A| - |A_1| + 1) + |B| - 2 \geqslant 0$.

5.2 Orientation of configuration spaces

We now say a few words about the orientation of (compactified) configuration spaces $\mathcal{C}^+_{A,B}$ and of their boundary strata of codimension one.

Following [AMM02], we consider the (left) principal G_2-bundle $\mathrm{Conf}^+_{A,B} \to C^+_{A,B}$ and define an orientation on the (open) configuration space $C^+_{A,B}$ in such a way that any trivialization of the G_2-bundle $\mathrm{Conf}^+_{A,B}$ is orientation-preserving.

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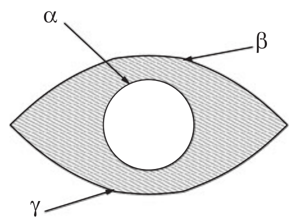

Figure 1. Kontsevich's eye.

Observe that: (i) the real, two-dimensional Lie group G_2 is oriented by the volume form $\Omega_{G_2} = db\, da$, where a general element of G_2 is denoted by (a, b) with $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$; and (ii) the real, $(2n+m)$-dimensional manifold $\mathrm{Conf}^+_{n,m}$ is oriented by the volume form $\Omega_{\mathrm{Conf}^+_{n,m}} = d^2 z_1 \cdots d^2 z_n dx_1 \cdots dx_m$, where $d^2 z_i = d\,\mathrm{Re}(z_i) d\,\mathrm{Im}(z_i)$ with z_i in $\mathbb{H}$ and x_j in $\mathbb{R}$.

We only recall, without going into any details, that there are three possible choices of global sections of $\mathrm{Conf}^+_{n,m}$, to which correspond three orientation forms on $C^+_{n,m}$ and on $\mathcal{C}^+_{n,m}$.

5.2.1 *Orientation of boundary strata of codimension one.* Recall the discussion at the end of § 5.1 on the boundary strata of codimension one of $\mathcal{C}^+_{A,B}$, for a finite subset A of $\mathbb{N}$ and a finite, ordered subset B of $\mathbb{N}$.

We are interested in determining the induced orientation on the two types of boundary strata (16) and (17). In fact, we want to compare the natural orientation of the boundary strata of codimension one induced from the orientation of $\mathcal{C}^+_{A,B}$ with the product orientation coming from the identifications (16) and (17).

Let us quote the following results from [AMM02, § I.2].

Lemma 5.1. *With the notation and conventions of § 5.1, we have that:*

(i) *for boundary strata of type (16),*

$$\Omega_{\partial_{A_1,B_1}\mathcal{C}^+_{A,B}} = (-1)^{j(|B_1|+1)-1}\Omega_{\mathcal{C}^+_{A_1,B_1}} \wedge \Omega_{\mathcal{C}^+_{A\smallsetminus A_1, B\smallsetminus B_1 \sqcup \{*\}}} \tag{18}$$

where j is the minimum of B_1;

(ii) *for boundary strata of type (17),*

$$\Omega_{\partial_{A_1}\mathcal{C}^+_{A,B}} = -\Omega_{\mathcal{C}_{A_1}} \wedge \Omega_{\mathcal{C}^+_{A\smallsetminus A_1 \sqcup \{*\}, B}}. \tag{19}$$

5.3 Explicit formulæ for the colored propagators

In this subsection we define and discuss the main properties of two-colored propagators and four-colored propagators, which will play a fundamental role in the constructions of §§ 6 and 7.

5.3.1 *The two-colored propagators.* First, we need an explicit description of the compactified configuration space $\mathcal{C}_{2,0}$, known as *Kontsevich's eye*. Figure 1 shows a picture of it, with the boundary strata of codimension one labelled by Greek letters.

We now describe the boundary strata of $\mathcal{C}_{2,0}$ of codimension one.

(i) The stratum labelled by α corresponds to $\mathcal{C}_2 = S^1$; intuitively, it describes the situation where the two points collapse to a single point in $\mathbb{H}$.

(ii) The stratum labelled by β corresponds to $\mathcal{C}_{1,1} \cong [0,1]$; it describes the situation where the first point goes to $\mathbb{R}$.

(iii) The stratum labelled by γ corresponds to $\mathcal{C}_{1,1} \cong [0,1]$; it describes the situation where the second point goes to $\mathbb{R}$.

For any two distinct points z and w in $\mathbb{H} \sqcup \mathbb{R}$, we set

$$\varphi^{+}(z,w) = \frac{1}{2\pi} \arg\left(\frac{z-w}{\overline{z}-w}\right) \quad \text{and} \quad \varphi^{-}(z,w) = \varphi^{+}(w,z).$$

Note that the real number $\varphi^{+}(z,w)$ represents the (normalized) angle between the geodesic from z to the point ∞ on the positive imaginary axis and the geodesic from z to w with respect to the hyperbolic metric of $\mathcal{H} \sqcup \mathbb{R}$, measured in the counterclockwise direction. Both functions are well-defined up to the addition of constant terms; therefore $\omega^{\pm} := d\varphi^{\pm}$ are well-defined 1-forms, which are obviously basic with respect to the action of G_2. In summary, $\omega^{\pm}$ are well-defined 1-forms on the open configuration space $C_{2,0}$.

Lemma 5.2. *The* 1*-forms* $\omega^{\pm}$ *extend to smooth* 1*-forms on Kontsevich's eye* $\mathcal{C}_{2,0}$*, satisfying the following properties:*

(i)
$$\omega^{\pm}|_{\alpha} = \pi_1^{*}(d\varphi), \tag{20}$$

where $d\varphi$ *denotes the (normalized) angle measured counterclockwise from the positive imaginary axis and* π_1 *is the projection from* $\mathcal{C}_2 \times \mathcal{C}_{1,0}$ *onto the first factor;*

(ii)
$$\omega^{+}|_{\beta} = 0 \quad \text{and} \quad \omega^{-}|_{\gamma} = 0. \tag{21}$$

Proof. First, observe that ω^{+} is the standard angle form of Kontsevich (see, e.g., [Kon03]); hence it is a smooth form on $\mathcal{C}_{2,0}$, enjoying the properties (20) and (21).

On the other hand, by definition we have $\omega^{-} = \tau^{*}\omega^{+}$, where τ is the involution of $\mathcal{C}_{2,0}$ which extends smoothly the involution $(z,w) \mapsto (w,z)$ on $\mathrm{Conf}_{2,0}$. Then, the smoothness of ω^{-} as well as properties (20) and (21) follow immediately. □

We refer to [CF04] for the physical origin of the two-colored Kontsevich propagators. Here we only mention that they arise from the Poisson sigma model in the presence of a brane (i.e. a coisotropic submanifold of the target Poisson manifold) dictating boundary conditions for the fields.

5.3.2 *The four-colored propagators.* We now describe the so-called four-colored propagators. For an explanation of their physical origin, which can be traced back to boundary conditions for the Poisson sigma model dictated by two branes (i.e. two coisotropic submanifolds of the target Poisson manifold), we refer once again to [CF04].

Here, we are mainly interested in the precise construction of the four-colored propagators and their properties. For this purpose, we seek an appropriate compactified configuration space to which the naïve definition of the four-colored propagators will extend smoothly.

The I-cube. We briefly describe the compactified configuration space $\mathcal{C}_{2,1}$ of two distinct points in the complex upper half-plane $\mathbb{H}$ and one point on the real axis $\mathbb{R}$. By construction, it is a smooth manifold with corners of real dimension three, called the *I-cube*; it is depicted in Figure 2. The I-cube's boundary stratification consists of 9 strata of codimension one, 20 strata of codimension two and 12 strata of codimension three. We will describe explicitly

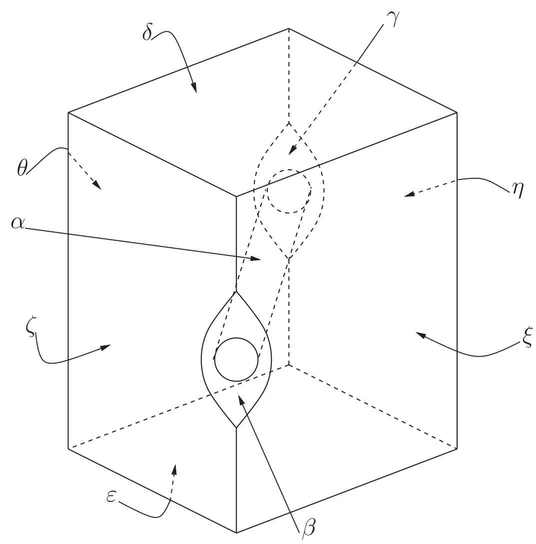

Figure 2. The I-cube $\mathcal{C}_{2,1}$.

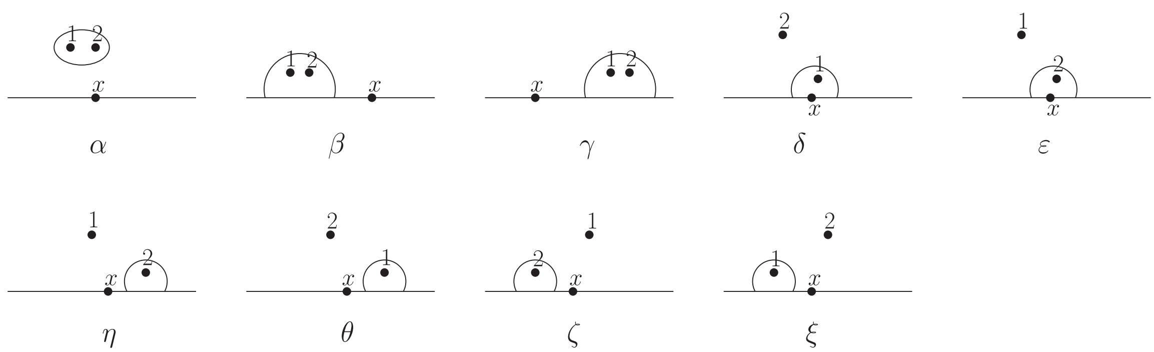

Figure 3. Boundary strata of the I-cube of codimension one.

only the boundary strata of codimension one; the boundary strata of higher codimensions can easily be characterized by inspecting the former strata.

Before giving a mathematical description of the codimension-one boundary strata of $\mathcal{C}_{2,1}$, we represent them pictorially in Figure 3. The boundary stratum labelled by α factors as $\mathcal{C}_2 \times \mathcal{C}_{1,1}$; since $\mathcal{C}_2 = S^1$ and $\mathcal{C}_{1,1}$ is a closed interval, α is a cylinder.

Remark 5.3. Consider the open configuration space $C_{1,1} \cong \{e^{it} : t \in (0, \pi)\}$. On this space, take the closed 1-form $dt/(2\pi)$; it extends smoothly to a closed 1-form ρ on the compactified configuration space $\mathcal{C}_{1,1}$, which vanishes on the two boundary strata of codimension one. These properties will play a central role in subsequent computations.

The boundary strata labelled by β and γ are both described by $\mathcal{C}_{2,0} \times \mathcal{C}^+_{0,2}$, the only difference being the position of the cluster corresponding to $\mathcal{C}_{2,0}$ relative to the point x on $\mathbb{R}$. Since $\mathcal{C}^+_{0,2}$ is zero-dimensional, the strata α and β are two copies of Kontsevich's eye $\mathcal{C}_{2,0}$.

The boundary strata labelled by δ and ε are both described by $\mathcal{C}_{1,1} \times \mathcal{C}_{1,1}$, distinguished according to which of the points labelled by 1 and 2 collapses to the point x on the real axis. Since $\mathcal{C}_{1,1}$ is a closed interval, δ and ε are both two squares.

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Finally, the boundary strata labelled by η, θ, ζ and ξ all factor as $\mathcal{C}^+_{1,2} \times \mathcal{C}_{1,0}$; they differ by which of points 1 and 2 goes to the real axis and whether it goes to the left or to the right of x. Since $\mathcal{C}_{1,0}$ is zero-dimensional, these boundary strata correspond to $\mathcal{C}^+_{1,2}$. The latter compactified configuration space is a hexagon, which can be verified easily by direct inspection of its boundary stratification.

Explicit formulae for the four-colored propagators. First of all, observe that there is a projection $\pi_{2,0}$ from $\mathcal{C}_{2,1}$ onto $\mathcal{C}_{2,0}$ which extends smoothly the obvious projection from $C_{2,1}$ onto $C_{2,0}$, forgetting the point x on the real axis. It therefore makes sense to set

$$\omega^{+,+} = \pi^*_{2,0}(\omega^+) \quad \text{and} \quad \omega^{-,-} = \pi^*_{2,0}(\omega^-).$$

Further, consider a triple (z, w, x) where z and w are two distinct points in $\mathbb{H}$ and x is a point on $\mathbb{R}$. Recall that the complex function $z \mapsto \sqrt{z}$ is a well-defined holomorphic function on $\mathbb{H}$, mapping $\mathbb{H}$ to the first quadrant $\mathbb{Q}^{+,+}$ of the complex plane; therefore, it makes sense to consider the 1-forms

$$\omega^{+,-}(z, w, x) = \frac{1}{2\pi} d \arg\left(\frac{\sqrt{z-x} - \sqrt{w-x}}{\sqrt{z-x} - \overline{\sqrt{w-x}}} \frac{\sqrt{z-x} + \overline{\sqrt{w-x}}}{\sqrt{z-x} + \sqrt{w-x}} \right),$$

$$\omega^{-,+}(z, w, x) = \frac{1}{2\pi} d \arg\left(\frac{\sqrt{z-x} - \sqrt{w-x}}{\sqrt{z-x} + \overline{\sqrt{w-x}}} \frac{\sqrt{z-x} - \overline{\sqrt{w-x}}}{\sqrt{z-x} + \sqrt{w-x}} \right).$$

Thus, $\omega^{+,-}$ and $\omega^{-,+}$ are smooth forms on the open configuration space $\mathrm{Conf}_{2,1}$. We recall that there is an action of the two-dimensional Lie group G_2 on $\mathrm{Conf}_{2,1}$. It is not difficult to verify that both of the 1-forms $\omega^{+,-}$ and $\omega^{-,+}$ are basic with respect to the action of G_2; hence they both descend to smooth forms on the open configuration spaces $C_{2,1}$.

In the following lemma, we use the convention that the point in $\mathbb{H}$ labelled by 1 (respectively, 2) corresponds to the initial (respectively, final) argument in $\mathbb{H}$ of the forms under consideration.

LEMMA 5.4. *The 1-forms $\omega^{+,+}$, $\omega^{+,-}$, $\omega^{-,+}$ and $\omega^{-,-}$ extend smoothly to the I-cube $\mathcal{C}_{2,1}$ and enjoy the following properties:*

(i)
$$\begin{aligned} \omega^{+,+}|_\alpha &= \pi_1^*(d\varphi), \quad \omega^{+,-}|_\alpha = \pi_1^*(d\varphi) - \pi_2^*(\rho), \\ \omega^{-,+}|_\alpha &= \pi_1^*(d\varphi) - \pi_2^*(\rho), \quad \omega^{-,-}|_\alpha = \pi_1^*(d\varphi), \end{aligned} \tag{22}$$

where π_i ($i = 1$ or 2) denotes the projection of the boundary stratum α onto the ith factor of the decomposition $\mathcal{C}_2 \times \mathcal{C}_{1,1}$ and ρ is the smooth 1-form on $\mathcal{C}_{1,1}$ discussed in Remark 5.3;

(ii)
$$\begin{aligned} &\omega^{+,+}|_\beta = \omega^+, \quad \omega^{+,-}|_\beta = \omega^+, \quad \omega^{-,+}|_\beta = \omega^-, \quad \omega^{-,-}|_\beta = \omega^-, \\ &\omega^{+,+}|_\gamma = \omega^+, \quad \omega^{+,-}|_\gamma = \omega^-, \quad \omega^{-,+}|_\gamma = \omega^+, \quad \omega^{-,-}|_\gamma = \omega^-, \end{aligned} \tag{23}$$

where we implicitly identify both boundary strata with Kontsevich's eye (see also § 5.3.1);

(iii)
$$\begin{aligned} \omega^{+,+}|_\delta &= \omega^{+,-}|_\delta = \omega^{-,+}|_\delta = 0, \\ \omega^{+,-}|_\varepsilon &= \omega^{-,+}|_\varepsilon = \omega^{-,-}|_\varepsilon = 0; \end{aligned} \tag{24}$$

(iv)
$$\begin{aligned} &\omega^{+,-}|_\eta = \omega^{-,-}|_\eta = 0, \quad \omega^{+,+}|_\theta = \omega^{-,+}|_\theta = 0, \\ &\omega^{-,+}|_\zeta = \omega^{-,-}|_\zeta = 0, \quad \omega^{+,+}|_\xi = \omega^{+,-}|_\xi = 0. \end{aligned} \tag{25}$$

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Proof. First of all, since the projection $\pi_{2,0} : \mathcal{C}_{2,1} \to \mathcal{C}_{2,0}$ is smooth, Lemma 5.2 from § 5.3.1 implies immediately that $\omega^{+,+}$ and $\omega^{-,-}$ are smooth 1-forms on $\mathcal{C}_{2,1}$. Lemma 5.2 also immediately yields properties (i), (ii), (iii) and (iv) for $\omega^{+,+}$ and $\omega^{-,-}$.

It remains to prove smoothness and properties (i)–(iv) for $\omega^{+,-}$ and $\omega^{-,+}$ on $\mathcal{C}_{2,1}$. We shall prove the statements for $\omega^{-,+}$; similar computations give the results for $\omega^{+,-}$.

In order to prove the assertions, we shall make use of local coordinates of $\mathcal{C}_{2,1}$ near the boundary strata of codimension one in all cases.

We begin by considering the boundary stratum labelled by α. Local coordinates of $\mathcal{C}_{2,1}$ near α are specified via

$$\mathcal{C}_2 \times \mathcal{C}_{1,1} \cong S^1 \times [0, \pi] \ni (\varphi, t) \mapsto [(e^{it}, e^{it} + \varepsilon e^{i\varphi}, 0)] \in \mathcal{C}_{2,1}, \quad \varepsilon > 0,$$

so that α is recovered as ε tends to 0. We have implicitly used local sections of $C_{1,1}$ and $C_{2,1}$: the point on $\mathbb{R}$ has been placed at 0, and the first point in $\mathbb{H}$ has been placed on a circle of radius 1 centered at 0.

Then, using the standard notation $[(z, w, x)]$ for a point in $\mathcal{C}_{2,1}$, we have

$$\sqrt{w-x} = \sqrt{z - x + \varepsilon e^{i\varphi}} = \sqrt{z-x} + \varepsilon \frac{1}{2\sqrt{|z-x|}} e^{i(\varphi - \frac{1}{2}t)} + \mathcal{O}(\varepsilon^2), \quad z = e^{it}, \quad x = 0.$$

Substituting the above into the rightmost expression in the definition of $\omega^{-,+}$ and taking the limit as ε tends to 0 gives

$$\omega^{-,+}|_\alpha = \frac{1}{2\pi}(d\varphi - dt) = \pi_1^* d\varphi - \pi_2^*(\rho),$$

where ρ is the smooth 1-form discussed in Remark 5.3. Note that in the last equality we have abused the notation $d\varphi$ so as to be consistent with the notation of Lemma 5.2.

