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On a Class of Fully Nonlinear Elliptic Equations Containing Gradient Terms on Compact Hermitian Manifolds

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Abstract. In this paper we study a class of second order fully nonlinear elliptic equations containing gradient terms on compact Hermitian manifolds and obtain *a priori* estimates under proper assumptions close to optimal. The analysis developed here should be useful to deal with other Hessian equations containing gradient terms in other contexts.

1 Introduction

In this paper we mainly focus on an equation containing gradient term of the form

$$\begin{cases} b_1(\mathfrak{g}[u])^{n-1} \wedge \omega + b_2(\mathfrak{g}[u])^{n-2} \wedge \omega^2 = \psi(\mathfrak{g}[u])^n, \\ \mathfrak{g}[u] = \sqrt{-1}\partial\overline{\partial}u + \chi(\,\cdot\,, du) > 0, \end{cases} \tag{1.1}$$

on a compact Hermitian manifold (M, ω) of complex dimension $n \geq 2$, where b_1 and b_2 are two nonnegative constants with $b_1 + b_2 > 0$, ψ is a smooth positive function on M, $\chi(z, \zeta)$, $(z, \zeta) \in T^*_{\mathbb{C}}M$, and is a smooth real $(1,1)$-form.

Suppose, in addition, that M has a smooth boundary ∂M and ψ is a smooth function on $\overline{M} := M \cup \partial M$. Then we consider the Dirichlet problem of equation (1.1) with the boundary value condition

$$u = \varphi \text{ on } \partial M, \quad \varphi \in C^\infty(\partial M). \tag{1.2}$$

When $b_2 = 0$, $b_1 > 0$, ψ is constant, and both ω and χ are Kähler, equation (1.1) arises naturally in the geometric problem that was posed by Donaldson [11] and Chen [8] in connection with the moment map and Mabuchi energy. The parabolic form, say J-flow, has been extensively studied, [8, 37, 45, 46]. Moreover, Song and Weinkove [37] found a necessary and sufficient solvability condition that is close to condition (1.7). This result was extended by some authors [14, 23, 38, 39].

When χ involves a gradient term, equation (1.1) differs from the standard equations on the complex manifolds, [1, 6–8, 11, 43, 47]. The equations containing gradient terms arise naturally in complex geometry and complex analysis, [15, 18, 25, 35, 44]. We refer the readers to [32–34, 41] for some recent related works.

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Differentiating from the real setting, it turns out to be a rather challenging task to derive *a priori* estimates for second derivatives for fully nonlinear elliptic equations containing gradient terms in the complex setting. The underlying reason is the two different types of complex derivatives. In our setting, the following structural condition of equation (1.1) plays a crucial role in overcoming this difficulty,

$$\liminf_{|\lambda|\to\infty} \frac{f_1\lambda_1^2}{\sum f_i} > \rho_0 \text{ in } \Big\{\lambda \in \Gamma_n : \inf_{\overline{M}} \psi \le f(\lambda) \le \sup_{\overline{M}} \psi\Big\}, \tag{1.3}$$

where f is the function in (2.1), $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, $|\lambda| = \sqrt{\sum \lambda_i^2}$, $\Gamma_n = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$, and ρ_0 is a positive constant.

We shall point out that for the generalized complex Monge–Ampère type equations

$$\sum_{k=1}^{n} b_k(\mathfrak{g}[u])^{n-k} \wedge \omega^k = \psi(\mathfrak{g}[u])^n$$

with $b_k \ge 0$ and $\sum_{k=1}^n b_k > 0$, condition (1.3) cannot be satisfied if $b_k > 0$ for some $k \ge 2$. Therefore, the main equation considered stops at the power $n-2$, and other terms of the form $b_k(\mathfrak{g}[u])^{n-k} \wedge \omega^k$ are not considered in this paper.

A central issue to solving equation (1.1) is to derive *a priori* (real) second estimates for admissible solutions. The estimates in this article rely heavily on the subsolution $\underline{u}$ defined as follows.

Definition 1.1 (Subsolution) A function $\underline{u} \in C^2(\overline{M})$ is called a $\mathcal{C}$-subsolution of equation (1.1) if for any nonzero $(1,0)$-form γ, one obtains

$$\lim_{t\to+\infty}\Big[\psi(\mathfrak{g}[\underline{u},t,\gamma])^n - b_1(\mathfrak{g}[\underline{u},t,\gamma])^{n-1}\wedge\omega - b_2(\mathfrak{g}[\underline{u},t,\gamma])^{n-2}\wedge\omega^2\Big] > 0, \tag{1.4}$$

where $\mathfrak{g}[\underline{u},t,\gamma] = \mathfrak{g}[\underline{u}] + t\sqrt{-1}\gamma\wedge\overline{\gamma}$, $\mathfrak{g}[\underline{u}] = \sqrt{-1}\partial\overline{\partial}\underline{u} + \chi(z, d\underline{u})$. Suppose in addition to $\partial M \ne \varnothing$, we say $\underline{u}$ is an admissible subsolution of Dirichlet problems (1.1)–(1.2) if it satisfies $\mathfrak{g}[\underline{u}] > 0$ and

$$\begin{aligned} \psi(\mathfrak{g}[\underline{u}])^n &\ge b_1(\mathfrak{g}[\underline{u}])^{n-1}\wedge\omega + b_2(\mathfrak{g}[\underline{u}])^{n-2}\wedge\omega^2 && \text{in } M,\\ \underline{u} &= \varphi && \text{on } \partial M. \end{aligned}$$

The $\mathcal{C}$-subsolution introduced by Székelyhidi [40] turns out to be a suitable condition to study certain Hessian equations on closed manifolds.

We state main results as follows. First, we shall present some notations used in this paper. In a local coordinate $(z_1,\dots,z_n)$, we write $\partial_i = \frac{\partial}{\partial z_i}$, $\overline{\partial}_i = \frac{\partial}{\partial \overline{z}_i}$,

$$\begin{gathered} \chi_{i\overline{j}} = \chi(z,du)(\partial_i,\overline{\partial}_j),\\ \chi_{i\overline{j}k} := \nabla_{\partial_k}(\chi_{i\overline{j}}) = \chi_{i\overline{j},k} + \chi_{i\overline{j},\zeta_\alpha}u_{\alpha k} + \chi_{i\overline{j},\overline{\zeta}_\alpha}u_{\overline{\alpha}k},\\ \vdots \end{gathered}$$

Moreover, we assume χ satisfies the following structural condition

$$\chi_{i\overline{j},\zeta_\alpha\zeta_\beta} = 0, \quad \chi_{i\overline{j},\zeta_\alpha\overline{\zeta}_\beta} = 0, \quad \chi_{i\overline{j},\overline{\zeta}_\alpha\overline{\zeta}_\beta} = 0. \tag{1.5}$$

Unlike the standard equation, deriving gradient estimates is extremely difficult even for the complex Monge–Ampère equation containing du. In our setting, we observe that equation (1.1) satisfies condition (3.5), which can be used to derive directly the gradient estimates.