We now consider the boundary strata labelled by β and γ. Local coordinates of $\mathcal{C}_{2,1}$ near β and near γ are specified via

$$\begin{aligned}\mathcal{C}_{2,0} \times \mathcal{C}_{0,2}^{+} \cong \mathcal{C}_{2,0} \times \{-1, 0\} &\ni ((i, i + \rho e^{i\varphi}), (-1, 0)) \\ &\mapsto [(-1 + \varepsilon i, -1 + \varepsilon(i + \rho e^{i\varphi}), 0))] \in \mathcal{C}_{2,1}\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_{2,0} \times \mathcal{C}_{0,2}^{+} \cong \mathcal{C}_{2,0} \times \{0, 1\} &\ni ((i, i + \rho e^{i\varphi}), (0, 1)) \\ &\mapsto [(1 + \varepsilon i, 1 + \varepsilon(i + \rho e^{i\varphi}), 0))] \in \mathcal{C}_{2,1},\end{aligned}$$

respectively, for $\rho, \varepsilon > 0$, where again β and γ are recovered as ε tends to 0. (Once again, we have made use of local sections of the interior of $\mathcal{C}_{2,0}$ and $\mathcal{C}_{2,1}$.)

Using the standard notation for a general point in (the interior of) $\mathcal{C}_{2,1}$, we have, near the boundary strata β and γ,

$$\sqrt{z-x} = \sqrt{y - x + \varepsilon \widetilde{z}} = i - \varepsilon \frac{i\widetilde{z}}{2} + \mathcal{O}(\varepsilon^2), \quad \sqrt{w-x} = \sqrt{y - x + \varepsilon \widetilde{w}} = i - \varepsilon \frac{i\widetilde{w}}{2} + \mathcal{O}(\varepsilon^2)$$

and

$$\sqrt{z-x} = \sqrt{y - x + \varepsilon \widetilde{z}} = 1 + \varepsilon \frac{\widetilde{z}}{2} + \mathcal{O}(\varepsilon^2), \quad \sqrt{w-x} = \sqrt{y - x + \varepsilon \widetilde{w}} = 1 + \varepsilon \frac{\widetilde{w}}{2} + \mathcal{O}(\varepsilon^2),$$

respectively, where $\tilde{z} = i$, $\tilde{w} = i + \rho e^{i\varphi}$, $y = -1$ for the stratum β, $y = 1$ for the stratum γ, and $x = 0$.

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By substituting the rightmost expressions of the above identities into $\omega^{+,-}$ and $\omega^{-,+}$ and letting ε tend to 0, we obtain (ii): in particular, the restrictions of $\omega^{+,-}$ and $\omega^{-,+}$ to β and γ are smooth 1-forms.

Next, we consider the boundary strata labelled by δ and ε. Local coordinates of $\mathcal{C}_{2,1}$ near δ and ε are specified via

$$\mathcal{C}_{1,1}\times\mathcal{C}_{1,1}\cong[0,\pi]\times[0,\pi]\ni(s,t)\mapsto[(\rho e^{is},e^{it},0)]\in\mathcal{C}_{2,1}$$

and

$$\mathcal{C}_{1,1}\times\mathcal{C}_{1,1}\cong[0,\pi]\times[0,\pi]\ni(s,t)\mapsto[(e^{is},\rho e^{it},0)]\in\mathcal{C}_{2,1},$$

respectively, such that δ and ε are recovered as ρ tends to 0.

Using again the standard notation for a point in (the interior of) $\mathcal{C}_{2,1}$, we then get

$$\sqrt{z-x}=\sqrt{\rho}\sqrt{\widetilde{z}}\quad\text{and}\quad\sqrt{w-x}=\sqrt{\rho}\sqrt{\widetilde{w}},$$

where $\widetilde{z}=e^{is}$ and $\widetilde{w}=e^{it}$. The remaining square roots do not contain ρ.

If we substitute the preceding expressions into $\omega^{+,-}$ and $\omega^{-,+}$ and let ρ tend to 0, we easily obtain

$$\omega^{-,+}|_\delta=\omega^{+,-}|_\delta=0\quad\text{and}\quad\omega^{-,+}|_\varepsilon=\omega^{+,-}|_\varepsilon=0,$$

which imply, in particular, that the restrictions of $\omega^{+,-}$ and $\omega^{-,+}$ to δ and ε are smooth 1-forms.

Finally, we consider the boundary stratum labelled by η. Local coordinates nearby are specified via

$$\mathcal{C}_{1,0}\times\mathcal{C}_{1,2}^{+}\cong\{i\}\times\mathcal{C}_{1,2}^{+}\ni(i,(z,0,1))\mapsto[(z,1+\varepsilon i,0)]\in\mathcal{C}_{2,1},$$

where η is recovered as ε tends to 0. Here, we have used global sections of $C_{1,1}$, $C_{1,2}^{+}$ and $C_{2,1}$, and used the action of G_2 to put the point in $\mathbb{H}$ to i, to put the first and second points on $\mathbb{R}$ to 0 and 1, and to put the point on $\mathbb{R}$ to 0 and the real part of the second point in $\mathbb{H}$ to 1.

Computations similar in spirit to those in the previous paragraphs enable us to compute explicit expressions for the restrictions of $\omega^{+,-}$ and $\omega^{-,+}$ to η. In particular, we see that $\omega^{+,-}$ and $\omega^{-,+}$ restrict to smooth 1-forms on $\mathcal{C}_{1,2}^{+}$, and we also get property (iv). $\square$

The four-colored propagators on the first quadrant. Note that the complex function $z\mapsto\sqrt{z}$ restricts to a holomorphic function on $\mathbb{H}\sqcup\mathbb{R}\smallsetminus\{0\}$, whose image is $\mathcal{Q}^{+,+}\sqcup\mathbb{R}^+\sqcup i\mathbb{R}^+$; the negative real axis is mapped to $i\mathbb{R}^+$, the positive real axis is mapped to itself, and $\mathbb{H}$ is mapped to $\mathcal{Q}^{+,+}$. Further, $z\mapsto\sqrt{z}$ is multi-valued when considered as a function on $\mathbb{C}$, with 0 as a branching point.

There is an explicit global section of the projection $\mathrm{Conf}_{2,1}\to C_{2,1}$, namely

$$C_{2,1}\ni[(z,w,x)]\mapsto\left(\frac{z-x}{|z-x|},\frac{w-x}{|z-x|},0\right)\in\mathrm{Conf}_{2,1}.$$

Setting $\widetilde{z}=(z-x)/|z-x|$ and $\widetilde{w}=(w-x)/|z-x|$, we get two points in $\mathbb{H}$; hence, upon taking $u=\sqrt{\widetilde{z}}$ and $v=\sqrt{\widetilde{w}}$, we see that u and v lie in $\mathcal{Q}^{+,+}$. We then find alternative descriptions of the four-colored propagators as follows:

$$\omega^{+,+}(u,v)=\frac{1}{2\pi}d\arg\left(\frac{u-v}{\overline{u}-v}\frac{u+v}{\overline{u}+v}\right),\quad\omega^{+,-}(u,v)=\frac{1}{2\pi}d\arg\left(\frac{u-v}{u-\overline{v}}\frac{u+\overline{v}}{u+v}\right),$$
$$\omega^{-,+}(u,v)=\frac{1}{2\pi}d\arg\left(\frac{u-v}{u+\overline{v}}\frac{u-\overline{v}}{u+v}\right),\quad\omega^{+,+}(u,v)=\frac{1}{2\pi}d\arg\left(\frac{u-v}{u-\overline{v}}\frac{u+v}{u+\overline{v}}\right).$$

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We observe that, by rescaling, the previous formulæ descend to the quotient of the configuration space of two points in $\mathcal{Q}^{+,+}$ with respect to the action of $G_1 \cong \mathbb{R}^+$.

In fact, the present description of the four-colored propagators is the original one; see [CF04]. We prefer to work with the previous (apparently more complicated) description, because it is better suited to dealing with compactified configuration spaces.

Finally, we remark that all previous formulæ are special cases of the main result in [Fer08], where general (super-)propagators for the Poisson sigma model in the presence of n branes, with $n \geqslant 1$, are explicitly produced.

6. L_∞-algebras and morphisms

In this section, we briefly discuss the concepts of L_∞-algebra and L_∞-morphism; moreover, we describe explicitly the two L_∞-algebras (which are actual genuine DG Lie algebras) which will be central in the constructions of § 7.

A DG Lie algebra $\mathfrak{g}$ is an object of $\mathtt{GrMod}_k$ endowed with an endomorphism $d_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ of degree 1 and a graded anti-symmetric bilinear map $[\bullet, \bullet] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ of degree 0 such that $d_\mathfrak{g}$ squares to 0 and

$$d_\mathfrak{g}([x, y]) = [d_\mathfrak{g}(x), y] + (-1)^{|x|}[x, d_\mathfrak{g}(y)],$$
$$(-1)^{|x||z|}[[x, y], z] + (-1)^{|x||y|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0$$

for any homogeneous elements x, y and z of $\mathfrak{g}$. The first identity above is the graded Leibniz rule, while the second is the graded Jacobi identity.

A formal pointed Q-manifold is an object V of $\mathtt{GrMod}_k$ such that $\mathrm{C}^+(V) \cong \mathrm{S}^+(V)$ is endowed with a codifferential Q. A morphism U between Q-manifolds (U, Q_U) and (V, Q_V) is a coalgebra morphism $\mathrm{C}^+(V) \to \mathrm{C}^+(V')$ of degree 0 that intertwines Q_U and Q_V.

Definition 6.1. An L_∞-structure on an object $\mathfrak{g}$ of the category $\mathtt{GrMod}_k$ is a Q-manifold structure on $\mathfrak{g}[1]$; the pair $(\mathfrak{g}, Q)$ is called an L_∞-algebra. Accordingly, a morphism F between L_∞-algebras $(\mathfrak{g}_1, Q_1)$ and $(\mathfrak{g}_2, Q_2)$ is a morphism between the corresponding pointed Q-manifolds.

The fact that Q is a coderivation on $\mathrm{S}^+(\mathfrak{g}[1])$ implies that Q is uniquely determined by its Taylor components $Q_n : \mathrm{S}^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$. An explicit formula for recovering Q from its Taylor components can be found in [Dol06, Kon03], for instance; here we just mention that it is similar in spirit to the formulæ appearing in the case of A_∞-structures, although the fact that we consider the symmetric algebra leads to shuffles arising.

Furthermore, the fact that an L_∞-morphism $F : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a coalgebra morphism implies that F is also uniquely determined by its Taylor components $F_n : \mathrm{S}^n(\mathfrak{g}_1[1]) \to \mathfrak{g}_2[1]$.

Remark 6.2. If $(\mathfrak{g}, d_\mathfrak{g}, [\bullet, \bullet])$ is a DG Lie algebra, then $\mathfrak{g}$ has the structure of an L_∞-algebra, which we now describe explicitly. All Taylor components of the coderivation vanish, except for Q_1 and Q_2, which are specified by

$$Q_1 = d_\mathfrak{g}, \quad Q_2(x_1, x_2) = (-1)^{|x_1|}[x_1, x_2] \quad \text{for } x_i \in \mathfrak{g}^{|x_i|} = (\mathfrak{g}[1])^{|x_i|-1}\ (i = 1, 2). \tag{26}$$

In fact, it is easy to verify that $Q^2 = 0$ is equivalent to the compatibility between $d_\mathfrak{g}$ and $[\bullet, \bullet]$ (the graded Leibniz rule) and the graded Jacobi identity.

We consider an L_∞-morphism $F: \mathfrak{g}_1 \to \mathfrak{g}_2$ between L_∞-algebras: the condition that F intertwine the codifferentials Q_1 and Q_2 can be rewritten as an infinite set of quadratic relations involving the Taylor coefficients of Q_1, Q_2 and F.

For example, assuming that $\mathfrak{g}_i$, $i = 1, 2$, are DG Lie algebras, the quadratic identities of order one and order two take the form

$$Q_2^1(F_1(x)) = F_1(Q_1^1(x)), \tag{27}$$

$$Q_2^2(F_1(x), F_1(y)) - F_1(Q_1^2(x,y)) \\ = F_2(Q_1^1(x), y) + (-1)^{|x|-1} F_2(x, Q_1^1(y)) - Q_2^1(F_2(x,y)) \quad \text{for } x, y \in \mathfrak{g}_1[1]. \tag{28}$$

Equation (27) is equivalent to stating that F_1 is a morphism of complexes, while (28) expresses the fact that F_1 is a morphism of graded Lie algebras up to a homotopy expressed by the Taylor component F_2.

More generally, we have the following proposition; we refer to [AMM02] for its proof.

Proposition 6.3. *Consider two DG Lie algebras $(\mathfrak{g}_1, d_1, [\bullet,\bullet]_1)$ and $(\mathfrak{g}_2, d_2, [\bullet,\bullet]_2)$, which we also view as L_∞-algebras as in Remark 6.2.*

Then, a coalgebra morphism $F: \mathrm{S}^+(\mathfrak{g}_1[1]) \to \mathrm{S}^+(\mathfrak{g}_2[1])$ is an L_∞-morphism if and only if it satisfies

$$Q_1'(F_n(\alpha_1, \dots, \alpha_n)) + \frac{1}{2} \sum_{I \sqcup J = \{1,\dots,n\}, I,J \neq \emptyset} \epsilon_\alpha(I,J) Q_2'(F_{|I|}(\alpha_I), F_{|J|}(\alpha_J))$$

$$= \sum_{k=1}^{n} \sigma_\alpha(k, 1, \dots, \hat{k}, \dots, n) F_n(Q_1(\alpha_k), \alpha_1, \dots, \widehat{\alpha_k}, \dots, \alpha_n) \\ + \frac{1}{2} \sum_{k \neq l} \sigma_\alpha(k, l, 1, \dots, \hat{k}, \dots, \hat{l}, \dots, n) F_{n-1} \\ \times (Q_2(\alpha_k, \alpha_l), \alpha_1, \dots, \widehat{\alpha_k}, \dots, \widehat{\alpha_l}, \dots, \alpha_n), \tag{29}$$

where $\epsilon_\alpha(I,J)$ denotes the sign associated to the shuffle relative to the decomposition $I \sqcup J = \{1, \dots, n\}$ and $\sigma_\alpha(\dots)$ denotes the sign associated to the permutation in $(\dots)$ (see § 2).

6.1 The DG Lie algebras $T_{\mathrm{poly}}(X)$ for $X = k^d$

Consider now a ground field k of characteristic 0 which contains $\mathbb{R}$ or $\mathbb{C}$; we further set $X = k^d$.

To X we associate the DG Lie algebra $T_{\mathrm{poly}}(X)$ of poly-vector fields on X with shifted degree. More precisely, the degree-p component $T_{\mathrm{poly}}^p(X)$, where $p \geqslant -1$, is given by $\Gamma(X, \wedge^{p+1} \mathrm{T}X) = \mathrm{S}(X^*) \otimes \wedge^{p+1}(X)$, with trivial differential and Schouten–Nijenhuis bracket determined by extending the Lie bracket between vector fields on X as a (graded) biderivation.

Hence $T_{\mathrm{poly}}(X)$ is an L_∞-algebra, whose Q-manifold structure is

$$Q_1 = 0, \quad Q_2(\alpha_1, \alpha_2) := -(-1)^{(k_1-1)(k_2)} [\alpha_2, \alpha_1]_{SN} = \alpha_1 \bullet \alpha_2 + (-1)^{k_1 k_2} \alpha_2 \bullet \alpha_1$$

for general elements $\alpha_1 \in T_{\mathrm{poly}}^{k_1-1}(X)$ and $\alpha_2 \in T_{\mathrm{poly}}^{k_2-1}(X)$, where the composition $\bullet$ is given by

$$\alpha_1 \bullet \alpha_2 = \sum_{l=1}^{k_1} (-1)^{l-1} \alpha_1^{i_1 \dots i_{k_1}} \partial_l \alpha_2^{j_1,\dots j_{k_2}} \partial_{i_1} \wedge \dots \wedge \widehat{\partial_{i_l}} \wedge \dots \wedge \partial_{i_{k_1}} \wedge \partial_{j_1} \wedge \dots \wedge \partial_{j_{k_2}}. \tag{30}$$

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6.2 The DG Lie algebra $\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A, B, K))$

We consider again the d-dimensional k-vector space $X = k^d$ and further assume X to be endowed with an inner product (therefore, we may safely assume here that $k = \mathbb{R}$ or $k = \mathbb{C}$). Consider two vector subspaces U and V of X such that, with respect to the previously introduced inner product, the following decomposition holds:

$$X = (U \cap V) \overset{\perp}{\oplus} (U^\perp \cap V) \overset{\perp}{\oplus} (U \cap V^\perp) \overset{\perp}{\oplus} (U + V)^\perp. \tag{31}$$

It follows immediately from (31) that

$$U = (U \cap V) \overset{\perp}{\oplus} (U \cap V^\perp), \quad V = (U \cap V) \overset{\perp}{\oplus} (U^\perp \cap V).$$

To X, U and V we may associate three graded vector spaces

$$\begin{aligned} A &= \Gamma(U, \wedge(\mathrm{N}U)) = \mathrm{S}(U^*) \otimes \wedge(X/U) = \mathrm{S}(U^*) \otimes \wedge(U^\perp \cap V) \otimes \wedge(U + V)^\perp, \\ B &= \Gamma(V, \wedge(\mathrm{N}V)) = \mathrm{S}(V^*) \otimes \wedge(X/V) = \mathrm{S}(V^*) \otimes \wedge(U \cap V^\perp) \otimes \wedge(U + V)^\perp, \\ K &= \Gamma(U \cap V, \wedge(\mathrm{T}X/(\mathrm{T}U + \mathrm{T}V))) = \mathrm{S}((U \cap V)^*) \otimes \wedge(U + V)^\perp, \end{aligned}$$

where $\mathrm{T}X$ and $\mathrm{N}U$ denote, respectively, the tangent bundle of X and the normal bundle of U in $\mathrm{T}X$.

Therefore A and B, endowed with the trivial differential, both admit a (trivial) A_∞-algebra structure. We now construct on K a non-trivial A_∞-A-B-bimodule structure.

We consider a set of linear coordinates $\{x_i\}$ on X which are adapted to the orthogonal decomposition (31) in the following sense: there are two non-disjoint subsets I_i and I_2 of $[d]$ such that

$$[d] = (I_1 \cap I_2) \sqcup (I_1 \cap I_2^c) \sqcup (I_1^c \cap I_2) \sqcup (I_1^c \cap I_2^c)$$

and $\{x_i\}$ is a set of linear coordinates on $U \cap V$, $U \cap V^\perp$, $U^\perp \cap V$ or $(U + V)^\perp$ if the index i belongs to $I_1 \cap I_2$, $I_1 \cap I_2^c$, $I_1^c \cap I_2$ or $I_1^c \cap I_2^c$, respectively.

To a general pair (n, m) of non-negative integers we associate the set $\mathcal{G}_{n,m}$ of admissible graphs of type (n, m): a general element Γ of $\mathcal{G}_{n,m}$ is a directed graph (i.e. every edge of Γ has an orientation) with n vertices of the first type and m vertices of the second type. We denote by $\mathrm{E}(\Gamma)$ and $\mathrm{V}(\Gamma)$ the sets of edges and vertices, respectively, of an admissible graph Γ.

Remark 6.4. We observe that, a priori, the admissible graphs considered here are permitted to have multiple edges (i.e. between any two distinct vertices there may be more than one edge) and loops (edges connecting a vertex of the first type to itself). As we shall see below, multiple edges and loops do not arise in the construction of the A_∞-A-B-bimodule structure on K, but they do arise in § 7 in the construction of a formality morphism.

We now take any pair of non-negative integers (m, n) and associate to it the compactified configuration space $\mathcal{C}^+_{0,m+1+n}$. We have $m + 1 + n$ ordered points on $\mathbb{R}$, one of which, namely the $(m + 1)$st point, plays a central role, whence the notation. For example, using the action of G_2 on $C^+_{0,m+1+n}$, we may put this 'central point' at $x = 0$.

Accordingly, we consider the set $\mathcal{G}_{0,m+1+n}$ of admissible graphs of type $(0, m + 1 + n)$: given any edge $e = (i, j)$ of a general admissible graph Γ, where the labels i and j, refer to, respectively, the initial and final points of e, we associate with it a projection $\pi_e : \mathcal{C}^+_{0,m+1+n} \to \mathcal{C}^+_{0,3} \subset \mathcal{C}_{2,1}$ or $\pi_e : \mathcal{C}^+_{0,m+1,n} \to \mathcal{C}^+_{2,0} \times \mathcal{C}_{1,1} \subset \mathcal{C}_{2,1}$.

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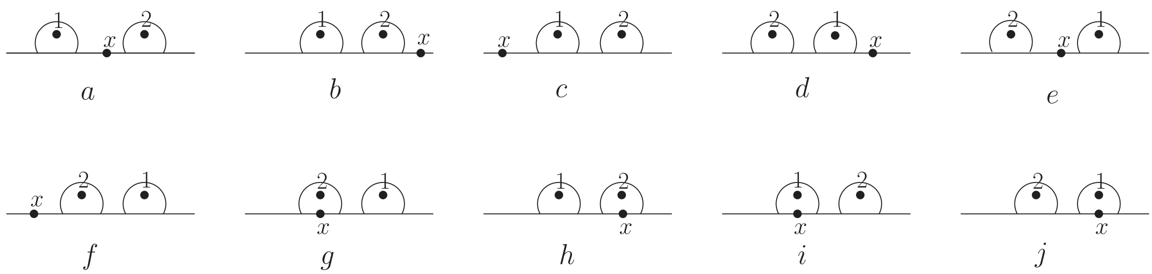

Figure 4. Codimension-two boundary strata of the I-cube needed to construct π_e.

In order to define the projection π_e precisely, we need to identify $\mathcal{C}^+_{0,3}$ and $\mathcal{C}^+_{2,0}\times\mathcal{C}_{1,1}$ with certain codimension-two boundary strata of the I-cube $\mathcal{C}_{2,1}$. It is clearest to do this pictorially; see Figure 4.

Thus, for any edge $e=(i,j)$ of Γ $(i,j=1,\dots,m+1+n)$, we have the following possibilities: (a) $1\leqslant i<m+1<j\leqslant m$; (b) $1\leqslant i<j\leqslant m$; (c) $m+1<i<j\leqslant m+1+n$; (d) $1\leqslant j<i\leqslant m$; (e) $1\leqslant j<m+1<i\leqslant m+1+n$; (f) $m+1<j<i<m+1+n$; (g) $m+1=j<i$; (h) $1\leqslant i<m+1=j$; (i) $m+1=i<j\leqslant m+1+n$; and (j) $1\leqslant j<m+1=i$. The labelling of these ten cases corresponds to the labelling of the codimension-two boundary strata in Figure 4. It should then be obvious how to define the projection π_e in all ten cases. We just note here that the vertices of the second type labelled i, j and $m+1$ correspond via the projection π_e to the vertices labelled 1, 2 and x, respectively, in Figure 4.

In this way, to every edge e of an admissible graph Γ in $\mathcal{G}_{0,m+1+n}$ we can associate an element ω_e^K of $\Omega^1(\mathcal{C}^+_{0,m+1+n})\otimes\mathrm{End}(\mathrm{T_{poly}}(X)^{\otimes m+1+n})$ via

$$\omega_e^K=\pi_e^*(\omega^{+,+})\otimes\tau_e^{I_1\cap I_2}+\pi_e^*(\omega^{+,-})\otimes\tau_e^{I_1\cap I_2^c}+\pi_e^*(\omega^{-,+})\otimes\tau_e^{I_1^c\cap I_2}+\pi_e^*(\omega^{-,-})\otimes\tau_e^{I_1^c\cap I_2^c},$$

where

$$\tau_e^I=\sum_{k\in I}1^{\otimes(i-1)}\otimes\iota_{dx_k}\otimes1^{\otimes(m-i)}\otimes1^{\otimes(j-1)}\otimes\partial_{x_k}\otimes1^{\otimes(m+1+n-j)}.$$

The degree of the operator τ_e^I is readily computed to be -1, because of the contraction operators.

To a general admissible graph Γ in $\mathcal{G}_{0,m+1+n}$, m general elements a_i of A, n general elements b_j of B and an element k of K we associate an element of K,

$$\mathcal{O}^K_\Gamma(a_1|\cdots|a_m|k|b_1|\cdots|b_n)=\mu^K_{m+1+n}\left(\int_{\mathcal{C}^+_{0,m+1+n}}\prod_{e\in\mathrm{E}(\Gamma)}\omega_e^K(a_1|\cdots|a_m|k|b_1|\cdots|b_n)\right),$$

where $\mu^K_{m+1+n}:T_{\mathrm{poly}}(X)^{\otimes m+1+n}\to K$ is the k-multi-linear map given by taking multiple products in $T_{\mathrm{poly}}(X)$ followed by restriction to K. Of course, we implicitly regard A, B and K as subalgebras of $T_{\mathrm{poly}}(X)$ with respect to the wedge product.

First, we note that the product over all edges of Γ does not depend on the ordering of the factors; in particular, ω_e^K is not only a smooth 1-form but also an endomorphism of $T_{\mathrm{poly}}(X)^{\otimes m+1+n}$ of degree -1, because of the contraction. Furthermore, since ω_e^K is a smooth 1-form on the compactified configuration space $\mathcal{C}^+_{0,m+1+n}$, the integral exists.

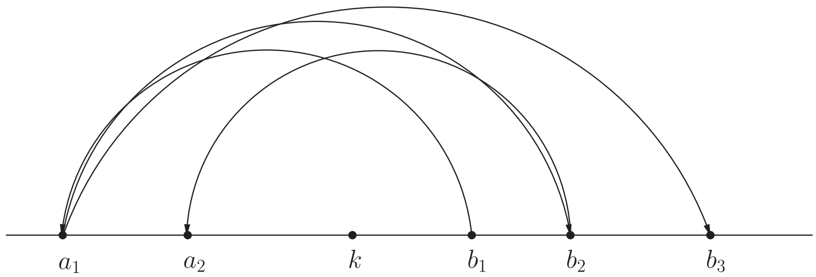

Figure 5. An admissible graph of type $(0,6)$ appearing in $d_K^{2,3}$.

Finally, we define the Taylor component $d_K^{m,n} : A[1]^{\otimes m} \otimes K[1] \otimes B[1]^{\otimes n} \to K[1]$ by

$$d_K^{m,n}(a_1|\cdots|a_m|k|b_1|\cdots|b_n) \\ = \sum_{\Gamma \in \mathcal{G}_{0,m+1+n}} \mathcal{O}_\Gamma^K(a_1|\cdots|a_m|k|b_1|\cdots|b_n) \quad \text{for } a_i \in A,\ b_j \in B,\ k \in K. \tag{32}$$

Observe that the map (32) has degree 1. For a general admissible graph Γ of type $(0, m+1+n)$, the operator $\mathcal{O}_\Gamma(a_1|\cdots|a_m|k|b_1|\cdots|b_n)$ is non-vanishing only if $|\mathrm{E}(\Gamma)| = m+n-1$, which is the dimension of $\mathcal{C}^+_{0,m+1+n}$. Since there is a contraction operator associated to each edge which lowers the degree by 1, it follows immediately that $d_K^{m,n}$ has degree 1; of course, if we omit the degree-shifting, then the degree of $d_K^{m,n}$ is equivalently $1-m-n$.

Lemma 5.4 from § 5.3.2 implies that the operator ω_e^K is non-trivial only if the edge e is as in case (a) or case (e), so that we have

$$\omega_e^K = \begin{cases} \pi_e^*(\omega^{+,-}) \otimes \tau^{I_1 \cap I_2^c} & \text{when } e \text{ is as in case } (a), \\ \pi_e^*(\omega^{-,+}) \otimes \tau^{I_1^c \cap I_2} & \text{when } e \text{ is as in case } (e). \end{cases}$$

Hence a general admissible graph of type $(0, m+1+n)$ appearing in formula (32) has the form shown in Figure 5.

In view of Remark 6.4, we observe that admissible graphs with multiple edges yield trivial contributions: specifically, if any two distinct vertices (both necessarily of the second type) are connected by more than one edge, then the corresponding integral weight must vanish since it contains the square of a 1-form $\omega^{+,-}$ or $\omega^{-,+}$.

Proposition 6.5. *For a field k of characteristic zero that contains $\mathbb{R}$ or $\mathbb{C}$, consider A, B and K as above.*

Then the Taylor components (32) endow K with an A_∞-A-B-bimodule structure, where A and B are viewed as GAs with their natural product; hence, in particular, A and B both have a (trivial) A_∞-algebra structure.

If we denote by d_A, d_B and d_K the A_∞-structures on A, B and K, respectively, as described in Proposition 6.5, then we may regard the formal sum $\gamma = d_A + d_B + d_K$ as a MCE for the graded Lie algebra $\widehat{\mathrm{C}}^\bullet(\mathtt{Cat}_\infty(A,B,K))$. Thus, the triple $(\widehat{\mathrm{C}}^\bullet(\mathtt{Cat}_\infty(A,B,K)), [\gamma, \bullet], [\bullet, \bullet])$ defines a DG Lie algebra, where $[\bullet, \bullet]$ denotes the Gerstenhaber bracket. In the case where

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d_K has only finitely many non-trivial Taylor components,[2] one can instead consider the DG Lie algebra $(\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A,B,K)),[\gamma,\bullet],[\bullet,\bullet])$.

Proof of Proposition 6.5. The Taylor components (32) define an A_∞-A-B-bimodule structure provided that the following identity holds for all m,n:

$$\sum_{j=1}^{m-1}(-1)^j d_K^{m-1,n}(a_1|\cdots|a_{j-1}|a_ja_{j+1}|a_{j+2}|\cdots|a_m|k|b_1|\cdots|b_n)$$
$$+\sum_{j=1}^{n-1}(-1)^{m+j+1}d_K^{m,n-1}(a_1|\cdots|a_m|k|b_1|\cdots|b_{j-1}|b_jb_{j+1}|b_{j+2}|\cdots|b_n)$$
$$+\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{(m-i+1)(i+j)+(1-i-j)\sum_{k=1}^{m-i}|a_k|}d_K^{m-i,n-j}$$
$$\times\big(a_1|\cdots|a_{m-i}|d_K^{i,j}(a_{m-i+1}|\cdots|a_m|k|b_1|\cdots|b_j)|b_{j+1}|\cdots|b_n\big)=0. \tag{33}$$

The proof of identity (33), based on Stokes' theorem, is in the same spirit of the proof of the main result in [Kon03]; that is, the quadratic relations in (33) are shown to be equivalent to quadratic relations between the corresponding integral weights, recalling (32).

For this purpose, we consider

$$\sum_{\widetilde{\Gamma}\in\mathcal{G}_{0,m+1+n}}\int_{\mathcal{C}^+_{0,m+1+n}} d\widetilde{\mathcal{O}}_{\widetilde{\Gamma}}(b_1|\cdots|b_m|k|a_1|\cdots|a_n)$$
$$=\sum_i\sum_{\widetilde{\Gamma}\in\mathcal{G}_{0,m+1+n}}\int_{\partial_i\mathcal{C}^+_{0,m+1+n}}\widetilde{\mathcal{O}}_{\widetilde{\Gamma}}(b_1|\cdots|b_m|k|a_1|\cdots|a_n)=0, \tag{34}$$

where the left summation in the right-hand side of (34) is over boundary strata of $\mathcal{C}^+_{0,m+1+n}$ of codimension one and

$$\widetilde{\mathcal{O}}_{\widetilde{\Gamma}}(a_1|\cdots|a_m|k|b_1|\cdots|b_n)=\mu^K_{m+1+n}\bigg(\prod_{e\in\mathrm{V}(\widetilde{\Gamma})}\omega^K_e(a_1|\cdots|a_m|k|b_1|\cdots|b_n)\bigg)$$
$$=\mu^K_{m+1+n}(\omega^K_{\widetilde{\Gamma}}(a_1|\cdots|a_m|k|b_1|\cdots|b_n))$$

is viewed as a smooth K-valued form on $\mathcal{C}^+_{0,m+1+n}$ of form-degree equal to $\mathrm{E}(\widetilde{\Gamma})$. We observe that, by construction, a contribution indexed by a graph $\widetilde{\Gamma}$ in $\mathcal{G}_{0,m+1+n}$ is non-trivial only if $\mathrm{E}(\widetilde{\Gamma})=m+n-2$.

Boundary strata of $\mathcal{C}^+_{0,m+1+n}$ of codimension one are all of type (16) (see § 4.1), with no points in $\mathbb{H}$. Furthermore, we distinguish the following three cases:

(i) $\partial_{\emptyset,B}\mathcal{C}^+_{0,m+1+n}\cong\mathcal{C}^+_{0,B}\times\mathcal{C}_{0,[m+1+n]\smallsetminus\{B\}\sqcup\{*\}}$, where B is an ordered subset of $[m]$ of consecutive elements;

(ii) $\partial_{\emptyset,B}\mathcal{C}^+_{0,m+1+n}\cong\mathcal{C}^+_{0,B}\times\mathcal{C}_{0,[m+1+n]\smallsetminus\{B\}\sqcup\{*\}}$, where B is an ordered subset of $\{m+1,\dots,n\}$ of consecutive elements;

(iii) $\partial_{\emptyset,B}\mathcal{C}^+_{0,m+1+n}\cong\mathcal{C}^+_{0,B}\times\mathcal{C}_{0,[m+1+n]\smallsetminus\{B\}\sqcup\{*\}}$, where B is an ordered subset of $[m+1+n]$ of consecutive elements which contains $m+1$.

[2] For example, this happens when $X=U\oplus V$.

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We begin by considering a general boundary stratum of type (i): it corresponds to the situation where $|B|$ consecutive points on $\mathbb{R}$, labelled by B, collapse to a single point on $\mathbb{R}$, which lies to the left of the special point labelled by $m+1$.

From (18) in Lemma 5.1 and Lemma 5.4(ii), we get

$$\int_{\partial_{\emptyset,B}\mathcal{C}^+_{0,m+1+n}} \omega^K_{\widetilde{\Gamma}} = (-1)^{j(|B|+1)+1}\bigg(\int_{\mathcal{C}^+_{0,B}} \omega^A_{\Gamma_B}\bigg)\bigg(\int_{\mathcal{C}^+_{0,[m+1+n]\smallsetminus\{B\}\sqcup\{*\}}} \omega^K_{\Gamma^B}\bigg), \tag{35}$$

where Γ_B is the subgraph of $\widetilde{\Gamma}$ whose edges have both endpoints belonging to B, Γ^B is the graph obtained from $\widetilde{\Gamma}$ by collapsing Γ_B to a single vertex, and j is the minimum of B.

The operator-valued form $\omega^A_{\Gamma_B}$ will be defined precisely later on; for the present computations we do not actually need its form. Recall the general form of an element of $\mathcal{G}_{0,m+1+n}$. Since all vertices labelled by B lie to the left of the vertex labelled by $m+1$, the degree of the form $\omega^A_{\Gamma_B}$ must equal 0 since the graph Γ_B does not contain any edge; hence, for dimensional reasons, the weight is non-vanishing only if $|B|=2$, i.e. $B=\{j,j+1\}$ for $1\leqslant j\leqslant m-1$, in which case it is equal to 1. As a consequence, Γ^B is an admissible graph in $\mathcal{G}_{0,m+n}$.

When moving a copy of the standard multiplication on $T_{\mathrm{poly}}(X)$ to act on the factors a_j and a_{j+1}, we do not get any other sign besides the one in identity (35) coming from the orientation, since the standard multiplication has degree 0. Therefore, the sum in identity (34) over codimension-one boundary strata of type (i) gives exactly the first term on the left-hand side of identity (33).

Second, we consider a general codimension-one boundary stratum of type (ii); it describes the situation where $|B|$ consecutive points on $\mathbb{R}$, labelled by B, collapse to a single point of $\mathbb{R}$, which lies to the right of the special point labelled by $m+1$.

Once again, we recall the orientation formulæ (18) from Lemma 5.1 to find a factorization of the form (35). We may now repeat almost verbatim the arguments used in the previous case: in particular, $|B|=2$ and the minimum j of B satisfies, by assumption, $m+1<j$, which we rewrite with an abuse of notation as $m+1+j$ for $1\leqslant j\leqslant n-1$. Thus, the sum in identity (34) over codimension-one boundary strata of type (ii) produces the second term on the left-hand side of identity (33).

It remains to discuss boundary strata of type (iii). In this case, the situation describes the collapse of $|B|$ consecutive points on $\mathbb{R}$, labelled by B, among which is the special point labelled by $m+1$, to a single point on $\mathbb{R}$, which will become the new special point.

Recalling the orientation formulæ (18) from Lemma 5.1, we find a factorization of the type (35).

First we observe that, in this case, the subgraph Γ_B is disjoint from $\Gamma\smallsetminus\Gamma_B$; this follows immediately from (24) and (25) in Lemma 5.4 and the discussion on the shape of admissible graphs appearing in formula (32) (in other words, there are no edges connecting Γ_B with its complement $\Gamma^B\smallsetminus\Gamma_B$). In particular, $\widetilde{\Gamma}$ factors as $\widetilde{\Gamma}=\Gamma_B\sqcup\Gamma^B$, and Γ_B and Γ^B are both admissible. We also observe that, in general, $|B|\geqslant 2$ in this case; that is, Γ_B can be non-empty.

The orientation sign is $j(|B|+1)+1$, where j is the minimum of B. Since $1\leqslant j\leqslant m$, we may rewrite this as $m-i+1$ for $i=1,\dots,m$. The maximum of B is bigger than or equal to $m+1$, hence we can write it as j, for $0\leqslant j\leqslant n$, shifting with respect to $m+1$.