Theorem 1.2 *Suppose that (1.5) holds and there exists a subsolution $\underline{u} \in C^2(\overline{M})$ of equation (1.1) in the sense of Definition 1.1. Then for any solution $u \in C^3(M) \cap C^1(\overline{M})$ of equation (1.1) with $\mathfrak{g}[u] > 0$, there is a uniform constant C depending on $|u|_{C^0(\overline{M})}$ and other known data under control, such that*

$$\max_{\overline{M}} |\nabla u| \le C(1 + \max_{\partial M} |\nabla u|).$$

Comparing to Székelyhidi [40], the bounds of gradient terms relax the restriction to the construction of the barrier functions for second estimates. Moreover, condition (1.3) guarantees we can control the bad terms arising from the gradient term contained in equation (1.1). Hence, we prove the following *a priori* second estimates.

Theorem 1.3 *Let $u \in C^4(M) \cap C^2(\overline{M})$ be a solution to equation (1.1) such that $\mathfrak{g}[u]$ is positive. Suppose, that (1.5) holds, and that at any fixed point $z \in M$, where $g_{i\bar{j}} = \delta_{ij}$ and $\mathfrak{g}_{i\bar{j}} = \lambda_i \delta_{ij}$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$,*

$$\sum_{\alpha > 1} \big| \mathfrak{Re}\{ f_\alpha \chi_{\alpha\bar{1},\zeta_\beta} \} \big| \le \rho(\lambda)\lambda_1 f_\beta, \quad \forall \beta, \tag{1.6}$$

where $\rho(\lambda) \colon \Gamma_n \to \mathbb{R}^+$ is a positive continuous function with $\rho(\lambda) \to 0$ as $|\lambda| \to +\infty$ and

$$f(\lambda) = -\sum_i \frac{d_1}{\lambda_i} - \sum_{i<j} \frac{d_2}{\lambda_i \lambda_j}, \quad d_1 = \frac{b_1}{n}, d_2 = \frac{2b_2}{n(n-1)}.$$

Then there is a uniform bounded positive constant C depending on $|u|_{C^1(\overline{M})}$, $|\psi|_{C^{1,1}(\overline{M})}$, $|\underline{u}|_{C^{1,1}(\overline{M})}$, and other data under control, such that $\sup_{\overline{M}} |\Delta u| \le C(1 + \sup_{\partial M} |\Delta u|)$, provided that there exists a subsolution $\underline{u} \in C^2(\overline{M})$ of equation (1.1) in the sense of Definition 1.1.

Condition (1.6) can be satisfied by many equations, for instance, the complex Monge–Ampère type equation on an annulus in the Kähler cone studied by Guan and Zhang [25, 26].

Suppose that M is closed and there is a function $\underline{u} \in C^2(M)$ such that $\mathfrak{g}[\underline{u}] > 0$ and

$$n\psi(\mathfrak{g}[\underline{u}])^{n-1} - (n-1)b_1(\mathfrak{g}[\underline{u}])^{n-2} \wedge \omega - (n-2)b_2(\mathfrak{g}[\underline{u}])^{n-3} \wedge \omega^2 > 0, \tag{1.7}$$

i.e., $\underline{u}$ is a $\mathcal{C}$-subsolution of equation (1.1). Then a uniform bound for $|u|_{C^0(M)}$ will be obtained by a modified argument in [40], which is inspired by Blocki's proof of Yau's zero-order estimates for complex Monge–Ampère equations on closed Kähler manifolds [2]. We also refer the reader to [47] for Yau's original proof by Moser's iteration. Combining that proof with Theorem 1.3 and Theorem 1.2, equation (1.1) becomes a uniform elliptic equation, so that we can use an argument in [22] to derive a uniform bound on the real Hessian. Then one can use the Evans–Krylov theorem [12, 28] and Schauder theory to derive higher order regularities.

To solve the equations on closed Hermitian manifolds via the continuity method, it is rather difficult to verify condition (1.7) along the continuity path [38,39]. Indeed, except for the function f satisfying $\lim_{t\to+\infty} f(\lambda_1,\dots,\lambda_n+t)=+\infty$, for all $\lambda\in\Gamma$, it is very difficult to solve general Hessian equations. We refer the reader to [40] for some related open problems.

Therefore, our main result on a closed Hermitian manifold is the establishment of the following *a priori* estimate.

Theorem 1.4 *Let (M,ω) be a closed Hermitian manifold and let u be a smooth solution to equation (1.1). Suppose conditions (1.5), (1.6), and (1.7) hold. Then there are uniform C^∞* a priori *estimates for u.*

When $n=2$, $b_1=0$, and $b_2=1$, condition (1.7) remains valid along the continuity path if $\mathfrak{g}[v]>0$, so one can solve the complex Monge–Ampère equation (1.1) on closed complex surfaces.

Corollary 1.5 *Let (M,ω) be a closed complex surface and let F be a smooth function on M. Assume $\chi=\chi(\cdot,\zeta,\overline{\zeta})$ satisfies $\chi(\cdot,0,0)>0$, and conditions (1.5) and (1.6) hold. Then there is a unique smooth function u and a unique constant b such that*

$$\begin{cases} (\mathfrak{g}[u])^2 = e^{F+b}\omega^2, \\ \mathfrak{g}[u]>0, \quad \sup\limits_M u = 0. \end{cases}$$

In contrast with the closed setting, the continuity method for the Dirichlet problem is much easier and is well understood. Next, we apply our estimates to treat the Dirichlet problem on the annulus in the Kähler cone.

Let (S,ξ,η,Φ,g) be a closed Sasakian manifold of dimension $2n-1$ with Sasakian structure (ξ,η,Φ,g), where ξ is a Reeb field on S, $\Phi(X)=\nabla_X\xi$, η is the contact 1-form with $\eta(X)=g(\xi,X)$. Then the metric cone $(C(S),\widetilde{g})=(S\times\mathbb{R}^+,r^2g+dr^2)$ is a Kähler manifold; here r is the coordinate on $\mathbb{R}^+$.

Inspired by the observations of Donaldson [10], Semmes [36], and Mabuchi [29] in Kähler geometry, Guan and Zhang [25] discovered that the geodesic equation connecting two points in the space of Sasakian metrics can be reduced to a Dirichlet problem of a homogeneous complex Monge–Ampère type equation containing radial derivatives on an annulus of the Kähler cone. Similar to the work of Chen [9] in the Kähler setting, Guan and Zhang dealt with the complex Monge–Ampère equation and obtained some useful properties of the space of Sasakian metrics [26]. However, their estimates depend on the assumption of *basic* data which is natural from the Sasakian geometry point of view.

Observing that $\overline{\omega}=\sqrt{-1}\partial\overline{\partial}r$ also determines a Kähler metric on the Kähler cone $C(S)=S\times\mathbb{R}^+$, assumption (1.6) in Theorem 1.3 automatically holds for equation (1.9). Hence the assumption of basic data can be removed for such (nondegenerate) equations.

Theorem 1.6 *Suppose that there exists a function $\underline{v} \in C^2(\overline{M})$, $\overline{M} = S \times [a,b]$, such that $\underline{v}|_{r=a} = \varphi_a$, $\underline{v}|_{r=b} = \varphi_b$, $\mathfrak{g}[\underline{v}] > 0$ and*

$$\begin{aligned}(1.8)\quad &\big[(n-1)b_1(\mathfrak{g}[\underline{v}])^{n-2} \wedge \overline{\omega} + (n-2)b_2(\mathfrak{g}[\underline{v}])^{n-3} \wedge \overline{\omega}^2\big] \wedge \sqrt{-1}\theta \\ &\qquad \wedge \overline{\theta} - n\psi(\mathfrak{g}[\underline{v}])^{n-1} \wedge \sqrt{-1}\theta \wedge \overline{\theta} < 0 \text{ in } M,\end{aligned}$$

where $\theta^n = dr + \sqrt{-1}r\eta$, $\overline{\theta}^n = dr - \sqrt{-1}r\eta$. Then there is a unique smooth solution u to

$$(1.9)\qquad \begin{cases} b_1(\mathfrak{g}[u])^{n-1} \wedge \overline{\omega} + b_2(\mathfrak{g}[u])^{n-2} \wedge \overline{\omega}^2 = \psi(\mathfrak{g}[u])^n, & \mathfrak{g}[u] > 0 \text{ in } M, \\ u|_{r=a} = \varphi_a, \quad u|_{r=b} = \varphi_b, \end{cases}$$

where $\mathfrak{g}[u] = \widetilde{\chi} + \sqrt{-1}\phi(r)(\partial\overline{\partial}u - \frac{\partial u}{\partial r}\partial\overline{\partial}r)$, $\phi\colon [a,b] \to \mathbb{R}^+$ is a positive and smooth function, $\varphi_a, \varphi_b \in C^\infty(S)$, ψ is a positive and smooth function on $\overline{M}$, and $\widetilde{\chi}$ is a smooth and real $(1,1)$-form on $\overline{M}$.

From Theorem 1.6, we know that the solvability of Dirichlet problem (1.9) is determined by the action of $\mathfrak{g}[\underline{v}]$ on the pullback contact bundle $P^*\mathcal{D}^{\mathbb{C}}$, where $P\colon C(S) \to S$ is the natural projection.

Remark 1.7 We find that Lagrangian phase equation (hypercritical phase)

$$(1.10)\qquad f(\lambda) = \sum \arctan \lambda_i = \psi, \quad \psi\colon \overline{M} \to \Big((n-1)\frac{\pi}{2}, n\frac{\pi}{2}\Big)$$

satisfies that there is a positive constant R_o (that may depend on σ) such that

$$f_{11} + \frac{f_1}{\lambda_1} \le 0 \text{ and } \lambda_1 f_1 \le \lambda_i f_i \text{ in } \Gamma \text{ for } i > 1, \lambda \in \partial\Gamma^\sigma, \forall \sigma \in [\inf_{\overline{M}} \psi, \sup_{\overline{M}} \psi],$$

where λ_1 is the largest eigenvalue, with

$$\lambda_1 \ge R_o, \quad f_i = \frac{\partial f}{\partial \lambda_i}, \quad f_{ij} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}, \quad \Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}.$$

This condition is a modification of the extra concavity condition (4.5) introduced in [40]. Notice that equation (1.10) is not satisfied for the original extra concavity condition and the related strong concavity condition in [13,14]. Moreover, we know that f satisfies the conditions of Lemma 2.1 and the two conditions (1.3) and (3.5), so Theorems 1.2 and 1.3 also hold for equation (1.10).

This paper is organized as follows. In Section 2 we present some notations and useful lemmas. In Section 3 we directly derive gradient estimates by the Bernstein method. In Section 4 we establish global *a priori* second order estimates for admissible solutions. In Section 5 our main content is to derive boundary estimates for second derivatives for solutions of Dirichlet problem (1.9) on an annulus of the Kähler cone. Moreover, some admissible subsolutions are constructed under the assumption that (1.8) holds.

2 Preliminaries

We rewrite equation (1.1) as follows.

$$F\big(\{\mathfrak{g}_{i\bar{j}}[u]\}\big) = f\big(\lambda(\mathfrak{g}[u])\big) = -\psi, \tag{2.1}$$

where $\lambda(\mathfrak{g}[u])$ are the eigenvalues of $\mathfrak{g}[u]$ with respect to ω, and

$$f(\lambda) = -\sum_i \frac{d_1}{\lambda_i} - \sum_{i<j} \frac{d_2}{\lambda_i\lambda_j}, \quad d_1 = \frac{b_1}{n}, d_2 = \frac{2b_2}{n(n-1)}.$$

Throughout this paper, we use derivatives with respect to the Chern connection ∇ of g, and in local coordinates $z = (z^1, \dots, z^n)$ we use notations such as

$$v_i = \nabla_{\frac{\partial}{\partial z^i}} v, \quad v_{ij} = \nabla_{\frac{\partial}{\partial z^j}} \nabla_{\frac{\partial}{\partial z^i}} v, \quad v_{i\bar{j}} = \nabla_{\frac{\partial}{\partial \bar{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} v, \dots .$$

Given a Hermitian matrix $A = \{a_{i\bar{j}}\}$, we write

$$F^{i\bar{j}}(A) = \frac{\partial F}{\partial a_{i\bar{j}}}(A), \quad F^{i\bar{j},k\bar{l}}(A) = \frac{\partial^2 F}{\partial a_{i\bar{j}}\partial a_{k\bar{l}}}(A).$$

We also write

$$\mathfrak{g}_{i\bar{j}} = \mathfrak{g}_{i\bar{j}}[u] = u_{i\bar{j}} + \chi_{i\bar{j}}(z, du), \quad \underline{\mathfrak{g}}_{i\bar{j}} = \mathfrak{g}_{i\bar{j}}[\underline{u}] = \underline{u}_{i\bar{j}} + \chi_{i\bar{j}}(z, d\underline{u}).$$

One can verify the following lemma.