Moreover, we get an additional sign $(1-i-j)(\sum_{k=1}^{m-i}|a_k|)$ when moving $\int_{\mathcal{C}^+_{0,B}}\omega^K_{\Gamma_B}$ through a_k, for $k=1,\dots,m-i$.

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Finally, the fact that Γ_B and Γ^B are disjoint implies that we can safely restrict the product of the B-factors in $\int_{\mathcal{C}^+_{0,B}} \omega^K_{\Gamma_B}(a_{m-i+1}|\cdots|a_m|k|b_1|\cdots|b_j)$ to K, since no derivative acts on it or departs from it. As a consequence, the sum in (34) over codimension-one boundary strata of type (iii) yields the third term on the left-hand side of identity (33).

(We now observe that the signs coming from orientations in the previous calculations agree with the signs in identity (33) up to a -1 sign overall, which is of no influence.) □

7. Formality for the Hochschild cochain complex of an A_∞-category

In this section we assume that d_K has finitely many non-trivial Taylor components.

We consider the A_∞-algebras A and B and the A_∞-A-B-bimodule K from § 6.2, to which we associate the A_∞-category $\mathtt{Cat}_\infty(A,B,K)$ and the corresponding Hochschild cochain complex $\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A,B,K))$. In particular, we are interested in the DG Lie algebra structure on $(\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A,B,K)),[\mu,\bullet],[\bullet,\bullet])$, where μ denotes the A_∞-A-B-bimodule structure on $\mathtt{Cat}_\infty(A,B,K)$.

We construct an L_∞-quasi-isomorphism $\mathcal{U}$ from the DG Lie algebra $(T_{\mathrm{poly}}(X),0,[\bullet,\bullet])$ to the DG Lie algebra $(\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A,B,K)),[\mu,\bullet],[\bullet,\bullet])$. The proof of the main result is divided into two parts: first, we construct $\mathcal{U}$ explicitly and prove, by means of Stokes' theorem, that $\mathcal{U}$ is an L_∞-morphism; second, we prove that $\mathcal{U}$ is a quasi-isomorphism. The proof of the second statement is a consequence of Keller's condition.

7.1 The explicit construction

We produce an explicit formula for the L_∞-quasi-isomorphism $\mathcal{U}$. First of all, by the results of § 6, constructing an L_∞-morphism from $T_{\mathrm{poly}}(X)$ to $\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A,B,K))$ is equivalent to constructing three distinct maps $\mathcal{U}_A$, $\mathcal{U}_B$ and $\mathcal{U}_K$, where

$$\mathcal{U}_A : T_{\mathrm{poly}}(X)\to \mathrm{C}^\bullet(A,A),\quad \mathcal{U}_B : T_{\mathrm{poly}}(X)\to \mathrm{C}^\bullet(B,B),$$
$$\mathcal{U}_K : T_{\mathrm{poly}}(X)\to \mathrm{C}^\bullet(A,B,K).$$

We fix an orthogonal decomposition (31) of X as in § 5.2, together with an adapted coordinate system $\{x_i\}$ in the sense of § 5.2; we also recall from §§ 5.3.1 and 5.3.2 the two-colored and four-colored propagators.

To a pair of non-negative integers (n,m) we associate the set $\mathcal{G}_{n,m}$ of admissible graphs of type (n,m); further, if $m\geqslant 1$ we write $(n,m)=(n,p+1+q)$ for some non-negative integers p and q.

To an admissible graph Γ in $\mathcal{G}_{n,m}$, general elements γ_i, $i=1,\dots,n$, of $T_{\mathrm{poly}}(X)$ and general elements a_j, $j=1,\dots,m$, of A we associate an element of A by the assignment

$$\mathcal{O}^A_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_m)=\mu^B_{n+m}\bigg(\int_{\mathcal{C}^+_{n,m}}\omega^A_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_m)\bigg), \tag{36}$$

where μ^A_{n+m} is the multiplication operator from $T_{\mathrm{poly}}(X)^{\otimes n+m}$ to $T_{\mathrm{poly}}(X)$ followed by restriction to A, viewed (in a non-canonical way) as a subalgebra of $T_{\mathrm{poly}}(X)$. Further, the $\Omega^{|\mathrm{E}(\Gamma)|}(\mathcal{C}^+_{n,m})$-valued endomorphism of $T_{\mathrm{poly}}(X)^{\otimes n+m}$ is defined by

$$\omega^A_\Gamma=\prod_{e\in\mathrm{E}(\Gamma)}\omega^A_e,\quad \omega^A_e=\pi^*_e(\omega^+)\otimes(\tau_e^{I_1\cap I_2}+\tau_e^{I_1\cap I_2^c})+\pi^*_e(\omega^-)\otimes(\tau_e^{I_1^c\cap I_2}+\tau_e^{I_1^c\cap I_2^c}), \tag{37}$$

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where π_e is the natural projection from $\mathcal{C}^+_{n,m}$ onto $\mathcal{C}_{2,0}$ or its boundary strata of codimension one (in fact, ω^+ and ω^- vanish on all strata of codimension two of $\mathcal{C}_{2,0}$, thanks to Lemma 5.2 of § 5.3.1) and the operator τ_e^I, for $I \subset [d]$, is as defined in § 5.2.

Once again, we observe that the product (37) is well-defined, since the two-colored propagators are 1-forms while τ_e^I is an endomorphism of $T_{\mathrm{poly}}(X)^{\otimes n+m}$ of degree -1. Further, since the dimension of $\mathcal{C}^+_{n,m}$ is $2n+m-2$, the element (36) is non-trivial precisely when $|\mathrm{E}(\Gamma)| = 2n+m-2$.

We then set

$$\mathcal{U}^n_A(\gamma_1|\cdots|\gamma_n)(a_1|\ldots|a_m) = (-1)^{(\sum_{i=1}^n |\gamma_i|-1)m} \sum_{\Gamma\in\mathcal{G}_{n,m}} \mathcal{O}^A_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_m). \tag{38}$$

Similar formulæ, with appropriate changes, specify the Taylor components $\mathcal{U}^n_B$, $n \geqslant 1$. Here we merely note that

$$\omega_e^B = \pi_e^*(\omega^+) \otimes (\tau_e^{I_1\cap I_2} + \tau_e^{I_1^c\cap I_2}) + \pi_e^*(\omega^-) \otimes (\tau_e^{I_1\cap I_2^c} + \tau_e^{I_1^c\cap I_2^c})$$

for an edge e of a general admissible graph Γ as above.

Finally, we define the Taylor components $\mathcal{U}^n_K$ via

$$\begin{aligned}&\mathcal{U}^n_K(\gamma_1|\cdots|\gamma_n)(a_1|\cdots|a_p|k|b_1|\cdots|b_q)\\ &\quad= (-1)^{(\sum_{i=1}^n |\gamma_i|-1)(p+q+1)} \sum_{\Gamma\in\mathcal{G}_{n,p+1+q}} \mathcal{O}^K_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_p|k|b_1|\cdots|b_q).\end{aligned} \tag{39}$$

Let us point out now, before entering into details, that: (i) formula (38) contains admissible graphs with multiple edges and no loops (i.e. whenever an admissible graph contains at least one loop, the corresponding contribution to formula (38) is set to be zero); and (ii) formula (39) contains admissible graphs with multiple edges and (possibly) loops.

Since in the usual constructions in deformation quantization multiple edges and loops are not present, we need to discuss separately how to deal with these possibilities.

If Γ is admissible and contains multiple edges, we consider a pair (i,j) of distinct vertices of Γ of the first type, such that the cardinality of the set $\mathrm{E}_{(i,j)} = \{e \in \mathrm{E}(\Gamma) : e = (i,j)\}$ is greater than 1. Then, to (i,j) we associate the smooth, operator-valued $|\mathrm{E}_{(i,j)}|$-form given by

$$\omega^A_{(i,j)} = \frac{1}{(|\mathrm{E}_{(i,j)}|)!} \prod_{e\in\mathrm{E}_{(i,j)}} \omega^A_e = \frac{(\omega^A_{(i,j)})^{|\mathrm{E}_{(i,j)}|}}{(|\mathrm{E}_{(i,j)}|)!}, \quad \omega^K_{(i,j)} = \frac{1}{(|\mathrm{E}_{(i,j)}|)!} \prod_{e\in\mathrm{E}_{(i,j)}} \omega^K_e = \frac{(\omega^K_{(i,j)})^{|\mathrm{E}_{(i,j)}|}}{(|\mathrm{E}_{(i,j)}|)!}$$

(when A is replaced by B, appropriate adjustments have to be made).

In particular, with an abuse of notation, we denote by ω^A_e and ω^K_e the normalized operator-valued forms associated to a (multiple) edge e of Γ in, respectively, (38) and (39). Of course, if the edge e appears only once in Γ, then ω^A_e and ω^K_e coincide with the respective standard expressions; otherwise, they are given by the formulæ above.

We now recall from § 5.3.2 the closed 1-form ρ on $\mathcal{C}_{1,1}$. The vertex v_ℓ of the first type, corresponding to a loop ℓ of Γ, specifies a natural projection $\pi_{v_\ell} : \mathcal{C}^+_{n,p+1+q} \to \mathcal{C}_{1,1}$, which extends to the corresponding compactified configuration spaces the projection onto the vertex v_ℓ and the special vertex $p+1$. We also consider the restricted divergence operator

$$\mathrm{div}^{(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)} = \sum_{k\in(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)} \iota_{dx_k}\partial_{x_k}$$

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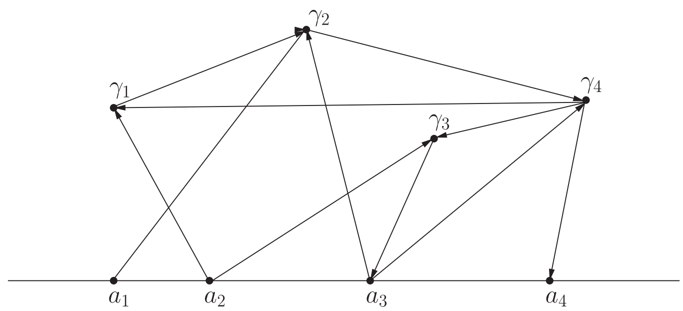

Figure 6. A general admissible graph of type $(4, 4)$ appearing in $\mathcal{U}_A$.

on $T_{\mathrm{poly}}(X)$. For $1 \leqslant r \leqslant n$, we denote by $\mathrm{div}_{(r)}^{(I_1 \cap I_2) \sqcup (I_1 \cap I_2)}$ the endomorphism of $T_{\mathrm{poly}}(X)^{\otimes(n+p+1+q)}$ of degree -1 given by

$$\mathrm{div}_{(r)}^{(I_1 \cap I_2) \sqcup (I_1^c \cap I_2^c)} = 1^{\otimes(r-1)} \otimes \mathrm{div}^{(I_1 \cap I_2) \sqcup (I_1^c \cap I_2^c)} \otimes 1^{(n-r+p+1+q)}.$$

Finally, for a loop ℓ of Γ, we set

$$\omega_\ell = \pi_{v_\ell}^*(\rho) \otimes \mathrm{div}_{(v_\ell)}^{(I_1 \cap I_2) \sqcup (I_1^c \cap I_2^c)}. \tag{40}$$

It is clear that ρ_ℓ is a closed 1-form on $\mathcal{C}_{n,p+1+q}^+$ which takes values in $\mathrm{End}(T_{\mathrm{poly}}(X)^{\otimes(n+p+1+q)})$ and is of degree -1; so ω_ℓ has total degree -1.

Remark 7.1. Loops are trivial when $U \oplus V = X$, because the restricted divergence operator vanishes by construction.

We want to examine in some detail the admissible graphs and their colorings that yield (possibly) non-trivial contributions to formulæ (38) and (39).

We begin with (38). In this case, we recall Lemma 5.2(ii), which tells us that the two-colored propagators ω^+ and ω^- vanish on the boundary strata β and γ, respectively. This, in turn, implies that the edges of an admissible graph Γ of type (n, m) whose initial (respectively, final) point lies in $\mathbb{R}$ are colored by propagators of type ω^- (respectively, ω^+). According to the definition of ω_e^A for e an edge of Γ, since there are elements of A associated to vertices of the second type, the above observation is in agreement with the fact that such elements may be differentiated only with respect to coordinates $\{x_i\}$ for i in $(I_1 \cap I_2) \sqcup (I_1 \cap I_2^c)$ and can be contracted only with respect to differentials of coordinates $\{x_i\}$ for i in $(I_1^c \cap I_2) \sqcup (I_1^c \cap I_2^c)$. To illustrate our analysis, an admissible graph of type (n, m) in this case is shown in Figure 6.

Similar arguments hold when A is replaced by B.

We now turn to formula (39); in particular, we consider an admissible graph Γ of type $(n, p+1+q)$.

The point $k+1$ on $\mathbb{R}$ plays a very special role in subsequent computations: in fact, with respect to the natural projections from $\mathcal{C}_{n,k+1+l}^+$ onto $\mathcal{C}_{2,1}$, it corresponds to the single point on $\mathbb{R}$ in $\mathcal{C}_{2,1}$.

Recall Lemma 5.4(iii) from § 5.3.2. As a consequence of this lemma, if e is an edge whose initial (respectively, final) point is $p+1$, then e is colored by the propagator $\omega^{-,-}$ (respectively, $\omega^{+,+}$), and, according to the definition of ω_e^K, this agrees with the fact that an element k of K

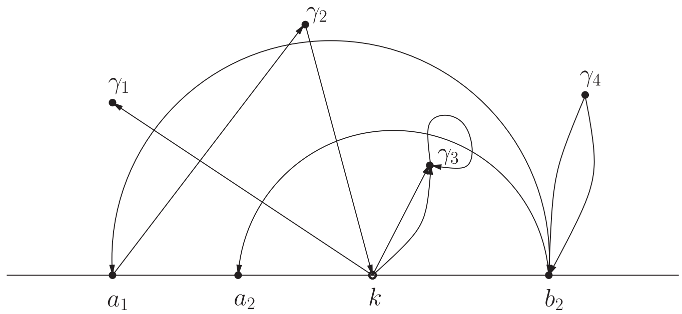

Figure 7. A general admissible graph of type $(4, 4)$ appearing in $\mathcal{U}_K$.

can be differentiated only with respect to coordinates $\{x_i\}$ for i in $I_1 \cap I_2$ and can be contracted only with respect to differentials of coordinates $\{x_i\}$ for i in $I_1^c \cap I_2^c$.

Following the very same arguments as in § 5.2, we deduce that Γ cannot contain any edge e that joins two vertices of the second type which both lie either on the left-hand side or on the right-hand side of $p+1$; similarly, there is no edge joining $p+1$ to any other vertex of the second type.

It is also clear that if Γ has a vertex of the first type with more than one loop attached to it, then the corresponding contribution to formula (39) vanishes, since it contains the square of the 1-form ρ on $\mathcal{C}_{1,1}$.

Finally, observe that if Γ has more than four multiple edges between the same two distinct vertices (which obviously would be of the first type), then the corresponding contribution to formula (39) is trivial. To be specific, since there is a sum of four distinct 1-forms associated to any edge, any power of at least five identical operator-valued forms must contain at least a square of one of the four-colored propagators.

The above analysis of contributions to formula (39) is summarized pictorially in Figure 7. Of course, we once again recall that loops do not appear in the special case where $U \oplus V = X$.

7.2 The main result

Now we are ready to state and prove the main result of the paper.

Theorem 7.2. *Consider $X = k^d$, and denote collectively by μ the A_∞-structure on the category $\mathtt{Cat}_\infty(A, B, K)$ defined as in § 6.2.*

The morphisms $\mathcal{U}_A^n$, $\mathcal{U}_B^n$ and $\mathcal{U}_K^n$, $n \geqslant 1$, are the Taylor components of an L_∞-quasi-isomorphism

$$\mathcal{U} : (T_{\mathrm{poly}}(X), 0, [\bullet, \bullet]) \to (\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A, B, K)), [\mu, \bullet], [\bullet, \bullet]).$$

Proof. First, note that $T_{\mathrm{poly}}(X)$ and $\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A, B, K))$ are L_∞-algebras via

$$\begin{aligned} &Q_1 = 0, \\ &Q_2(\gamma_1, \gamma_2) = (-1)^{|\gamma_2|}[\gamma_1, \gamma_2] \quad \text{for } \gamma_i \in (T_{\mathrm{poly}}(W)[1])_{|\gamma_i|},\ i = 1, 2 \\ &Q_1' = [\mu, \bullet], \\ &Q_2'(\phi_1, \phi_2) = (-1)^{|\phi_1|}[\phi_1, \phi_2] \quad \text{for } \phi_i \in (\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A, B, K))[1])_{|\phi_i|},\ i = 1, 2. \end{aligned} \tag{41}$$

For simplicity, set $\mathcal{U}^n = \mathcal{U}_B^n + \mathcal{U}_K^n + \mathcal{U}_A^n$.

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The conditions for $\mathcal{U}$ to be an L_∞-morphism translate into the semi-infinite family of relations

$$[\mu, \mathcal{U}^n(\gamma_1|\cdots|\gamma_n)] + \frac{1}{2}\sum_{I\sqcup J=\{1,\ldots,n\};I,J\neq\emptyset} \epsilon_\gamma(I,J)Q_2'(\mathcal{U}^{|I|}(\gamma_I),\mathcal{U}^{|J|}(\gamma_J))$$
$$= \frac{1}{2}\sum_{k\neq l}\sigma_\gamma(k,l,1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n)\mathcal{U}^{n-1}(Q_2(\gamma_k,\gamma_l),\gamma_1,\ldots,\widehat{\gamma_k},\ldots,\widehat{\gamma_l},\ldots,\gamma_n). \quad (42)$$

For each index set $I=\{i_1,\ldots,i_I\}\subseteq\{1,\ldots,n\}$ of cardinality $|I|$, we denote by γ_I the element $(\gamma_{i_1},\ldots,\gamma_{i_I})\in C^{+|I|}(T_{\mathrm{poly}}(W)[1])$. Similarly, we define γ_J.

The infinite set of identities (42) comprises three different infinite sets of identities, which correspond to the three projections of (42) onto A, B and K. It is easy to verify that the projections onto A or B of (42) define infinite sets of identities that correspond to the identities satisfied by L_∞-morphisms from $T_{\mathrm{poly}}(X)$ to $\mathrm{C}^\bullet(A,A)$ or $\mathrm{C}^\bullet(B,B)$, which were proved in [CF07] (in a slightly different form).

Thus, it remains to prove identity (42) for the K-component.