Lemma 2.1 *For any $\sigma \in (\sup_{\partial\Gamma} f, \sup_\Gamma f)$, there is a $\kappa_0 > 0$ depending on σ such that $\sum_{i=1}^n f_i(\lambda) \geq \kappa_0$ for $\lambda \in \partial\Gamma_n^\sigma := \{\lambda \in \Gamma_n : f(\lambda) = \sigma\}$.*

Building on the work of Guan [21], Székelyhidi [40] introduced the $\mathcal{C}$-subsolution satisfying (1.4) and proved some properties of the $\mathcal{C}$-subsolution. Applying Székelyhidi's results to equation (2.1), one has the following lemma.

Lemma 2.2 ([40]) *Suppose, in addition to $n \geq 2$, that there exists a subsolution $\underline{u} \in C^2(\overline{M})$ in the sense of Definition 1.1. Then there exist positive constants R_0, ε with the following property. If $|\lambda| \geq R_0$, then we have either*

(i) $F^{i\bar{j}}(\underline{\mathfrak{g}}_{i\bar{j}} - \mathfrak{g}_{i\bar{j}}) \geq \varepsilon \sum F^{i\bar{j}} g_{i\bar{j}}$ *or*

(ii) $F^{i\bar{j}} \geq \varepsilon (F^{p\bar{q}} g_{p\bar{q}}) g^{i\bar{j}}$.

3 Gradient Estimates

Using the Bernstein method, we will prove Theorem 1.2 in this section.

Proof of Theorem 1.2 Our method is similar to that used in [3, 23]. We consider $\phi = Ae^\eta$, $\eta = A[\underline{u} - u - \inf_{\overline{M}}(\underline{u} - u)]$, where A is a positive constant to be determined.

Suppose that $e^\phi |\nabla u|^2$ achieves its maximum at an interior $p \in M$. Set $W = |\nabla u|^2$. We choose local holomorphic coordinates $z = (z^1, \dots, z^n)$, such that at p, $g_{i\bar{j}} = \delta_{ij}$, $\mathfrak{g}_{i\bar{j}} = \lambda_i \delta_{ij}$, $F^{i\bar{j}} = f_i \delta_{ij}$. Then at p,

$$\frac{W_i}{W} + \phi_i = 0, \quad \frac{W_{\bar{i}}}{W} + \phi_{\bar{i}} = 0, \quad \frac{W_{i\bar{i}}}{W} - \frac{|W_i|^2}{W^2} + \phi_{i\bar{i}} \leq 0. \tag{3.1}$$

In what follows the computations are done at p. We compute

$$
\begin{aligned}
W_i &= \sum_k (u_{ki}u_{\bar{k}} + u_k u_{i\bar{k}}),\\
W_{i\bar{i}} &= \sum_k |u_{ki}|^2 + 2\sum_k \mathfrak{Re}(u_{i\bar{i}k}u_{\bar{k}}) + \sum_{k,l} R_{i\bar{i}k\bar{l}}u_l u_{\bar{k}} + \sum_k |u_{i\bar{k}}\\
&\quad - \sum_l T^k_{il}u_{\bar{l}}|^2 - \sum_k |\sum_l T^k_{il}u_{\bar{l}}|^2.
\end{aligned}
$$

Therefore,

$$
|W_i|^2 \le |\nabla u|^2 \sum_k |u_{ki}|^2 - 2|\nabla u|^2 \sum_k \mathfrak{Re}(u_k u_{i\bar{k}}\phi_{\bar{i}}). \tag{3.2}
$$

Differentiating equation (2.1), one gets

$$
F^{i\bar{i}}(u_{i\bar{i}k} + \chi_{i\bar{i},\zeta_\alpha}u_{\alpha k} + \chi_{i\bar{i},\bar{\zeta}_\alpha}u_{\bar{\alpha}k}) = -\psi_k - F^{i\bar{i}}\chi_{i\bar{i},k}.
$$

Thus the crucial assumption (1.5) implies that

$$
\mathcal{L}(W) \ge F^{i\bar{i}}|u_{ki}|^2 - C|\nabla u|\Big(1 + \sum F^{i\bar{i}}\Big) - C|\nabla u|^2 \sum F^{i\bar{i}},
$$

where $\mathcal{L}$ is the linearized operator of equation (2.1)

$$
\mathcal{L}v = F^{i\bar{j}}v_{i\bar{j}} + F^{i\bar{j}}\chi_{i\bar{j},\zeta_\alpha}v_\alpha + F^{i\bar{j}}\chi_{i\bar{j},\bar{\zeta}_\alpha}v_{\bar{\alpha}} \quad \text{for } v \in C^2(M).
$$

Next, $\phi_i = \phi\eta_i, \phi_{i\bar{i}} = \phi(|\eta_i|^2 + \eta_{i\bar{i}})$. Using the Cauchy–Schwarz inequality and the crucial assumption (1.5) again, one obtains

$$
2\phi^{-1}\mathfrak{Re}(u_k u_{i\bar{k}}\phi_{\bar{i}}) \ge 2\mathfrak{Re}(\mathfrak{g}_{i\bar{k}}u_k\eta_{\bar{i}}) - \frac{1}{2}|\nabla u|^2|\eta_i|^2 - C|\nabla u|^2,
$$

and

$$
\mathcal{L}\phi = \phi\mathcal{L}\eta + \phi F^{i\bar{i}}|\eta_i|^2, \tag{3.3}
$$

where we use assumption (1.5). Then by (3.1), (3.2), (3.3), we have

$$
\begin{aligned}
|\nabla u|^2\Big(\frac{1}{2}F^{i\bar{i}}|\eta_i|^2 + \mathcal{L}\eta\Big) - \frac{C}{\phi}|\nabla u|\Big(1 + \sum F^{i\bar{i}}\Big) - C|\nabla u|^2 \sum F^{i\bar{i}}&\\
- \frac{C|\nabla u|^2}{\phi}\sum F^{i\bar{i}} \le -2\mathfrak{Re}(F^{i\bar{i}}\mathfrak{g}_{i\bar{i}}u_i\eta_{\bar{i}}).&
\end{aligned} \tag{3.4}
$$

We need to control the term of the right-hand side. Applying the Cauchy–Schwarz inequality, one obtains $-2\sum\mathfrak{Re}(F^{i\bar{i}}\mathfrak{g}_{i\bar{i}}u_i\eta_{\bar{i}}) \le \frac{1}{4}|\nabla u|^2F^{i\bar{i}}|\eta_i|^2 + 4\sum F^{i\bar{i}}\mathfrak{g}^2_{i\bar{i}}$.