Observe first that

$$[\mu,\mathcal{U}^n(\gamma_1|\cdots|\gamma_n)] = \mu\bullet\mathcal{U}^n(\gamma_1|\cdots|\gamma_n) - (-1)^{\sum_{i=1}^n|\gamma_i|+2-n}\mathcal{U}^n(\gamma_1|\cdots|\gamma_n)\bullet\mu.$$

By setting $\mathcal{U}^0=\mu$, recalling the higher compositions $\bullet$ from § 3.1 and the product $\bullet$ on $T_{\mathrm{poly}}(X)$, and finally projecting identity (42) down onto K, we find

$$\sum_{I\sqcup J=[n]}\epsilon_\gamma(I,J)(\mathcal{U}_K^{|I|}(\gamma_I)\bullet\mathcal{U}_B^{|J|}(\gamma_J)+\mathcal{U}_K^{|I|}(\gamma_I)\bullet\mathcal{U}_K^{|J|}(\gamma_J)+\mathcal{U}_K^{|I|}(\gamma_I)\bullet\mathcal{U}_A^{|J|}(\gamma_J))$$
$$=\sum_{k\neq l}\sigma_\gamma(k,l,1,\ldots,\hat{k},\ldots,\hat{l},\ldots,n)\mathcal{U}_K^{n-1}(\gamma_k\bullet\gamma_l,\gamma_1,\ldots,\widehat{\gamma_k},\ldots,\widehat{\gamma_l},\ldots,\gamma_n). \quad (43)$$

The proof of identity (43) relies on Stokes' theorem. For any two non-negative integers p and q, we consider the following identity for elements of $\mathrm{Hom}(T_{\mathrm{poly}}(X)^{\otimes(n+p+1+q)},K)$:

$$\sum_{\widetilde{\Gamma}\in\mathcal{G}_{n,p+1+q}}\int_{\mathcal{C}^+_{n,p+1+q}} d\widetilde{\mathcal{O}}^K_{\widetilde{\Gamma}} = \sum_i\sum_{\widetilde{\Gamma}\in\mathcal{G}_{n,p+1+q}}\int_{\partial_i\mathcal{C}^+_{n,p+1+q}}\widetilde{\mathcal{O}}^K_{\widetilde{\Gamma}}=0, \quad (44)$$

where the left summation in the right-hand side of (44) is over codimension-one boundary strata of $\mathcal{C}^+_{n,p+1+q}$ and

$$\widetilde{\mathcal{O}}^K_{\widetilde{\Gamma}} = \mu^K_{n+p+1+q}\circ\prod_{e\in\mathrm{V}(\widetilde{\Gamma})}\omega^K_e = \mu^K_{n+p+1+q}\circ\omega^K_{\widetilde{\Gamma}},$$

regarded as a smooth K-valued form on $\mathcal{C}^+_{n,p+1+q}$ of form-degree equal to $|\mathrm{E}(\widetilde{\Gamma})|$. Then, by construction, a contribution indexed by a graph $\widetilde{\Gamma}$ in $\mathcal{G}_{n,p+1+q}$ is non-trivial only if $|\mathrm{E}(\widetilde{\Gamma})|=2n+p+q-2$.

Boundary strata of $\mathcal{C}^+_{n,p+1+q}$ of codimension one are either of type (16) or of type (17) given in § 5.1. We have the following cases and subcases:

(i) $\partial_A\mathcal{C}^+_{n,p+1+q}\cong\mathcal{C}_A\times\mathcal{C}^+_{[n]\smallsetminus A\sqcup\{*\},p+1+q}$, where A is a subset of $[n]$ with $|A|\geqslant 2$;

(ii$_1$) $\partial_{A_1,B_1}\mathcal{C}^+_{n,p+1+q}\cong\mathcal{C}^+_{A_1,B_1}\times\mathcal{C}^+_{[n]\smallsetminus A_1,[p+1+q]\smallsetminus B_1\sqcup\{*\}}$, where A_1 is a subset of $[n]$ with $|A_1|\geqslant 1$ and B_1 is an ordered subset of consecutive elements of $[p]$ with $|B_1|\geqslant 1$;

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(ii$_2$) $\partial_{A_1,B_1}\mathcal{C}^+_{n,p+1+q} \cong \mathcal{C}^+_{A_1,B_1} \times \mathcal{C}^+_{[n]\smallsetminus A_1,[p+1+q]\smallsetminus B_1\sqcup\{*\}}$, where A_1 is a subset of $[n]$ with $|A_1| \geqslant 1$ and B_1 is an ordered subset of consecutive elements of $\{p+2,\dots,p+q+1\}$ with $|B_1| \geqslant 1$;

(ii$_3$) $\partial_{A_1,B_1}\mathcal{C}^+_{n,p+1+q} \cong \mathcal{C}^+_{A_1,B_1} \times \mathcal{C}^+_{[n]\smallsetminus A_1,[p+1+q]\smallsetminus B_1\sqcup\{*\}}$, where A_1 is a subset of $[n]$ with $|A_1| \geqslant 1$ and B_1 is an ordered subset of consecutive elements of $[p+1+q]$ which contains $p+1$ and has $|B_1| \geqslant 1$.

We begin by considering a general boundary stratum of type (i); it corresponds to the situation where points in $\mathbb{H}$, labelled by A, collapse to a single point which is again in $\mathbb{H}$.

For a general admissible graph $\widetilde{\Gamma}$ of type $(n, p+1+q)$ as in identity (44), we find the following factorization by using Lemma 5.4(i) and recalling the orientations (19) from Lemma 5.1:

$$\int_{\partial_A\mathcal{C}^+_{n,p+1+q}} \omega^K_{\widetilde{\Gamma}} = -\bigg(\int_{\mathcal{C}_A} \omega^K_{\Gamma_A}\bigg)\bigg(\int_{\mathcal{C}^+_{[n]\smallsetminus A\sqcup\{*\},p+1+q}} \omega^K_{\Gamma^A}\bigg), \tag{45}$$

where Γ_A is the subgraph of $\widetilde{\Gamma}$ whose edges have both endpoints in A and Γ^A is the graph obtained by collapsing the subgraph Γ_A to a point.

We now focus on the first factor on the right-hand side of (45).

Recalling Lemma 5.4(i) from § 5.3.2, the restriction to $\mathcal{C}_A$ of ω^K_e, when e is an edge of the subgraph Γ_A (not counted with multiplicities in the case of a multiple edge), can be rewritten as

$$\omega^K_e\big|_{\mathcal{C}_A} = (\pi_e^*(d\varphi) - \pi^*_{v_A}(\rho)) \otimes \tau_e^{[d]} + \pi^*_{v_A}(\rho) \otimes \tau_e^{(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)} = \widetilde{\omega}_e + \rho_{v_A,e},$$

where π_e is the (smooth extension to compactified configuration spaces of the) natural projection from $\mathcal{C}_A$ onto $\mathcal{C}_2$ and π_{v_A}, with v_A denoting the vertex corresponding to the collapse of the subgraph Γ_A, is the (smooth extension to compactified configuration spaces of the) natural projection from $\mathcal{C}^+_{[n]\smallsetminus A\sqcup\{v_A\},p+1+q}$ onto $\mathcal{C}_{1,1}$. Of course, when $U \oplus V = X$, the second term in the rightmost expression vanishes; see also Remark 7.1.

Therefore, we can write

$$\int_{\mathcal{C}_A} \omega^K_{\Gamma_A} = \int_{\mathcal{C}_A} \prod_{e\in \mathrm{E}(\Gamma_A)} (\widetilde{\omega}_e + \rho_{v_A,e}) \prod_{\ell \text{ loop of } \Gamma_A} \omega_\ell, \tag{46}$$

where the contributions to multiple edges are normalized as above.

We now observe that the form part of any loop contribution and of any operator-valued form $\rho_{v_A,e}$ is simply ρ evaluated at the vertex corresponding to the collapse; hence there can be at most one such contribution and, in particular, if Γ_A contains more than one loop, the corresponding boundary contribution vanishes.

First, let us assume Γ_A to be loop-free. By the previous argument, we can rewrite the right-hand side of (46) as

$$\int_{\mathcal{C}_A} \omega^K_{\Gamma_A} = \int_{\mathcal{C}_A} \prod_{e\in \mathrm{E}(\Gamma_A)} \widetilde{\omega}_e + \sum_{e\in \mathrm{E}(\Gamma_A)} \bigg(\int_{\mathcal{C}_A} \prod_{e'\neq e} \widetilde{\omega}_{e'}\bigg)\rho_{v_A,e}.$$

The two integral contributions on the right-hand side vanish if $|A| \geqslant 3$, either because of dimensional reasons or by virtue of Kontsevich's lemma. Therefore, we need only consider the case where $|A| = 2$. The integral contributions are non-trivial in this case only if the degree of the integrand equals 1, which can happen only when Γ_A has at most two edges. The contributions are shown graphically in Figure 8.

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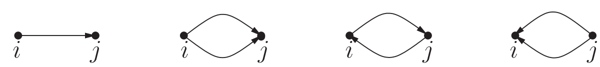

Figure 8. The four possible loop-free subgraphs Γ_A that yield non-trivial boundary contributions of type (i).

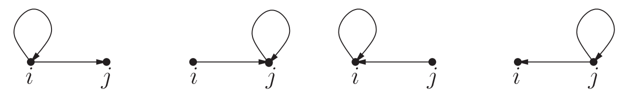

Figure 9. The four possible subgraphs Γ_A containing one loop which yield non-trivial boundary contributions of type (i).

The contribution from the first graph, in view of the previous expression, is given by

$$\int_{\mathcal{C}_2} \omega^K_{\Gamma_A} = \left(\int_{S_1} d\varphi\right) \otimes \tau_e^{[d]} = \tau_e^{[d]},$$

as the integral over $\mathcal{C}_2 = S^1$ of the second term in $\widetilde{\omega}_e$ is basic with respect to the fiber integration.

Taking into account the fact that the second graph has two multiple edges and thus recalling the normalization factor 2, we find that the contribution from this graph is

$$\int_{\mathcal{C}_2} \omega^K_{\Gamma_A} = \pi^*_{v_A}(\rho) \otimes \tau_e^{[d]} \tau_e^{(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)},$$

where $e = (i, j)$. The very same computations yield, for the fourth graph,

$$\int_{\mathcal{C}_2} \omega^K_{\Gamma_A} = \pi^*_{v_A}(\rho) \otimes \tau_e^{[d]} \tau_e^{(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)},$$

where $e = (j, i)$ in this case.

Finally, the third graph yields the contribution

$$\int_{\mathcal{C}_2} \omega^K_{\Gamma_A} = \pi^*_{v_A}(\rho) \otimes \tau_{e_1}^{[d]} \tau_{e_2}^{(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)} + \pi^*_{v_A}(\rho) \otimes \tau_{e_2}^{[d]} \tau_{e_1}^{(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)},$$

where $e_1 = (i, j)$ and $e_2 = (j, i)$.

Now, assume that the subgraph Γ_A has exactly one loop. In this case, the right-hand side of (46) can be rewritten as

$$\int_{\mathcal{C}_A} \omega^K_{\Gamma_A} = \left(\int_{\mathcal{C}_A} \prod_{e\in \mathrm{E}(\Gamma_A)} \widetilde{\omega}_e\right)\omega_\ell,$$

because the 1-form associated to the loop is basic with respect to the projection onto $\mathcal{C}_A$. Again, by invoking dimensional reasons or Kontsevich's lemma, we deduce that the above contribution is non-trivial only if $|A| = 2$; in this case, the subgraph Γ_A yields non-trivial contributions only if it is as shown in Figure 9.

We write down explicitly only the contribution coming from the first graph:

$$\int_{\mathcal{C}_2} \omega^K_{\Gamma_A} = \pi^*_{v_A}(\rho) \otimes \mathrm{div}_{(v_A)}^{(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)} \tau_e^{[d]},$$

where $e = (i, j)$ and, by the construction of ω_ℓ, $v_\ell = v_A$.

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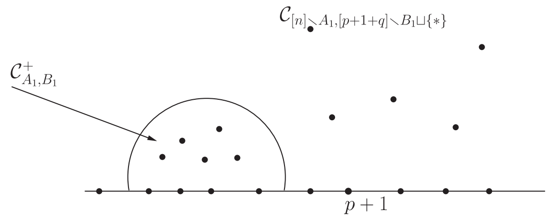

Figure 10. A general configuration of points in a boundary stratum of $\mathcal{C}^+_{n,p+1+q}$ of type (ii$_1$).

We now recall the sign conventions discussed previously, which imply that sign issues can be dealt with in this framework exactly as in [CF07, proof of Theorem A.7]. Note that: (i) the endomorphism $\tau_e^{[d]}$, which appears in all contributions, leads to the Schouten–Nijenhuis bracket between the poly-vector fields associated to the two distinct vertices of Γ_A; and (ii) the contributions that involve the restricted divergence and the endomorphism $\tau_e^{(I_1^c\cap I_2)\sqcup(I_1\cap I_2^c)}$ sum up, by Leibniz's rule, to the restricted divergence applied to the Schouten–Nijenhuis bracket between the aforementioned poly-vector fields.

Thus, the sum in (44) involving boundary strata of type (i) contributes to the right-hand side of identity (43) in all cases.

Next, we consider boundary strata of type (ii$_1$). Such strata describe the collapse of points in $\mathbb{H}$ labelled by A_1 and consecutive points on $\mathbb{R}$ labelled by B_1, where the maximum of B_1 lies on the left of the special point labelled by $p+1$, to a single point in $\mathbb{R}$ (with the point resulting from the collapse obviously lying to the left of $p+1$). This situation is illustrated in Figure 10. Using Lemma 5.4(ii) for the restriction of four-colored propagators on the boundary stratum β of $\mathcal{C}_{2,1}$ and recalling the orientations (18) from Lemma 5.1, we get the factorization

$$\int_{\partial_{A_1,B_1}\mathcal{C}^+_{n,p+1+q}}\omega^K_{\widetilde{\Gamma}}=(-1)^{j(|B_1|+1)+1}\left(\int_{\mathcal{C}^+_{A_1,B_1}}\omega^A_{\Gamma_{A_1,B_1}}\right)\left(\int_{\mathcal{C}^+_{[n]\smallsetminus A_1,[p+1+q]\smallsetminus B_1\sqcup\{*\}}}\omega^K_{\Gamma^{A_1,B_1}}\right),\quad(47)$$

where Γ_{A_1,B_1} denotes the subgraph of $\widetilde{\Gamma}$ whose edges have both endpoints labelled by $A_1\sqcup B_1$ and Γ^{A_1,B_1} denotes the graph obtained by collapsing $\Gamma_{A,B}$ to a single point.

Observe first that, owing to Lemma 5.4(iv), Γ_{A_1,B_1} cannot have edges connecting vertices labelled by $A_1\sqcup B_1$ to vertices on $\mathbb{R}$ that are not labelled by $A_1\sqcup B_1$ and which lie to the left of the vertex labelled by p. It follows that Γ_{A_1,B_1}, as well as Γ^{A_1,B_1}, is an admissible graph.

Second, notice that if Γ_{A_1,B_1} has at least one loop, then the corresponding contribution must vanish, because the 1-form ρ vanishes on the codimension-one boundary strata of $\mathcal{C}_{1,1}$.

Once again, the sign conventions for the higher compositions $\bullet$ that we previously elucidated in § 3.1 imply that all signs arising in this situation are the same as those appearing in [CF07, proof of Theorem A.7], with appropriate modifications for the different algebraic setting. Owing to the appearance of operators of the form $\omega_{\Gamma^B_{A_1,B_1}}$ in identity (47), the sum in (44) over all boundary strata of type (ii$_1$) yields the first term on the left-hand side of identity (43).

Now, we turn to boundary strata of type (ii$_2$). In this case, a boundary stratum describes the collapse of points in $\mathbb{H}$ labelled by A_1 and consecutive points on $\mathbb{R}$ labelled by B_1, where the minimum of B_1 lies on the right of $p+1$, to a single point on $\mathbb{R}$. (Clearly, the point resulting from the collapse lies to the right of $p+1$.) This situation is illustrated in Figure 11.

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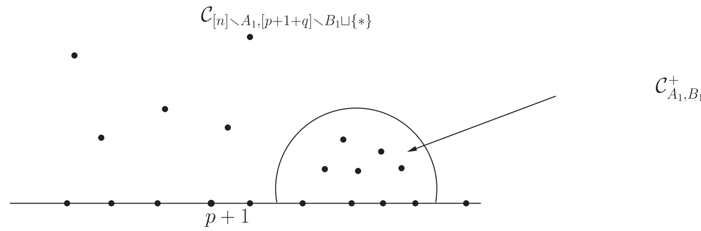

FIGURE 11. A general configuration of points in a boundary stratum of $\mathcal{C}^+_{n,p+1+q}$ of type (ii$_2$).

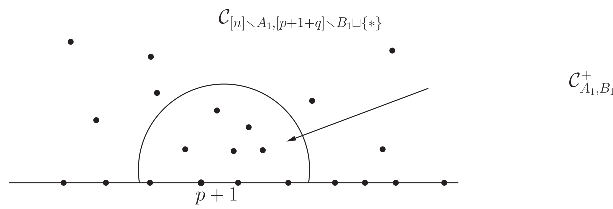

FIGURE 12. A general configuration of points in a boundary stratum of $\mathcal{C}^+_{n,p+1+q}$ of type (ii$_3$).

Here we use Lemma 5.4(ii), for dealing with the restriction of four-colored propagators on the boundary stratum γ of $\mathcal{C}_{2,1}$, together with the orientations (18) from Lemma 5.1, to obtain the factorization

$$\int_{\partial_{A_1,B_1}\mathcal{C}^+_{n,p+1+q}} \omega^K_{\widetilde{\Gamma}} = (-1)^{j(|B_1|+1)+1}\left(\int_{\mathcal{C}^+_{A_1,B_1}} \omega^B_{\Gamma_{A_1,B_1}}\right)\left(\int_{\mathcal{C}^+_{[n]\smallsetminus A_1,[p+1+q]\smallsetminus B_1\sqcup\{*\}}} \omega^K_{\Gamma^{A_1,B_1}}\right), \quad (48)$$

with the same notation as in (47).

Once again, because of Lemma 5.4(iv), the subgraph Γ_{A_1,B_1} cannot have edges connecting vertices of Γ_{A_1,B_1} to vertices on $\mathbb{R}$ that lie to the right of p; hence Γ_{A_1,B_1} and Γ^{A_1,B_1} are both admissible graphs.

As was already noted for boundary strata of type (ii$_1$), if the subgraph Γ_{A_1,B_1} contains at least one loop, the corresponding contribution vanishes by the same arguments as above.

Needless to repeat, the sign conventions for the corresponding higher compositions $\bullet$ from § 3.1 imply that all signs arising in this situation are tantamount to the signs (with appropriate modifications) in [CF07, proof of Theorem A.7]. Owing to the presence of the form $\omega_{\Gamma^A_{A_1,B_1}}$ in identity (47), the sum in (44) over all boundary strata of type (ii$_2$) yields the third term on the left-hand side of identity (43).

Finally, we consider boundary strata of type (ii$_3$). A stratum of this type describes the collapse of points in $\mathbb{H}$ labelled by A_1 and points on $\mathbb{R}$ labelled by B_1, which this time contains the special point $p+1$, to a single point in $\mathbb{R}$, which then becomes the new special point. This situation is illustrated in Figure 12.