We claim that there is a positive constant c_0 such that

$$
\sum F^{i\bar{i}}\mathfrak{g}^2_{i\bar{i}} \le c_0\Big(1 + \sum F^{i\bar{i}}\Big). \tag{3.5}
$$

Recall that

$$
\sum F^{i\bar{i}}\mathfrak{g}^2_{i\bar{i}} = nd_1 + (n-1)d_2\sum_i \mathfrak{g}^{i\bar{i}}, \quad \sum F^{i\bar{i}} = d_2\sum_{j\ne i}\mathfrak{g}^{i\bar{i}}\mathfrak{g}^{i\bar{i}}\mathfrak{g}^{j\bar{j}} + d_1\sum(\mathfrak{g}^{i\bar{i}})^2.
$$

If $d_2 = 0$, then it is trivial. Suppose $d_2 > 0$. If $d_1 > 0$, $\sum_{i<j}\frac{d_2}{\lambda_i\lambda_j} + \sum\frac{d_1}{\lambda_i} = \psi$, then

$$
\sum F^{i\bar{i}}\mathfrak{g}^2_{i\bar{i}} \le nd_1 + \frac{(n-1)d_2}{d_1}\psi.
$$

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Let $\lambda_\varsigma \le \lambda_i$ for $i = 1, \dots, n$. If $d_1 = 0$, $\sum_{i<j} d_2 \mathfrak{g}^{i\bar{i}} \mathfrak{g}^{j\bar{j}} = \psi$, $f(\lambda) = -\sum_{i<j} \frac{d_2}{\lambda_i \lambda_j}$, then

$$\frac{\sum f_i}{\sum f_i \lambda_i^2} = \frac{\sum_i \sum_{j\neq i} \frac{1}{\lambda_i^2 \lambda_j}}{(n-1)\sum \frac{1}{\lambda_i}} \ge \sum_i \sum_{j\neq i} \frac{1}{(n-1)n} \frac{\lambda_\varsigma}{\lambda_i^2 \lambda_j} \ge \frac{1}{(n-1)n} \sum_{j\neq\varsigma} \frac{1}{\lambda_\varsigma \lambda_j}$$
$$\ge \frac{1}{(n-1)n^2} \sum_{i<j} \frac{1}{\lambda_i \lambda_j} = \frac{\psi}{(n-1)n^2 d_2}.$$

We complete the proof of the claim.

By Lemma 2.1, $\sum F^{i\bar{i}} \ge \kappa_0$ for some $\kappa_0 > 0$. Suppose that $|\lambda| \ge R_0$, where R_0 is the constant in Lemma 2.2. In Lemma 2.2(i), it holds that

$$\frac{A\varepsilon\kappa_0}{1+\kappa_0}\big(1 + \sum F^{i\bar{i}}\big) - \Big(\frac{C}{\phi|\nabla u|} + \frac{4c_0}{|\nabla u|^2}\Big)\big(1 + \sum F^{i\bar{i}}\big) - C\big(1 + \frac{1}{\phi}\big) \sum F^{i\bar{i}} \le 0.$$

This gives us a bound $|\nabla u| \le C$, when A is chosen to be large.

In Lemma 2.2(ii), $F^{i\bar{i}} \ge \varepsilon \sum F^{k\bar{k}} \ge \varepsilon\kappa_0$. By $\mathfrak{g}^{i\bar{i}}\psi \ge F^{i\bar{i}}$ one has $\mathfrak{g}_{i\bar{i}} \le (\varepsilon\kappa_0)^{-1}\psi$ and there is a positive a constant C_{R_0} such that $\sum F^{i\bar{i}} \mathfrak{g}_{i\bar{i}} \le C_{R_0}$. By (3.4), one derives the bound of $|\nabla u| \le C$ again.

If $|\lambda| < R_0$, the proof is similar to that of (ii), and we omit it here. ∎

4 Global Second Estimates

In this section, we will derive global second order estimates for admissible solutions.

Proof of Theorem 1.3 We use the method used in [40] to treat the third order terms arising from nontrivial torsion tensors. Denote the eigenvalues of the matrix $A = \{A^i_j\} = \{g^{i\bar{q}} \mathfrak{g}_{j\bar{q}}\}$ by $(\lambda_1, \dots, \lambda_n)$, where $\mathfrak{g}_{i\bar{j}} = u_{i\bar{j}} + \chi_{i\bar{j}}$, $\lambda_1 : \overline{M} \to \mathbb{R}$ is the largest eigenvalue and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ at each point, and g is the Hermitian metric. We want to apply the maximum principle to H, *i.e.*, $H := \lambda_1 e^\phi$, where the test function ϕ is chosen later. Suppose H achieves its maximum at interior point $p_0 \in M$.

Since the eigenvalues of A need not be distinct at the point p_0, H may be only continuous. To circumvent this difficulty we use a perturbation argument [40]. To do this, we choose local coordinates $z = (z_1, \dots, z_n)$ around p_0, such that at p_0,

$$g_{i\bar{j}} = \delta_{ij}, \quad \mathfrak{g}_{i\bar{j}} = \delta_{ij}\lambda_i, \quad F^{i\bar{j}} = \delta_{ij} f_i.$$

Let B be a diagonal matrix B^p_q with real entries satisfying $B^1_1 = 0$, $B^n_n > 2B^2_2$, and such that $B^n_n < B^{n-1}_{n-1} < \cdots < B^2_2 < 0$ are small. Then we define the matrix $\widetilde{A} = A + B$ with eigenvalues $\widetilde{\lambda} = (\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_n)$. At the origin, $\widetilde{\lambda}_1 = \lambda_1 = \mathfrak{g}_{1\bar{1}}$, $\widetilde{\lambda}_i = \lambda_i + B^i_i$ if $i \ge 2$ and the eigenvalues of $\widetilde{A}$ define C^2-functions near the origin.