We make use of Lemma 5.4(iii), for the restriction of four-colored propagators on the boundary strata δ and ε of $\mathcal{C}_{2,1}$, together with the orientations (18) from Lemma 5.1, to come to the factorization

$$\int_{\partial_{A_1,B_1}\mathcal{C}^+_{n,p+1+q}} \omega^K_{\widetilde{\Gamma}} = (-1)^{j(|B_1|+1)+1}\left(\int_{\mathcal{C}^+_{A_1,B_1}} \omega^K_{\Gamma_{A_1,B_1}}\right)\left(\int_{\mathcal{C}^+_{[n]\smallsetminus A_1,[p+1+q]\smallsetminus B_1\sqcup\{*\}}} \omega^K_{\Gamma^{A_1,B_1}}\right), \quad (49)$$

where we have used the same notation as in (47) and (48).

Observe that in this case, Γ_{A_1,B_1} once again cannot have edges connecting vertices of Γ_{A_1,B_1} to vertices on $\mathbb{R}$, because of Lemma 5.4(iv). Further, the only incoming (respectively, outgoing) edges of Γ_{A_1,B_1} are labelled by propagators of the form $\omega^{+,+}$ (respectively, $\omega^{-,-}$), owing to Lemma 5.4(iii). In particular, Γ_{A_1,B_1}, as well as Γ^{A_1,B_1}, is an admissible graph.

Thanks to the previously discussed sign conventions for the higher compositions $\bullet$ in § 3.1, all signs arising in this situation are the same as those appearing in [CF07, proof of Theorem A.7]. Owing to the presence of operators of the form $\omega_{\Gamma^K_{A_1,B_1}}$ in identity (47), the sum in (44) over all boundary strata of type (ii$_1$) yields the second term on the left-hand side of identity (43). □

7.3 Proof that the L_∞-morphism $\mathcal{U}$ is an L_∞-quasi-isomorphism

So far, we have proved only that the morphism constructed in § 7.1 is an L_∞-morphism. It remains to show that $\mathcal{U}$ is, in fact, an L_∞-quasi-isomorphism; equivalently, we have to prove that its first Taylor component $\mathcal{U}_1$ is a quasi-isomorphism.

Observe that the L_∞-morphism $\mathcal{U}$ fits into the following commutative diagram of L_∞-algebras.

$$\begin{array}{ccccc} & & (\mathrm{C}^\bullet(A,A),\,[d_A,\bullet],\,[\bullet,\bullet]) & & \\ & \nearrow^{\mathcal{U}_A} & & \nwarrow^{\mathrm{p}_A} & \\ (T_{\mathrm{poly}}(X),\,0,\,[\bullet,\bullet]) & & \xrightarrow{\quad\mathcal{U}\quad} & & (\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A,B,K)),\,[\mu,\bullet],\,[\bullet,\bullet]) \\ & \searrow_{\mathcal{U}_B} & & \swarrow_{\mathrm{p}_B} & \\ & & (\mathrm{C}^\bullet(B,B),\,[d_B,\bullet],\,[\bullet,\bullet]) & & \end{array} \tag{50}$$

The relative formality theorem of [CF07] implies that $\mathcal{U}_A^1$ and $\mathcal{U}_B^1$ are L_∞-quasi-isomorphisms. Hence, if we can prove that the projections p_A and p_B are quasi-isomorphisms (in particular, L_∞-quasi-isomorphisms), the invertibility property of L_∞-quasi-isomorphisms would imply that $\mathcal{U}$ is also a quasi-isomorphism.

By Theorem 4.9, it suffices to prove that the derived left-action L_A and the derived right-action R_B are quasi-isomorphisms.

We will prove that the derived left-action L_A is a quasi-isomorphism; the proof for R_B relies on the same arguments (with appropriate modifications).

7.3.1 $\mathrm{S}(Y^*)$ *as a (relative) quadratic algebra.* We consider a finite-dimensional graded k-vector space Y (endowed with the discrete topology) with a fixed direct sum decomposition $Y = X_1 \oplus X_2$ into (finite-dimensional) graded subspaces X_1 and X_2.

We further consider the symmetric algebra $\mathrm{S}(Y^*)$ as an object of $\mathtt{GrMod}_k$: owing to the decomposition $Y = X_1 \oplus X_2$, we have $\mathrm{S}(Y^*) \cong \mathrm{S}(X_1^*) \otimes \mathrm{S}(X_2^*)$, therefore $\mathrm{S}(Y^*)$ has the structure of a left $\mathrm{S}(X_1^*)$-module. Conversely, $\mathrm{S}(X_1^*)$ has the structure of a left $\mathrm{S}(Y^*)$-module, with respect to the natural projection from $\mathrm{S}(Y^*)$ onto $\mathrm{S}(X_1^*)$.

We now set, for the sake of simplicity, $A_0 = \mathrm{S}(X_1^*)$ and $A_1 = \mathrm{S}(X_1^*) \otimes X_2^*$: A_1 is a free A_0-module in a natural way. We further have the obvious identification $\mathrm{T}_{A_0}(A_1) \cong \mathrm{S}(X_1^*) \otimes \mathrm{T}(X_2^*)$, where $\mathrm{T}_{A_0}(A_1)$ denotes the tensor algebra over A_0 of A_1, as objects of the category $\mathtt{GrMod}_k$.

Further, we consider

$$R = \{1\otimes v_1^*\otimes v_2^* - (-1)^{|v_1^*||v_2^*|}1\otimes v_2^*\otimes v_1^* : v_i^*\in X_2^*, i=1,2\}\subset \mathrm{S}(X_1^*)\otimes (X_2^*)^{\otimes 2}\cong A_1\otimes_{A_0} A_1,$$

and, by abuse of notation, we denote by R also the two-sided ideal in $\mathrm{T}_{A_0}(A_1)$ spanned by R.

It is then quite easy to verify that

$$\mathrm{T}_{A_0}(A_1)/R\cong \mathrm{S}(X_1^*)\otimes \mathrm{S}(X_2^*)\cong \mathrm{S}(Y^*),$$

whence it follows that $A=\mathrm{S}(Y^*)$ is a quadratic A_0-algebra; A is bigraded with respect to the internal grading (as an object of $\mathtt{GrMod}_k$) and the Koszul grading, which is the grading associated to the piece $\mathrm{S}(X_2^*)$, viewed as an object of $\mathtt{GrMod}_k$.

The algebra $A^!$, the quadratic dual of A, can also be computed explicitly: since $A^! = \mathrm{T}_{A_0}(A_1^\vee)/R^\perp$, where $A_1^\vee$ is the dual (over A_0) of A_1 and $R^\perp$ is the (two-sided ideal in $T_{A_0}(A_1^\vee)$ generated by the) annihilator of R in $A_1^\vee\otimes_{A_0} A_1^\vee$, and since Y is finite-dimensional, we have

$$A^!\cong \mathrm{S}(X_1^*)\otimes \Lambda(X_2)\cong \mathrm{S}(X_1^*)\otimes \mathrm{S}(X_2[-1])\cong \mathrm{S}(X_1^*\oplus X_2[-1]),$$

where the exterior algebra $\Lambda(X_2)$ of X_2 is defined by mimicking the standard definition in the category $\mathtt{GrMod}_k$, and where the second isomorphism is explicitly defined by the so-called *décalage* isomorphism. Again, $A^!$ is bigraded with respect to the grading as an object of $\mathtt{GrMod}_k$ and with respect to the Koszul grading.

Finally, by means of A and $A^!$, we may compute the Koszul complex of A: since $\mathrm{K}^{-n}(A) = A\otimes_{A_0}(A_n^!)^\vee$, where again $(A_n^!)^\vee$ denotes the dual over A_0 of $A_n^!$, we obtain

$$\mathrm{K}^\bullet(A)\cong \mathrm{S}(Y^*)\otimes \mathrm{S}(X_2^*[1])\cong \mathrm{S}(Y^*\oplus X_2^*[1]),$$

with the natural formula for the Koszul differential.

7.3.2 *The Koszul complex of* $\mathrm{S}(Y^*)$. We now inspect more carefully the Koszul complex $\mathrm{K}^\bullet(A)$ (viewed as a cohomological complex) of the algebra $A=\mathrm{S}(Y^*)$.

First of all, we discuss the gradings of $\mathrm{K}^\bullet(A)$. The shift by 1 of the grading of X_2^* induces the *cohomological grading*, which is concentrated in $\mathbb{Z}_{\leqslant 0}$. Alternatively, we may view (the graded vector space of the complex) $\mathrm{K}^\bullet(A)$ as the (graded vector space of the) relative de Rham complex of Y with respect to X_2, and the cohomological grading is the opposite of the natural grading of the relative de Rham complex as a complex.

Then, for each $n\geqslant 0$, $\mathrm{K}^{-n}(A)$ is naturally an object of $\mathtt{GrMod}_k$, and the corresponding grading is called the *total grading*. Furthermore, the total grading can be written as the sum of the cohomological grading and the internal grading.

For example, $\mathrm{K}^\bullet(A)$ is generated by x_i, y_j and θ_k, where $\{\theta_k\}$ denotes a basis of $X_2^*[1]$ associated to a basis $\{y_j\}$ of X_2^*; by definition, $|\theta_j| = |y_j| - 1$. Thus, a general element $x_{i_1}\cdots x_{i_p}y_{j_1}\cdots y_{j_q}\theta_{k_1}\cdots\theta_{k_r}$ of $\mathrm{K}^\bullet(A)$ has total degree

$$\sum_{s=1}^p |x_{i_s}| + \sum_{t=1}^q |y_{j_t}| + \sum_{u=1}^r |\theta_{k_u}| - r,$$

cohomological degree $-r$ and internal degree

$$\sum_{s=1}^p |x_{i_s}| + \sum_{t=1}^q |y_{j_t}| + \sum_{u=1}^r |\theta_{k_u}|.$$

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The Koszul differential d is defined with reference to the previous basis as $d = y_j\partial_{\theta_j}$, where ∂_{θ_j} denotes the derivation with respect to θ_j acting from the left with total degree 1, cohomological degree 1 and internal degree 0.

The Koszul complex $\mathrm{K}^\bullet(A)$ is endowed with a distinct differential $d_{\mathrm{dR}} = \theta_j\partial_{y_j}$, where the differential ∂_{y_j} acts from the left with total degree -1, cohomological degree -1 and internal degree 0.

The operator $\mathrm{L}_{\mathrm{rel}} = [d_{\mathrm{dR}}, d]$, where $[\,,]$ is the commutator in $\mathrm{End}(\mathrm{K}^\bullet(A))$ with respect to internal degree, has total degree 0, cohomological degree 0 and internal degree 0. On generators, $\mathrm{L}_{\mathrm{rel}}$ is expressed by $\mathrm{L}_{\mathrm{rel}}(x_i) = 0$, $\mathrm{L}_{\mathrm{rel}}(y_j) = y_j$ and $\mathrm{L}_{\mathrm{rel}}(\theta_j) = \theta_j$, and it is extended to general elements by means of the Leibniz rule.

The homotopy formula $\mathrm{L}_{\mathrm{rel}} = [d_{\mathrm{dR}}, d]$ implies, via a direct computation, that the Koszul complex of the quadratic algebra $A = \mathrm{S}(Y^*)$ is a resolution of $A_0 = \mathrm{S}(X_1^*)$ as a left module over A; therefore A and $A^! = \mathrm{S}(X_1^* \oplus X_2[-1])$ are quadratic Koszul algebras over A_0.

7.3.3 *Relative Koszul duality.* We consider the category ${}_A\mathtt{GrMod}$, for A as before, of $\mathbb{Z}$-graded topological, complete left A-modules, with inner morphisms spaces specified via $\mathrm{Hom}_{A-}(V,W) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_{A-}(V,W)$, and the homogeneous component $\mathrm{Hom}^n_{A-}(V,W)$ of degree n is defined in a way similar to inner homomorphisms space of the category $\mathtt{GrMod}_k$, see also § 2. The previous arguments imply that A_0 and $\mathrm{K}^\bullet(A)$ are objects of ${}_A\mathtt{GrMod}$.

We then consider a bigraded object in the category ${}_A\mathtt{GrMod}$

$$\mathrm{Ext}^{(p,q)}_{A-}(A_0, A_0) = \mathrm{H}^q(\mathrm{RHom}^p_{A-}(A_0, A_0)), \tag{51}$$

where $\mathrm{RHom}^n_{A-}(A_0, A_0)$ denotes the right derived functor of $\mathrm{Hom}^n_{A-}(\bullet, A_0)$ as an element of the derived category of ${}_A\mathtt{GrMod}$. Explicitly, it is a complex, obtained by plugging in the functor $\mathrm{RHom}^n_{A-}(\bullet, A_0)$ any free or projective resolution of A_0 in ${}_A\mathtt{GrMod}$.

In particular, $\mathrm{Ext}^{(p,q)}_{A-}(A_0, A_0)$ can be computed by means of the Koszul resolution $\mathrm{K}^\bullet(A)$, whence

$$\mathrm{Ext}^{(p,q)}_{A-}(A_0, A_0) = \mathrm{H}^q(\mathrm{Hom}^p_{A-}(\mathrm{K}^{-\bullet}(A), A_0), d),$$

where, by abuse of notation, d denotes the differential induced by the Koszul differential d by composition on the right. The Koszul differential d acts trivially by a direct computation, whence the cohomology of the previous complex identifies with the complex itself:

$$\mathrm{Ext}^{(p,q)}n_{A-}(A_0, A_0) = \mathrm{Hom}^p_{A-}(\mathrm{K}^{-q}(A), A_0) \cong (A^!_q)_p,$$

where the index p, respectively q, in the rightmost term refers to the internal, respectively Koszul, grading.

We note that Ext_{A-} groups admit the *Yoneda product*, a pairing of Koszul and internal degrees 0:

$$\mathrm{Ext}^{(m_1,n_1)}_{A-}(A_0, A_0) \otimes \mathrm{Ext}^{(m_2,n_2)}_{A-}(A_0, A_0) \to \mathrm{Ext}^{(m_1+m_2,n_1+n_2)}_{A-}(A_0, A_0).$$

We consider a representative α of an element of $\mathrm{Ext}^{(m_1,n_1)}_{A-}(A_0, A_0) \cong (A^!_{m_1})_{n_1}$. More explicitly, α acts by multiplication with respect to A_0 and by derivations on $\mathrm{S}^{m_1}(X_2^*[1])$, finally setting coordinates on X_2^* to 0. Furthermore, α can be lifted to an element α_n of $\mathrm{Hom}^{m_1}_{A-}(\mathrm{K}^{-n_1-n}(A), \mathrm{K}^{-n}(A))$ acting by contraction.

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We now consider two elements α and β of $\mathrm{Ext}_{A-}^{(m_1,n_1)}(A_0, A_0)$ and $\mathrm{Ext}_{A-}^{(m_2,n_2)}(A_0, A_0)$, respectively. The Yoneda product between them is represented by the composition of contractions

$$\alpha \otimes \beta \mapsto (-1)^{(m_1+n_1)(m_2+n_2)} \beta \circ \alpha_n = \beta\alpha,$$

viewed as an element of $\mathrm{Hom}_A^{m_1+m_2}(\mathrm{K}^{-n_1-n_2}(A), A_0) \cong (A^!_{n_1+n_2})_{m_1+m_2}$. Therefore, the Yoneda product is represented by the opposite product in $A^!$.

The arguments so far can be summarized in the following theorem.

Theorem 7.3. *For a finite-dimensional graded vector space Y that admits a decomposition $Y = X_1 \oplus X_2$, there is an isomorphism*

$$\mathrm{Ext}^{\bullet}_{\mathrm{S}(Y^*)-}(\mathrm{S}(X_1^*), \mathrm{S}(X_1^*))^{\mathrm{op}} \cong \mathrm{S}(X_1^* \oplus X_2[-1])$$

of bigraded algebras with respect to the Koszul and internal gradings.

Of course, the arguments above, with suitable modifications, hold also when left modules are replaced by right modules.

7.3.4 *Proof of Keller's condition.* It is clear that the GAs A and B and the graded vector space K from § 6.2 fit into the setting of § 7.3.1; here we consider K as an A-B-bimodule, where the actions are given by multiplication followed by restriction.

We recall the A_∞-A-B-bimodule structure from § 6.2.

Lemma 7.4. *For the structure maps (32) from § 6.2, the following triviality conditions hold:*

$$d_K^{0,n} = d_K^{m,0} = 0 \quad \text{if } n, m \geqslant 2.$$

Furthermore, $d_K^{0,1}$ and $d_K^{1,0}$ endow K with the structures of, respectively, a right B-module and a left A-module, with actions simply given by multiplication followed by restriction. In particular, the A_∞-A-B-structure on K restricts to the above left A- and right B-module structures.

Proof. Recall from § 6.2 the construction of (32): if, say, we consider the Taylor component

$$d_K^{0,n}(k|b_1|\cdots|b_n) = \sum_{\Gamma \in \mathcal{G}_{0,1+n}} \mu_{1+n}^K \left(\int_{\mathcal{C}^+_{0,1+n}} \omega_\Gamma^K(k|b_1|\cdots|b_n) \right),$$

the discussion on admissible graphs in § 6.2 implies that a general admissible graph Γ in the above sum has no edges. The corresponding integral is thus non-trivial only if the dimension of the corresponding configuration space is zero, which happens exactly when $n = 1$.

In such a case, $d_K^{0,1}$ is simply given by multiplication followed by restriction to K, since there is no integral contribution. □

We observe that Lemma 7.4 implies that the left A_∞-module structure on K, coming from the A_∞-A-B-bimodule structure by restriction, is the standard one; similarly for the right A_∞-module structure. On the other hand, the A_∞-A-B-bimodule structure is *not* the standard one. In particular, if we take the bar–cobar construction on K, then for the left A-module structure we get a resolution of K, and likewise for the right B-module structure; however, we do *not* get a resolution of K as an A-B-bimodule.

Lemma 7.4 implies, in particular, that the cohomology of $\underline{\mathrm{End}}_{-B}(K)$ coincides with $\mathrm{Ext}^{\bullet}_{-B}(K, K)$, the latter being the derived functor of $\mathrm{Hom}_{-B}(\bullet, K)$ in the category $\mathtt{Mod}_{-B}$. It is also clear that the graded algebra structure on $\underline{\mathrm{End}}_{-B}(K)$ induces the opposite of the

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Figure 13. The only two admissible graphs that contribute to $\mathrm{L}_A^1(x_i)$.

Yoneda product on $\mathrm{Ext}^\bullet_{-B}(K,K)$; see, for example, [Rin63] for a direct computational approach to the Yoneda product.

We know from § 4.1 that L_A is an A_∞-algebra morphism from A to $\underline{\mathrm{End}}_{-B}(K)$; in particular, since the cohomology of the A_∞-algebra A coincides with A itself, L_A descends to a morphism of GAs from A to $\mathrm{Ext}^\bullet_{-B}(K,K)^{\mathrm{op}}$, where the product on $\mathrm{Ext}^\bullet_{-B}(K,K)^{\mathrm{op}}$ is the opposite of the Yoneda product.

Proposition 7.5. *Let A, B and K be as in § 6.2, with the corresponding A_∞-algebra structures and A_∞-A-B-bimodule structure. Then the derived left-A-action L_A is a quasi-isomorphism.*

Proof. By the previous arguments, L_A descends to a morphism of GAs from A to $\mathrm{Ext}^\bullet_{-B}(K,K)$; using the notation from § 6.2, the GA A is generated by the commuting variables $\{x_i\}$, with i in $(I_1\cap I_2)\sqcup(I_1\cap I_2^c)$, and the anti-commuting variables $\{\partial_{x_i}\}$, with i in $(I_1^c\cap I_2)\sqcup(I_1^c\cap I_2^c)$.