Notice that $\widetilde{H} = \widetilde{\lambda}_1 e^\phi$ achieves its maximum at the same point p_0 (we can assume $\lambda_1(p_0) = \widetilde{\lambda}_1(p_0) > 1$). In what follows, we use derivatives with respect to the Chern connection of g and the computations will be given at the origin p_0. Then we have

$$\frac{\widetilde{\lambda}_{1,k}}{\lambda_1} + \phi_k = 0, \qquad \frac{\widetilde{\lambda}_{1,k\bar{k}}}{\lambda_1} - \frac{|\widetilde{\lambda}_{1,k}|^2}{\lambda_1^2} + \phi_{k\bar{k}} \le 0. \tag{4.1}$$

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By straightforward calculations, one obtains

$$\widetilde{\lambda}_{1,k} = \mathfrak{g}_{1\bar{1}k} - (B_1^1)_k. \tag{4.2}$$

Moreover, (see [40])

$$\begin{aligned}\widetilde{\lambda}_{1,k\bar{k}} = \mathfrak{g}_{1\bar{1}k\bar{k}} + \sum_{p>1} \frac{|\mathfrak{g}_{p\bar{1}k}|^2 + |\mathfrak{g}_{1\bar{p}k}|^2}{\widetilde{\lambda}_1 - \widetilde{\lambda}_p} - (B_1^1)_{k\bar{k}} \\ + 2\mathfrak{Re} \sum_{p>1} \frac{\mathfrak{g}_{p\bar{1}k}(B_p^1)_{\bar{k}} + \mathfrak{g}_{1\bar{p}k}(B_1^p)_{\bar{k}}}{\widetilde{\lambda}_1 - \widetilde{\lambda}_p} + \widetilde{\lambda}_1^{pq,rs}(B_q^p)_k (B_s^r)_{\bar{k}},\end{aligned}$$

where

$$\widetilde{\lambda}_1^{pq,rs} = (1-\delta_{1p})\frac{\delta_{1q}\delta_{1r}\delta_{ps}}{\widetilde{\lambda}_1 - \widetilde{\lambda}_p} + (1-\delta_{1r})\frac{\delta_{1s}\delta_{1p}\delta_{rq}}{\widetilde{\lambda}_1 - \widetilde{\lambda}_r}.$$

We need to estimate $\widetilde{\lambda}_1 - \widetilde{\lambda}_p$ near the origin p_0 for $p > 1$. Since $\Gamma_n \subseteq \Gamma \subset \Gamma_1$, letting B be sufficiently small, we can assume $\sum \widetilde{\lambda}_i > 0$ (otherwise we are done). Then $|\widetilde{\lambda}_i| \le (n-1)\widetilde{\lambda}_1$ for all i, and so $(\widetilde{\lambda}_1 - \widetilde{\lambda}_p)^{-1} \ge (n\widetilde{\lambda}_1)^{-1}$. As in [40], one obtains

$$\widetilde{\lambda}_{1,k\bar{k}} \ge \mathfrak{g}_{1\bar{1}k\bar{k}} + \frac{1}{2n\lambda_1}\sum_{p>1}(|\mathfrak{g}_{p\bar{1}k}|^2 + |\mathfrak{g}_{1\bar{p}k}|^2) - C_0.$$

We assume $\lambda_1 \ge R_0$, where R_0 is the constant in Lemma 2.2. Note that $0 < f_i\lambda_i \le \psi$ for $i = 1, \dots, n$. If $f_i = F^{i\bar{i}} \ge \varepsilon \sum F^{k\bar{k}} \ge \varepsilon\kappa_0$ for $i = 1, \dots, n$, then it immediately gives a bound $\lambda_1 = \mathfrak{g}_{1\bar{1}} \le C$. So we only need to consider the case

$$F^{i\bar{j}}\big(\underline{\mathfrak{g}}_{i\bar{j}} - \mathfrak{g}_{i\bar{j}}\big) \ge \varepsilon \sum F^{k\bar{l}} g_{k\bar{l}} \ge \frac{\kappa_0\varepsilon}{1+\kappa_0}\big(1 + \sum F^{k\bar{l}} g_{k\bar{l}}\big).$$

By straightforward calculations, one has

$$\chi_{i\bar{j}k} = \chi_{i\bar{j},k} + \chi_{i\bar{j},\zeta_\alpha} u_{\alpha k} + \chi_{i\bar{j},\bar{\zeta}_\alpha} u_{\bar{\alpha}k} \tag{4.3}$$

and

$$\begin{aligned}\chi_{i\bar{i}k\bar{k}} = {} & \chi_{i\bar{i},k\bar{k}} + \chi_{i\bar{i},\zeta_\alpha} R_{k\bar{k}\alpha\bar{l}} u_l + 2\mathfrak{g}_{k\bar{k}} \mathfrak{Re}\big\{\chi_{i\bar{i},k\zeta_k} - \chi_{i\bar{i},\zeta_\alpha} T^k_{k\alpha}\big\} \\ & + 2\mathfrak{Re}\big\{\chi_{i\bar{i},\zeta_\alpha} \mathfrak{g}_{k\bar{k}\alpha}\big\} - 2\mathfrak{Re}\big\{\chi_{i\bar{i},\zeta_\alpha}(\chi_{i\bar{i},\alpha} - T^l_{k\alpha}\chi_{l\bar{k}}) + \chi_{i\bar{i},k\bar{\zeta}_\alpha}\chi_{\alpha\bar{k}}\big\} \\ & + 2\mathfrak{Re}\big\{\chi_{i\bar{i},\zeta_\alpha\bar{k}} u_{\alpha k}\big\} - 2\mathfrak{Re}\big\{\chi_{i\bar{i},\zeta_\alpha}\chi_{k\bar{k},\zeta_\beta} u_{\beta\alpha} + \chi_{i\bar{i},\zeta_\alpha}\chi_{k\bar{k},\bar{\zeta}_\beta} u_{\alpha\bar{\beta}}\big\}.\end{aligned}$$

Differentiating equation (2.1) twice (using the covariant derivative), we have at p_0

$$F^{k\bar{k}}\mathfrak{g}_{k\bar{k}l} = -\psi_l, \quad F^{k\bar{k}}\mathfrak{g}_{k\bar{k}1\bar{1}} + F^{i\bar{j},l\bar{m}}\mathfrak{g}_{i\bar{j}1}\mathfrak{g}_{l\bar{m}\bar{1}} = -\psi_{1\bar{1}}. \tag{4.4}$$

It is easy to verify that

$$f_{11} + \frac{f_1}{\lambda_1} \le 0, \quad \frac{f_1 - f_i}{\lambda_i - \lambda_1} \ge \frac{f_i}{\lambda_1}, \quad i > 1, \tag{4.5}$$

so we obtain

$$-F^{i\bar{j},l\bar{m}}\mathfrak{g}_{i\bar{j}1}\mathfrak{g}_{l\bar{m}\bar{1}} \ge \sum_{i>1}\frac{f_1 - f_i}{\lambda_i - \lambda_1}|\mathfrak{g}_{i\bar{1}1}|^2 \ge \sum_{k>1}\frac{1}{\lambda_1}F^{k\bar{k}}|\mathfrak{g}_{k\bar{1}1}|^2;$$

see Székelyhidi [40]. We shall point out that (4.5) is the extra concavity condition introduced by Székelyhidi [40]. Then

$$(4.6)\quad F^{k\bar k}\mathfrak{g}_{1\bar 1 k\bar k} \ge \sum_{k>1}\frac{1}{\lambda_1}F^{k\bar k}|\mathfrak{g}_{k\bar 1 1}|^2 + F^{i\bar i}(u_{1\bar 1 i\bar i} - u_{i\bar i 1\bar 1}) - 2\mathfrak{Re}\{F^{i\bar i}\chi_{i\bar i,\zeta_\alpha}\mathfrak{g}_{1\bar 1\alpha}\} - 2\mathfrak{Re}\{F^{i\bar i}\chi_{i\bar i,\zeta_\alpha\bar 1}u_{\alpha 1}\} + 2\mathfrak{Re}\{F^{i\bar i}\chi_{1\bar 1,\zeta_\alpha\bar i}u_{\alpha i}\} - C - C\lambda_1\sum F^{i\bar i}.$$