On the other hand, as a corollary of Theorem 7.3, there is an isomorphism of GAs $\mathrm{Ext}^\bullet_{-B}(K,K)\cong A$. Specifically, $B=\mathrm{S}(Y^*)$ for

$$Y^*=(U\cap V)^*\oplus(U^\perp\cap V)^*\oplus(U\cap V^\perp)[-1]\oplus(U+V)^\perp[-1],$$

and we set $X_1=(U\cap V)\oplus((U+V)^\perp)^*[-1]$ and $X_2=(U^\perp\cap V)\oplus(U\cap V^\perp)^*[-1]$.

We will now prove that L_A is the identity map of A by evaluating L_A on the generators of A.

First, consider x_i for i in $I_1\cap I_2$: the Taylor components of $\mathrm{L}_A^1(x_i)$ are given by

$$\begin{aligned}\mathrm{L}_A^1(x_i)^m(k|b_1|\cdots|b_n)&=d_K^{1,n}(x_i|k|b_1|\cdots|b_n)\\&=\sum_{\Gamma\in\mathcal{G}_{0,1+1+n}}\mu^K_{1+1+n}\bigg(\int_{\mathcal{C}^+_{0,1+1+n}}\omega^K_\Gamma(x_i|k|b_1|\cdots|b_n)\bigg).\end{aligned}$$

An admissible graph Γ that yields a non-trivial contribution to the previous expression has at most one edge. Since $n=|\mathrm{E}(\Gamma)|\geqslant 1$, we have only two possibilities: either (i) Γ has two vertices of the second type and no edge, or (ii) Γ has three vertices of the second type and one edge. These possibilities are depicted in Figure 13.

In case (ii), we get

$$\mathrm{L}_A^1(x_i)^1(k|b_1)=d_K^{1,1}(x_i|1|b_1)=\bigg(\int_{\mathcal{C}^+_{0,3}}\omega^{+,-}\bigg)(-1)^{|k|}k(\iota_{dx_i}b_1)|_K,$$

and since b_1 contains poly-vector fields that are normal with respect to V, the contraction with respect to dx_i annihilates b_1. Thus, we are left with case (i), whence, immediately,

$$\mathrm{L}_A^1(x_i)^0(k)=x_i\,k.$$

We next consider x_i for i in $I_1\cap I_2^c$. Again, we have to deal with only $\mathrm{L}_A^1(x_i)^0$ and $\mathrm{L}_A^1(x_i)^1$. In the former case, the contribution is trivial because $\mathrm{L}_A^1(x_i)_0(k)$ is simply the restriction on K

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FIGURE 14. The only two admissible graphs that contribute to $\mathrm{L}_A^1(\partial_i)$.

of the product $x_i\, k$. We are left with $\mathrm{L}_A^1(x_i)^1(k|b_1)$: by construction,

$$\mathrm{L}_A^1(x_i)^1(k|b_1) = \left(\int_{\mathcal{C}_{3,0}^+} \omega^{+,-}\right)(-1)^{|k|} k(\iota_{dx_i} b_1)|_K = (-1)^{|k|} k(\iota_{dx_i} b_1)|_K,$$

because the integral can be computed explicitly (e.g. by choosing a section of $C_{0,3}^+$ which fixes the middle vertex to 0 and the leftmost one to -1 and using the explicit formulæ for the four-colored propagators from § 5.3.2) and is equal to 1.

Now consider $\partial_i = \partial_{x_i}$ for i in $I_1^c \cap I_2$. We have

$$\begin{aligned}\mathrm{L}_A^1(\partial_i)^n(k|b_1|\cdots|b_n) &= d_K^{1,n}(\partial_i|k|b_1|\cdots|b_n)\\ &= \sum_{\Gamma\in\mathcal{G}_{0,1+1+n}} \mu_{1+1+n}^K\left(\int_{\mathcal{C}_{0,1+1+n}^+} \omega_\Gamma^K(\partial_i|k|b_1|\cdots|b_n)\right).\end{aligned}$$

The arguments of § 6.2 give that the admissible graphs in the previous formula can have at most one edge; thus, only two graphs can possibly contribute non-trivially, namely (i) the only graph with two vertices of the second type and no edge, or (ii) the only graph with three vertices of the second type and one edge. These possibilities are depicted in Figure 14.

For $|\mathrm{E}(\Gamma)| = 0$, there is only one graph with two vertices of the second type and no edges, and its corresponding contribution vanishes since we restrict to K. On the other hand, for $|\mathrm{E}(\Gamma)| = 1$ we have only one graph with three vertices of the second type and one edge, whose contribution is

$$\mathrm{L}_A^1(\partial_i)^1(k|b_1) = \left(\int_{\mathcal{C}_{3,0}^+} \omega^{-,+}\right) k(\partial_i(b_1))|_K = k(\partial_i b_1)|_K,$$

where the integral can be computed explicitly (e.g. by choosing a section of $C_{0,3}^+$ which fixes the middle vertex to 0 and the leftmost one to -1).

Finally, consider ∂_i for i in $I_1^c \cap I_2^c$. By the same arguments as above, we need only consider $\mathrm{L}_A^1(\partial_i)^0$ and $\mathrm{L}_A^1(\partial_i)^1$. We first look at $\mathrm{L}_A^1(\partial_i)^1$: the computation in the previous case implies that $\mathrm{L}_A^1(\partial_i)^1(k|b_1)$ vanishes, since b_1 does not depend on the variables $\{x_i\}$ for i in $I_1^c \cap I_2^c$. Thus, we are left with $\mathrm{L}_A^1(\partial_i)^0$, which, by construction, is simply left multiplication by ∂_i.

In the previous computations, $\mathrm{L}_A^1(\bullet)$ was regarded as an element of either $\mathrm{Hom}(K[1], K[1])$ or $\mathrm{Hom}(K[1] \otimes B[1], K[1])$; more precisely, in all four cases we viewed $\mathrm{L}_A^1(\bullet)$ as a representative of a cocycle in $\mathrm{Ext}^\bullet_{-B}(K, K)$ with respect to the bar resolution of K as a right A-module. To correctly identify $\mathrm{L}_A^1(\bullet)$ with an element of A, we still need a chain map from the bar resolution of K to the Koszul resolution of K as a right B-module, because of § 7.3.3; in particular, we need the components from $\mathcal{B}_0^B(K) = K \otimes B$ to $\mathrm{K}^0(B) = B$ and from $\mathcal{B}_1^B(K) = K \otimes B \otimes B$ to $\mathrm{K}^{-1}(B)$. (Note that the abstract existence of such a chain map is automatically guaranteed by standard arguments from homological algebra; the same arguments imply that such a chain map is homotopically invertible.)

Since K is a subalgebra of B, the map $\mathcal{B}_0^B(K) \to \mathrm{K}^0(B)$ is obviously given by multiplication. The map $\mathcal{B}_1^B(K) \to \mathrm{K}^{-1}(B)$ is a consequence of the Poincaré lemma in a linear graded manifold;

more explicitly,

$$\mathcal{B}_1^B(K) \ni (k|b_1|b_2) \mapsto (-1)^{|k|} k\left(dy_i \int_0^1 (\partial_{y_i} b_1)(ty)\, dt\right) b_2 \in \mathrm{K}^{-1}(B),$$

where $\{y_i\}$ denotes a set of linear graded coordinates (associated to the chosen coordinates $\{x_i\}$ on X) of the graded vector space X_2. In the above formula, we have hidden linear graded coordinates on X_1, because they are left untouched by integration or derivation. Graded derivations and the corresponding contraction operators act from left to right.

From the previous computations we see that the $\mathrm{L}_A^1(x_i)$, i in $I_1 \cap I_2^c$, act non-trivially only on elements of the form $(k|\partial_i|b_2)$ for i in $I_1 \cap I_2^c$, while the $\mathrm{L}_A^1(\partial_i)$, i in $I_1^c \cap I_2$, act non-trivially only on elements of the form $(k|x_i|b_2)$ for i in $I_1^c \cap I_2$. The image of such an element in $\mathcal{B}_1^B(K)$ under the previous map is $(-1)^{|k|} k\, dy_i\, b_2$, where now y_i is a standard coordinate if i is in $I_1^c \cap I_2$ or a coordinate of degree -1 if i is in $I_1 \cap I_2^c$.

Then, upon setting $b_2 = 1$, the computations in § 7.3.3 imply the desired claim. □

The same arguments, with appropriate modifications, give that $\mathrm{R}_B : B \to \underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}$ is also a quasi-isomorphism; in fact, the same kind of computations as in the proof of Proposition 7.5 show that R_B equals the identity map on B, identifying the cohomology of $\underline{\mathrm{End}}_{A-}(K)^{\mathrm{op}}$ with $\mathrm{Ext}_{A-}(K,K)^{\mathrm{op}}$ in the category of left A-modules. Thus, Keller's condition of § 4.3 for the A_∞-algebras A and $\underline{\mathrm{End}}_{-B}(K)^{\mathrm{op}}$ is verified, from which we can deduce that the projection p_B in diagram (50) is a quasi-isomorphism by virtue of Theorem 4.9; similarly, the projection p_A is a quasi-isomorphism, and then the commutativity of diagram (50) implies that $\mathcal{U}$ is also a quasi-isomorphism.

Equivalently, the Taylor component $\mathcal{U}^1$ is a quasi-isomorphism of Hochschild–Kostant–Rosenberg (HKR) type from $T_{\mathrm{poly}}(X)$ to the cohomology of the Hochschild cochain complex $(\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A,B,K)), [\mu, \bullet])$, where μ is the structure of the A_∞-category on $\mathtt{Cat}_\infty(A,B,K)$ described in § 6.2. From the discussion in § 3.1, the HKR quasi-isomorphism has three components: $\mathcal{U}_A^1$, $\mathcal{U}_B^1$ and $\mathcal{U}_K^1$. All three components can be described explicitly in terms of admissible graphs; the components $\mathcal{U}_A^1$ and $\mathcal{U}_B^1$ were already described explicitly in [CF07] in the framework of a formality result for graded manifolds.

The third component $\mathcal{U}_K^1 : (T_{\mathrm{poly}}(X), 0) \to (\mathrm{C}^\bullet(A,B,K), [d_K, \bullet])$, on the other hand, is new. By construction,

$$\mathcal{U}_K^1(\gamma)(a_1|\cdots|a_m|k|b_1|\cdots|b_n) = \sum_{\Gamma \in \mathcal{G}_{1,m+1+n}} \mathcal{O}_\Gamma^K(\gamma|a_1|\cdots|a_m|k|b_1|\cdots|b_n).$$

Since the dimension of the configuration space $\mathcal{C}_{1,m+1+n}^+$ equals $m+n+1$, only those admissible graphs Γ in $\mathcal{G}_{1,m+1+n}$ which have $|\mathrm{E}(\Gamma)| = m+n+1$ can possibly yield non-trivial contributions to the previous sum. Such graphs can be of two types: (i) HKR-graphs, where there are no edges between vertices of the second type (and hence all edges connect the sole vertex of the first type, corresponding to the multi-vector field γ, to vertices of the second type); or (ii) HKR-A_∞-graphs, which contain (possibly multiple) edges connecting vertices of the second type, edges connecting the sole vertex of the first type to vertices of the second type, and at most one loop at the sole vertex of the first type.

We observe that for an admissible graph Γ of type $(1, m+1+n)$ to yield a non-trivial contribution to the previous expression, the only vertex of the first type must be at least bivalent (i.e. having at least two edges departing from or coming into this vertex). For similar reasons,

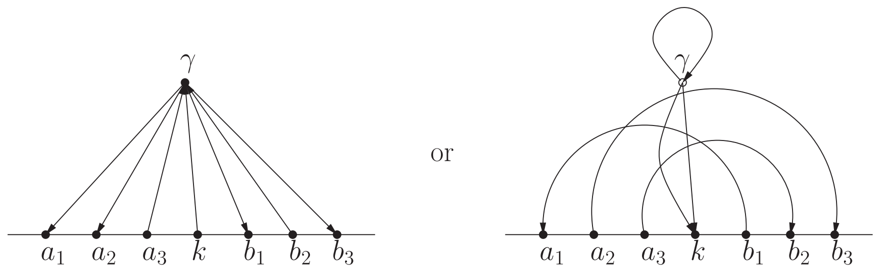

FIGURE 15. Two possible admissible graphs of type $(1, 7)$ contributing to $\mathcal{U}_K^1$.

there can be no 0-valent vertex of the second type (i.e. a vertex of the second type that is not the initial or final point of any edge).

Pictorially, the component $\mathcal{U}_K^1$ of the HKR-type quasi-isomorphism $\mathcal{U}$ in Theorem 7.2 is a sum of the two types of graphs shown in Figure 15.

8. Maurer–Cartan elements, deformed A_∞-structures and Koszul algebras

In § 7, we constructed an L_∞-quasi-isomorphism $\mathcal{U}$ from $T_{\text{poly}}(X)$ to $\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A, B, K))$.

Consider a formal parameter $\hbar$; the ring $k_\hbar = k[[\hbar]]$ is a complete topological ring with respect to the $\hbar$-adic topology. Accordingly, we denote by $T^\hbar_{\text{poly}}(X)$ the trivial deformation $T_{\text{poly}}(X)[[\hbar]]$, where the Schouten–Nijenhuis bracket is extended to $T^\hbar_{\text{poly}}(X)$ in a $k_\hbar$-linear fashion; we also let $A_\hbar$, $B_\hbar$ and $K_\hbar$ denote, respectively, the trivial $k_\hbar$-deformations of A, B and K as in § 6.2, where the GA structures on A and B as well as the A_∞-A-B-bimodule structure are extended $k_\hbar$-linearly to the respective algebras and modules.

In this framework, an $\hbar$-dependent MCE of $T^\hbar_{\text{poly}}(X)$ is defined to be an $\hbar$-dependent polynomial bivector $\pi_\hbar$ that satisfies the Maurer–Cartan equation $[\pi_\hbar, \pi_\hbar] = 0$. The $\hbar$-formal Poisson bivector $\pi_\hbar$ is assumed to be of the form $\pi_\hbar = \hbar\pi_1 + \mathcal{O}(\hbar^2)$; in particular, the Maurer–Cartan equation translates into a (possibly) infinite set of equations for the components π_n, $n \geqslant 1$, where, for instance, π_1 is a standard Poisson bivector on X.

Since $\mathcal{U}$ is an L_∞-morphism, the image of $\pi_\hbar$ with respect to (the $k_\hbar$-linear extension of) $\mathcal{U}$ is also an MCE of $\mathrm{C}^\bullet(\mathtt{Cat}_\infty(A, B, K))$; that is,

$$\mathcal{U}(\pi_\hbar) = \sum_{n \geqslant 1} \frac{1}{n!} \mathcal{U}^n(\underbrace{\pi_\hbar | \cdots | \pi_\hbar}_{n \text{ times}}).$$

Again, the MCE $\mathcal{U}(\pi_\hbar)$ splits into three components, which we denote by $\mathcal{U}_A(\pi_\hbar)$, $\mathcal{U}_B(\pi_\hbar)$ and $\mathcal{U}_K(\pi_\hbar)$ and view as elements of $\mathrm{C}^1(A_\hbar, A_\hbar)$, $\mathrm{C}^1(B_\hbar, B_\hbar)$ and $\mathrm{C}^1(A_\hbar, B_\hbar, K_\hbar)$, respectively.

8.1 Deformation quantization of quadratic Koszul algebras

Before entering into the details, we point out that, in the present subsection, the category $\mathtt{GrMod}_k$ is the standard category of graded vector spaces: thus, (inner) spaces of morphisms have to be understood in the usual sense, as well as tensor and symmetric algebras of graded vector spaces. The topology involved, unless otherwise specified, is now the trivial one. The only relevant change in our constructions caused by this substitution is the fact that $\mathcal{U}$ is only an L_∞-morphism: this is still sufficient for our purposes, as we are concerned only with deformations of the A_∞-bimodule structures on A, B and K.

Assume that we are in the framework of § 6.2, now with $U = \{0\}$ and $V = X$, so that $A = \wedge(X)$, $B = \mathrm{S}(X^*)$ and $K = k$. Here A and B are once again regarded as GAs, while K is endowed with the (non-trivial) A_∞-A-B-bimodule structure described in § 6.2.

Furthermore, Theorem 7.3 from § 7.3.3 yields the well-known Koszul duality between A and B:

$$\mathrm{Ext}^\bullet_{A-}(K, K) = B \quad \text{and} \quad \mathrm{Ext}^\bullet_{-B}(K, K) = A,$$

where K is viewed, respectively, as a left A-module and a right B-module, as a consequence of Lemma 7.4 in § 7.3.4.

The Koszul complex of A in the category ${}_A\mathtt{GrMod}$ can be identified with the deRham complex of X, with differential given by contraction with respect to the Euler field of X. This can be readily verified by repeating the arguments of § 7.3.1 in the present situation; in particular, the Koszul complex is acyclic, whence it follows that A and B are Koszul algebras over k.

We recall that for a non-negatively graded algebra A over a field $K = A_0$ (or, more generally, over a semisimple ring $K = A_0$), the property of being Koszul is equivalent to the existence of a (projective or free) resolution of K in the category of graded right A-modules whose component of cohomological degree p is concentrated in internal degree p (here 'internal' refers to the grading in the category $\mathtt{GrMod}_k$).

For our purposes, we are interested in another criterion for a non-negatively graded algebra to be Koszul, namely: A is a Koszul algebra if and only if the $\mathrm{Ext}^\bullet_{A-}(K, K)$-groups are concentrated in bidegree $(i, -i)$ for $i \geqslant 0$. Observe that the Koszul property implies that A is quadratic; see, for example, [BGS96, Sho08] for details.

For a detailed discussion of Koszul algebras, we refer to [BGS96]. Nevertheless, to aid the reader in understanding the upcoming computations, here we shall develop the above criterion further.

The graded bar resolution of K in the category of graded left A-modules, denoted by $\mathcal{B}^{A,+}_\bullet(K)$, is defined as follows:

$$\mathcal{B}^{A,+}_p(K) = A \otimes A_+^{\otimes p} \otimes K,$$

where $A_+ = \bigoplus_{n \geqslant 1} A_n$ and the tensor products should be understood over the ground field k; the differential is a slight modification of the standard bar differential.

By the definition of the category ${}_A\mathtt{GrMod}$, we have

$$\mathrm{Hom}_{A-}(\mathcal{B}^{A,+}_p(K), K) = \bigoplus_{q \in \mathbb{Z}} \mathrm{hom}_{A-}(\mathcal{B}^{A,+}_p(K), K[q]). \tag{52}$$

The differential on $\mathcal{B}^{A,+}_\bullet(K)$ has homological degree 1 and Koszul degree 0, where now the Koszul grading refers to the non-negative degree on $\mathcal{B}^{A,+}_\bullet(K)$ coming from the grading of A. By duality, $\mathrm{Hom}_{A-}(\mathcal{B}^{A,+}_p(K), K)$ has a differential of bidegree $(1, 0)$, where the first and second gradings are the cohomological and Koszul gradings, respectively.