Recall that $u_{1\bar 1 i\bar i} - u_{i\bar i 1\bar 1} = R_{i\bar i 1\bar p}u_{p\bar 1} - R_{1\bar 1 i\bar p}u_{p\bar 1} + 2\mathfrak{Re}\{\overline{T}^j_{1i}u_{i\bar j 1}\} + T^p_{i1}\overline{T}^q_{i1}u_{p\bar q}$. From (4.1), (4.2), (4.3), and (4.6), we know that

$$(4.7)\quad 0 \ge \mathcal{L}\phi - F^{k\bar k}\frac{|\widetilde{\lambda}_{1,k}|^2}{\lambda_1^2} + \sum_{k>1}F^{k\bar k}\frac{|\mathfrak{g}_{k\bar 1 1}|^2}{\lambda_1^2} + 2\mathfrak{Re}\Big\{F^{i\bar i}\overline{T}^1_{1i}\frac{\widetilde{\lambda}_{1,i}}{\lambda_1}\Big\} + \frac{2}{\lambda_1}\mathfrak{Re}\Big\{F^{i\bar i}\chi_{1\bar 1,\zeta_\alpha\bar i}u_{\alpha i}\Big\} - \frac{C}{\lambda_1}|u_{\alpha 1}|\sum F^{i\bar i} - C\Big(1+\sum F^{i\bar i}\Big).$$

Thus, the estimates of the differences $|\mathfrak{g}_{k\bar 1 1}|^2 - |\widetilde{\lambda}_{1,k}|^2$ for $k \ge 2$ are crucial; $\mathfrak{g}_{k\bar 1 1} = \widetilde{\lambda}_{1,k} + \tau_k - \chi_{1\bar 1,\zeta_\alpha}u_{\alpha k} + \chi_{k\bar 1,\zeta_\beta}u_{\beta 1}$, where

$$\tau_k := \chi_{k\bar 1,1} - \chi_{1\bar 1,k} + \chi_{k\bar 1,\zeta_{\bar\alpha}}u_{\bar\alpha 1} - \chi_{1\bar 1,\zeta_{\bar\alpha}}u_{\bar\alpha k} + T^l_{k1}u_{l\bar 1} + (B^1_1)_k.$$

Thus

$$(4.8)\quad |\mathfrak{g}_{k\bar 1 1}|^2 \ge |\widetilde{\lambda}_{1,k}|^2 - |\tau_k - \chi_{1\bar 1,\zeta_\alpha}u_{\alpha k}|^2 + 2\mathfrak{Re}\Big\{\widetilde{\lambda}_{1,\bar k}(\tau_k - \chi_{1\bar 1,\zeta_\alpha}u_{\alpha k})\Big\} + \frac{1}{2}\Big|\sum_\beta \chi_{k\bar 1,\zeta_\beta}u_{\beta 1}\Big|^2 + 2\mathfrak{Re}\sum_\beta\{\widetilde{\lambda}_{1,\bar k}\chi_{k\bar 1,\zeta_\beta}u_{\beta 1}\}\quad \text{for } k\ge 2.$$

Set $\phi = A_2|\nabla u|^2 + \Psi(u-\underline{u})$, $A_2 > 0$, where Ψ shall satisfy $\Psi' < 0$ and $\Psi'' > 0$. From (4.1) one knows $\widetilde{\lambda}_{1,\bar k} = -\lambda_1\phi_{\bar k} = -\lambda_1\big\{A_2(u_{\bar i}u_{i\bar k} + u_i u_{\bar i\bar k}) + \Psi'(u-\underline{u})_{\bar k}\big\}$ and

$$\sum_\beta 2\mathfrak{Re}\Big\{\widetilde{\lambda}_{1,\bar k}\chi_{k\bar 1,\zeta_\beta}u_{\beta 1}\Big\} \ge -\sum_\beta 2\lambda_1\mathfrak{Re}\Big\{\Psi'\chi_{k\bar 1,\zeta_\beta}u_{\beta 1}(u-\underline{u})_{\bar k}\Big\} - \frac{1}{3}A_2\lambda_1^2(|u_{ik}|^2 + |u_{\bar i k}|^2) - 3nA_2|\nabla u|^2\Big|\sum_\beta\chi_{k\bar 1,\zeta_\beta}u_{\beta 1}\Big|^2.$$

Let $A_2 \le \frac{1}{12n}K^{-1}$, where $K = 1 + \sup_M|\nabla u|^2 + \sup_M|\nabla(u-\underline{u})|^2$. Absorbing the terms $-|\tau_k - \chi_{1\bar 1,\zeta_\alpha}u_{\alpha k}|^2 + 2\mathfrak{Re}\big\{\widetilde{\lambda}_{1,\bar k}(\tau_k - \chi_{1\bar 1,\zeta_\alpha}u_{\alpha k})\big\}$ in (4.8) by

$$\frac{A_2\lambda_1^2}{6}(|u_{ki}|^2 + |u_{k\bar i}|^2),$$

for $k \ge 2$, one has

$$(4.9)\quad F^{k\bar k}\frac{(|\mathfrak{g}_{k\bar 1 1}|^2 - |\widetilde{\lambda}_{1,k}|^2)}{\lambda_1^2} \ge \frac{1}{4}F^{k\bar k}\frac{|\sum_\beta\chi_{k\bar 1,\zeta_\beta}u_{\beta 1}|^2}{\lambda_1^2} - \frac{1}{2}A_2F^{k\bar k}\big(|u_{ik}|^2 + |u_{\bar i k}|^2\big) - C\Big(1 + \frac{|u_{\alpha k}|}{\lambda_1}\Big)F^{k\bar k}\Big|\frac{\widetilde{\lambda}_{1,k}}{\lambda_1}\Big| - C\sum F^{i\bar i} - \frac{2\Psi'}{\lambda_1}F^{k\bar k}\mathfrak{Re}\Big\{\sum_\beta\chi_{k\bar 1,\zeta_\beta}u_{\beta 1}(u-\underline{u})_k\Big\}.$$