Hence, we have a natural bigrading on $\mathrm{Ext}^\bullet_{A-}(K, K)$ inherited from identity (52). Further, since K is concentrated in Koszul degree 0, $K[q]$ is concentrated in degree $-q$. Since, by construction, $\mathcal{B}^{A,+}_p(K)$ is concentrated in Koszul degree greater than or equal to $p \geqslant 0$, it follows immediately that, in general, $-q \geqslant p$; that is, $\mathrm{Ext}^\bullet_{A-}(K, K) = \bigoplus_{p+q \leqslant 0} \mathrm{ext}^p_{A-}(K, K[q])$.

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In particular, assuming that A is a Koszul algebra, the same arguments leading to the bigrading of $\mathrm{Ext}^\bullet_{A-}(K,K)$ yield the following condition on the bigrading:

$$\mathrm{Ext}^\bullet_{A-}(K,K)=\bigoplus_{p+q=0}\mathrm{Ext}^{p,q}_{A-}(K,K)\overset{!}{=}\bigoplus_{p+q=0}\mathrm{ext}^p_{A-}(K,K[q]). \tag{53}$$

For a proof of the converse statement, we refer again to [BGS96].

Lemma 7.4 implies that the cohomology of $\underline{\mathrm{End}}_{A-}(K)$ can be identified with $\mathrm{Ext}^\bullet_{A-}(K,K)$ in the category ${}_A\mathtt{GrMod}$ and, similarly, the cohomology of $\underline{\mathrm{End}}_{-B}(K)$ can be identified with $\mathrm{Ext}^\bullet_{-B}(K,K)$ in the category $\mathtt{GrMod}_B$.

For computational reasons, we choose a set of linear coordinates $\{x_i\}$, $i=1,\dots,d$, on X. Thus, A is generated by $\{x_i\}$ and B is generated by $\{\partial_{x_i}=\partial_i\}$, for $i=1,\dots,d$.

The chain map from $\mathcal{B}^B_\bullet(K)$ to $\mathrm{K}^{-\bullet}(B)$ used in the proof of Proposition 7.5 simplifies considerably; in particular, the image of $(1|b_1|b_2)$ in $\mathcal{B}^B_1(K)$ equals $-(\int_0^1(\partial_i b_1)(tx)b_2(x)\,dt)\,dx_i$ in $\mathrm{K}^{-1}(B)$.

Proposition 8.1. *The derived left-action* L_A *descends to an isomorphism from* A *to* $\bigoplus_{p\geqslant 0}\mathrm{Ext}^{(p,-p)}_{-B}(K,K)$.

Proof. Adapting the arguments in the proof of Proposition 7.5 to the present situation, we find that

$$\mathrm{L}^1_A(\partial_i)^n(1|b_1|\cdots|b_n)=\begin{cases}(\partial_{x_i}b_1)(0) & \text{if } n=1,\\ 0 & \text{otherwise.}\end{cases} \tag{54}$$

Viewing $1\otimes x_i\otimes 1$ as an element of Koszul degree 1 in $\mathcal{B}^{B,+}_1(K)$ and recalling the previous discussion on the bigrading on $\mathrm{Ext}^\bullet_{-B}(K,K)$, the previous computation implies, in particular, that the image of L_A is contained in $\mathrm{Ext}^{(1,-1)}_{-B}(K,K)$. Using the fact that L_A is an algebra morphism and the above criterion for Koszulness, we deduce that B is a Koszul algebra.

Finally, using the chain map from the bar resolution to the Koszul resolution of K in ${}_B\mathtt{GrMod}$, we find that $\mathrm{L}^1_A(\partial_i)_1=-\iota_{\partial_i}$, where the expression on the right-hand side is viewed as a B-linear morphism from $\mathrm{K}_1(B)$ to K. □

We remark that, by repeating these arguments verbatim, it is possible to prove that R_B is an algebra isomorphism from B to $\mathrm{Ext}^\bullet_{A-}(K,K)\cong A$ and that the image of $Y^*=A_1$ with respect to R_B is contained in the piece of $\mathrm{Ext}^\bullet_{A-}(K,K)$ of bidegree $(1,-1)$.

We now consider an $\hbar$-formal quadratic Poisson bivector on X, along with the corresponding MCE $\mathcal{U}(\pi_\hbar)$ with components $\mathcal{U}_A(\pi_\hbar)$, $\mathcal{U}_B(\pi_\hbar)$ and $\mathcal{U}_K(\pi_\hbar)$.

It is easy to verify that $\mathcal{U}_A(\pi_\hbar)$ and $\mathcal{U}_B(\pi_\hbar)$ define associative products on $A_\hbar$ and $B_\hbar$, respectively; for example, $\mathcal{U}_A(\pi_\hbar)$ can be written explicitly as

$$\mathcal{U}_A(\pi_\hbar)^m(\underbrace{\bullet|\cdots|\bullet}_{m\text{ times}})=\sum_{n\geqslant 1}\frac{1}{n!}\sum_{\Gamma\in\mathcal{G}_{n,m}}\mathcal{O}^A_\Gamma(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{n\text{ times}}|\underbrace{\bullet|\cdots|\bullet}_{m\text{ times}}),$$

borrowing notation from § 7.1.

For a general admissible graph Γ of type (n,m), we have

$$\mathcal{O}^A_\Gamma(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{n\text{ times}}|\underbrace{\bullet|\cdots|\bullet}_{m\text{ times}})=\mu^A_{n+m}\bigg(\int_{\mathcal{C}^+_{n,m}}\omega^A_\Gamma(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{n\text{ times}}|\underbrace{\bullet|\cdots|\bullet}_{m\text{ times}})\bigg).$$

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The integral in the right-hand side of the previous expression is non-trivial only if the degree of the integrand equals $2n+m-2$, the dimension of $\mathcal{C}^+_{n,m}$.

Now, the degree of the integrand is $2n$ because we restrict to A by means of the multiplication operator μ^A_{n+m} and no edge in this situation can depart from a vertex of the second type if the corresponding contribution to the previous expression is non-trivial; this forces $m=2$.

Thus, we may consider $\mu_A + \mathcal{U}_A(\pi_\hbar)$, where μ_A is the $K_\hbar$-linear extension of the product on A to $A_\hbar$. It is easy to verify that this defines an associative product $\star_A$ on $A_\hbar$, which, for $\hbar=0$, reduces to the standard product on A. Similar arguments give that $\mathcal{U}_B(\pi_\hbar)$ defines an associative product $\star_B$ on $B_\hbar$, which reduces, for $\hbar=0$, to the standard product on B.

Furthermore, the expressions $d^{m,n}_{K_\hbar} = d^{n,m}_K + \mathcal{U}_K(\pi_\hbar)^{m,n}$, with non-negative integers m and n, define an A_∞-$A_\hbar$-$B_\hbar$-bimodule structure on $K_\hbar$, which reduces, for $\hbar=0$, to the A_∞-A-B-bimodule structure on K described in § 6.2.

Lemma 8.2. *The Taylor components $d^{m,n}_{K_\hbar}$ satisfy the following triviality conditions:*

$$d^{m,0}_{K_\hbar} = d^{0,n}_{K_\hbar} = 0 \quad \text{if either } m=n=0 \text{ or } m,n \geqslant 2.$$

Proof. We take, for example, a Taylor component $d^{0,n}_{K_\hbar}$ with $n \geqslant 0$. To be explicit,

$$d^{0,n}_{K_\hbar}(1|b_1|\cdots|b_n) = \sum_{l\geqslant 0}\frac{1}{l!}\sum_{\Gamma\in\mathcal{G}_{l,1+n}} \mathcal{O}^K_\Gamma(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{l \text{ times}}|1|b_1|\cdots|b_n),$$

with the same notation as above.

For a general admissible graph Γ of type $(l, 1+n)$, we have

$$\mathcal{O}^K_\Gamma = \mu^K_{l+1+n}\left(\int_{\mathcal{C}^+_{l,1+n}} \omega^K_\Gamma\right).$$

Such an operator gives a non-trivial contribution to $d^{0,n}_{K_\hbar}$ only if $|\mathrm{E}(\Gamma)| = 2l+n-1$, where $2n+l$ is the dimension of $\mathcal{C}^+_{l,1+n}$. Now, a general vertex of Γ of the first type has at most two outgoing edges, while a general vertex of the second type has no outgoing edges, and since we restrict to K, it follows that $|\mathrm{E}(\Gamma)| = 2l$, whence $n=1$. Similar arguments prove the assertion for $d^{m,0}_{K_\hbar}$ when $m \geqslant 2$ or $m=0$. □

We now discuss the grading on the deformed algebras $A_\hbar$ and $B_\hbar$. Recall that the corresponding undeformed algebras possess a natural grading.

Lemma 8.3. *The natural gradings of A and B are preserved by the associative products $\star_A$ and $\star_B$, respectively.*

Proof. Consider a general non-trivial summand in

$$\mathcal{U}_A(\pi_\hbar)^2(a_1|a_2) = \sum_{n\geqslant 1}\frac{1}{n!}\sum_{\Gamma\in\mathcal{G}_{n,2}} \mathcal{O}^A_\Gamma(\underbrace{\bullet|\cdots|\bullet}_{m \text{ times}}|a_1|a_2)$$

associated to an admissible graph Γ of type $(n,2)$.

By the same arguments used in the proof of Lemma 8.2, such a graph has the property $|\mathrm{E}(\Gamma)| = 2n$, which, by construction of the operator $\mathcal{O}^A_\Gamma$, implies that $\mathcal{O}^A_\Gamma$ contains exactly $2n$ derivations. Since the polynomial degree of the element

$$(\underbrace{\bullet|\cdots|\bullet}_{m \text{ times}}|a_1|a_2) \text{ equals } 2n + \deg(a_1) + \deg(a_2),$$

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the claim follows directly, where $\deg(\bullet)$ denotes the polynomial degree and we recall that $\pi_\hbar$ is a quadratic bivector.

Similar arguments establish the claim for $B_\hbar$. $\square$

As a consequence of Lemma 8.2, $K_\hbar$ has the structure of a left $A_\hbar$- and a right $B_\hbar$-module, and the degree-zero component of both $B_\hbar$ and $A_\hbar$ can be identified with $K_\hbar$, where the degree is specified by Lemma 8.3. Hence, the cohomology of $\underline{\mathrm{End}}_{A_\hbar-}(K_\hbar)^{\mathrm{op}}$ can be identified with $\mathrm{Ext}^\bullet_{A_\hbar-}(K_\hbar, K_\hbar)^{\mathrm{op}}$ in the category ${}_{A_\hbar}\mathtt{GrMod}$, and the product on $\underline{\mathrm{End}}_{A_\hbar-}(K_\hbar)^{\mathrm{op}}$ descends, via a direct computation, to the opposite of the Yoneda product on $\mathrm{Ext}^\bullet_{A_\hbar-}(K_\hbar, K_\hbar)$, where $(A_\hbar, \star_A)$ and $(B_\hbar, \star_B)$ are GAs in view of Lemma 8.3. Similarly, the cohomology of $\underline{\mathrm{End}}_{-B_\hbar}(K_\hbar)$ can be identified with $\mathrm{Ext}^\bullet_{-B_\hbar}(K_\hbar, K_\hbar)$ in $\mathtt{GrMod}_{B_\hbar}$, and composition descends, again, to the opposite of the Yoneda product.

Remark 8.4. We have been very sketchy in the definition of, for instance, $\underline{\mathrm{End}}_{-B_\hbar}(K_\hbar)$. In fact, we defined it as a graded vector space, as the direct sum of the homogeneous components of the $\hbar$-trivial deformation $\underline{\mathrm{End}}_{-B}(K)_\hbar$, where the product is extended $\hbar$-linearly and continuously with respect to $\hbar$-adic topology, but whose differential is now the graded commutator with the deformed differential $d_{K_\hbar, B_\hbar}$. Thus, in the previous identification, we should also write $\mathrm{Ext}^\bullet_{A-}(K,K)^{\mathrm{op}}_\hbar$, where the product is similarly extended $\hbar$-linearly and continuously with respect to the $\hbar$-adic topology; however, we keep the previous notation.

By the arguments of § 4.1, the Taylor components $d^{m,n}_{K_\hbar}$, for non-negative integers m and n, define the derived left-action $\mathrm{L}_{A_\hbar}$ of A_∞-algebras from $A_\hbar$ to $\underline{\mathrm{End}}_{-B_\hbar}(K_\hbar)$ and, similarly, the derived right-action $\mathrm{R}_{B_\hbar}$; $\mathrm{L}_{A_\hbar}$ descends to an algebra morphism from $A_\hbar$ to $\mathrm{Ext}^\bullet_{-B_\hbar}(K_\hbar, K_\hbar)$.

Lemma 8.3 yields a bigrading on $\mathrm{Ext}^\bullet_{-B_\hbar}(K_\hbar, K_\hbar)$ and on $\mathrm{Ext}^\bullet_{-B_\hbar}(K_\hbar, K_\hbar)$ in the respective categories by the previous arguments.

LEMMA 8.5. *The derived left-action $\mathrm{L}_{A_\hbar}$ maps $A_\hbar$ to $\bigoplus_{p\geqslant 0}\mathrm{Ext}^{(p,-p)}_{-B_\hbar}(K_\hbar, K_\hbar)$.*

Proof. First, we consider, for $n \geqslant 1$,

$$\mathrm{L}^1_{A_\hbar}(\partial_i)^n(1|b_1|\cdots|b_n) = d^{1,n}_{K_\hbar}(\partial_i|1|b_1|\cdots|b_n) = \sum_{l\geqslant 0}\frac{1}{l!}\sum_{\Gamma\in\mathcal{G}_{l,1+n+1}}\mathcal{O}^K_\Gamma(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{l\text{ times}}|\partial_i|1|b_1|\cdots|b_n)$$

with b_j in $B_\hbar$, $j = 1, \ldots, n$, using the same notation as above.

We consider a general admissible graph Γ of type $(l, 1+n+1)$ with $l \geqslant 0$ and $n \geqslant 0$. Its contribution is

$$\mathcal{O}^K_\Gamma(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{l\text{ times}}|\partial_i|1|b_1|\cdots|b_n) = \mu^K_{n+2}\left(\int_{\mathcal{C}^+_{l,1+1+n}}\omega^K_\Gamma(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{l\text{ times}}|\partial_i|1|b_1|\cdots|b_n)\right).$$

The degree of the integrand equals $|\mathrm{E}(\Gamma)|$, which, as we have seen from previous discussions, equals $2l+1$; since the dimension of $\mathcal{C}^+_{l,n+2}$ is $2l+n$, the previous integral is non-trivial only if $n = 1$.

Thus, it just remains to consider

$$\mathrm{L}^1_{A_\hbar}(\partial_i)^1(1|b_1) = \sum_{l\geqslant 0}\frac{1}{l!}\sum_{\Gamma\in\mathcal{G}_{l,1+1+1}}\mathcal{O}^K_\Gamma(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{l\text{ times}}|\partial_i|1|b_1) \quad \text{for } b_1 \in A_\hbar.$$

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For a general admissible graph Γ in $\mathcal{G}_{l,1+1+1}$, we consider the element

$$\mathcal{O}_\Gamma^K(\underbrace{\pi_\hbar|\cdots|\pi_\hbar}_{l\text{ times}}|\partial_i|1|b_1)$$

of $K_\hbar$; by construction, it is non-vanishing only if its polynomial degree with respect to $\{x_j\}$ is zero. The arguments in the proof of Lemma 8.3 imply that its degree in the symmetric part is $\deg(a_1)-1$, which is equal to zero only if $\deg(a_1)=1$, i.e. a_1 is a monomial of degree one.

The claim follows. □

Further, it follows immediately from the previous discussions that

$$\mathrm{L}_{A_\hbar}|_{\hbar=0}=\mathrm{L}_A \quad\text{and}\quad \mathrm{Ext}^\bullet_{-B_\hbar}(K_\hbar,K_\hbar)|_{\hbar=0}=\mathrm{Ext}^\bullet_{-B}(K,K).$$

We also observe that all deformed structures are obviously $\hbar$-linear; in particular, the differential on $\underline{\mathrm{End}}_{-B_\hbar}(K_\hbar)$ is $\hbar$-linear.

Summarizing the previous results, we have an $\hbar$-linear morphism $\mathrm{L}_{A_\hbar}$ of DG algebras from $(A_\hbar,0,\star_A)$ to $(\underline{\mathrm{End}}_{-B_\hbar}(K_\hbar),[d_{K_\hbar,B_\hbar},\bullet],\circ)$, which restricts to a quasi-isomorphism when $\hbar=0$; then, a standard perturbative argument with respect to $\hbar$ implies that $\mathrm{L}_{A_\hbar}$ is also a quasi-isomorphism, i.e. Keller's condition is verified for $\mathrm{L}_{A_\hbar}$.

By virtue of Lemmata 8.3 and 8.5, Keller's condition implies that $\mathrm{Ext}^\bullet_{-B_\hbar}(K_\hbar,K_\hbar)$ is concentrated in bidegrees $(p,-p)$ for $p\geqslant 1$, whence it follows that $B_\hbar$ is a Koszul algebra over $K_\hbar$.

On the other hand, the same arguments imply the validity of Keller's condition for $\mathrm{R}_{B_\hbar}$, and this, in turn, implies that $A_\hbar$ is a Koszul algebra.

Theorem 8.6. *Consider the d-dimensional vector space $X=k^d$ and an $\hbar$-formal quadratic Poisson bivector $\pi_\hbar=\hbar\pi_1+\mathcal{O}(\hbar^2)$ on X. Set $A=\wedge(X)$, $B=\mathrm{S}(X^*)$ and $K=k$, with the A_∞-structures discussed in § 6.2.*

Then the MCE $\pi_\hbar$ defines, by means of the L_∞-morphism $\mathcal{U}$ of Theorem 7.2, graded algebra structures on $A_\hbar$ and $B_\hbar$, as well as an A_∞-$A_\hbar$-$B_\hbar$-bimodule structure on $K_\hbar$ that deforms A and B to Koszul algebras $A_\hbar$ and $B_\hbar$, which are again Koszul dual to each other; that is,

$$\mathrm{Ext}^\bullet_{A_\hbar-}(K_\hbar,K_\hbar)^{\mathrm{op}}\cong B_\hbar \quad\textit{and}\quad \mathrm{Ext}^\bullet_{-B_\hbar}(K_\hbar,K_\hbar)^{\mathrm{op}}\cong A_\hbar$$

in the respective categories.

Remark 8.7. Theorem 8.6 gives an alternative proof of the main result of [Sho08]. The main differences are the following: (i) we make use of Kontsevich's formality result in the framework examined in [CF07]; and (ii) instead of deforming Koszul's complex of A and B to a resolution of $A_\hbar$ and $B_\hbar$, we consider already at the classical level (i.e. when $\hbar=0$) a non-trivial A_∞-A-B-bimodule structure on $K=k$, which we later deform by means of a quadratic MCE in $T^\hbar_{\mathrm{poly}}(X)$.

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