By straightforward calculations and (4.4),

$$\mathcal{L}\phi = A_2\mathcal{L}(|\nabla u|^2) + \Psi'\mathcal{L}(u-\underline{u}) + \Psi'' F^{i\bar{i}}|(u-\underline{u})_i|^2$$

and

$$\begin{aligned}\mathcal{L}(|\nabla u|^2) &\geq F^{i\bar{i}}\big(|u_{ki}|^2 + |u_{k\bar{i}}|^2\big) - C_\epsilon\big(1+\sum F^{i\bar{i}}\big) - \epsilon f_i\lambda_i^2 \\ &\geq \frac{7}{8}F^{i\bar{i}}\big(|u_{ki}|^2 + |u_{k\bar{i}}|^2\big) - C\big(1+\sum F^{i\bar{i}}\big).\end{aligned} \tag{4.10}$$

Note that both $2\mathfrak{Re}\{F^{i\bar{i}}\chi_{1\bar{1},\zeta_\alpha\bar{i}}u_{\alpha i}\}$ in (4.7) and $F^{k\bar{k}}|u_{\alpha k}\widetilde{\lambda}_{1,k}|$ in (4.9) can be absorbed by $\lambda_1 F^{i\bar{i}}(|u_{ki}|^2+|u_{k\bar{i}}|^2)$. Therefore by (4.7)–(4.10)

$$\begin{aligned}0 \geq\ & \frac{1}{4}A_2F^{k\bar{k}}(|u_{ik}|^2+|u_{\bar{i}k}|^2) + \Psi'\mathcal{L}(u-\underline{u}) + \Psi''F^{i\bar{i}}|(u-\underline{u})_i|^2 \\ & - CF^{k\bar{k}}\Big|\frac{\widetilde{\lambda}_{1,k}}{\lambda_1}\Big| - C\frac{|u_{\alpha 1}|}{\lambda_1}\sum F^{i\bar{i}} - C\big(1+\sum F^{i\bar{i}}\big) \\ & - \frac{2\Psi'}{\lambda_1}\sum_{k>1}\sum_\beta F^{k\bar{k}}\mathfrak{Re}\big\{\chi_{k\bar{1},\zeta_\beta}u_{\beta 1}(u-\underline{u})_k\big\} + \frac{1}{4\lambda_1^2}\sum_{k>1}F^{k\bar{k}}\Big|\sum_\beta \chi_{k\bar{1},\zeta_\beta}u_{\beta 1}\Big|^2,\end{aligned}$$

where we use the fact that $0 < f_1 \leq \frac{\psi}{\lambda_1}$ and the constant A_2 is small to control $-F^{1\bar{1}}\frac{|\widetilde{\lambda}_{1,1}|^2}{\lambda_1^2}$.

Next let us verify that equation (1.1) satisfies condition (1.3). If $d_2 = 0$, $d_1 > 0$, $f(\lambda) = -\sum \frac{d_1}{\lambda_i}$, then $f_i\lambda_i^2 = f_1\lambda_1^2 = d_1$, $\sum f_i = \sum \frac{d_1}{\lambda_i^2}$. Note that $\sum \frac{d_1}{\lambda_i} = \psi$. By the Cauchy–Schwarz inequality, $\frac{1}{nd_1}\psi^2 \leq \sum f_i \leq \frac{1}{d_1}\psi^2$; thus (1.3) holds for $d_2 = 0$. Suppose that $d_2 > 0$. Then

$$f_1\lambda_1^2 = d_1 + \sum_{j>1}\frac{d_2}{\lambda_j}, \quad \frac{f_1\lambda_1^2}{\sum f_i} \geq \frac{f_1\lambda_1^2}{nf_n} \geq \frac{f_1\lambda_1^2\lambda_n}{n\psi} \geq \frac{d_2}{n\psi},$$

where we use $f_i\lambda_i \leq \psi$ for $i = 1,\dots,n$. We complete the proof of the claim.

Let us control the bad terms containing $\sum_\beta \chi_{k\bar{1},\zeta_\beta}u_{\beta 1}$ by the assumptions in (1.6). Suppose that $\sum_\beta |\Psi' u_{\beta 1}| \leq \lambda_1$. Then

$$-\frac{2\Psi'}{\lambda_1}\sum_\beta\sum_{k>1}F^{k\bar{k}}\mathfrak{Re}\big\{\chi_{k\bar{1},\zeta_\beta}u_{\beta 1}(u-\underline{u})_k\big\} \geq -C\sum F^{i\bar{i}}.$$

Otherwise $\sum_\beta |\Psi' u_{\beta 1}| \geq \lambda_1$. If assumption (1.6) holds, then

$$\begin{aligned}&\frac{A_2}{12}F^{k\bar{k}}(|u_{ik}|^2+|u_{\bar{i}k}|^2) - \frac{2\Psi'}{\lambda_1}\sum_\beta\sum_{k>1}F^{k\bar{k}}\mathfrak{Re}\big\{\chi_{k\bar{1},\zeta_\beta}u_{\beta 1}(u-\underline{u})_k\big\} \\ &\quad \geq \frac{A_2}{12}F^{\beta\bar{\beta}}|u_{1\beta}|^2 - C|\Psi'|\rho(\lambda)\sum_\beta F^{\beta\bar{\beta}}|u_{1\beta}| - C|\Psi'|\rho(\lambda)\sum F^{i\bar{i}} \geq -\sum F^{i\bar{i}},\end{aligned}$$

provided that $\lambda_1 \gg 1$.

Therefore the crucial assumptions in (1.6) imply that

$$0 \geq \frac{1}{6} A_2 F^{k\bar{k}} (|u_{ik}|^2 + |u_{\bar{i}k}|^2) + \Psi' \mathcal{L}(u - \underline{u}) + \Psi'' F^{i\bar{i}} |(u-\underline{u})_i|^2 \\ - C F^{k\bar{k}} |\frac{\widetilde{\lambda}_{1,k}}{\lambda_1}| - C \frac{|u_{\alpha 1}|}{\lambda_1} \sum F^{i\bar{i}} - C\big(1 + \sum F^{i\bar{i}}\big),$$

absorbing some bad terms using the first term and the third term of the right-hand side of previous inequality. At p_0, identity (4.1) yields that

$$F^{k\bar{k}} |\frac{\widetilde{\lambda}_{1,k}}{\lambda_1}| = F^{k\bar{k}} |A_2 (u_{ik} u_{\bar{i}} + u_{\bar{i}k} u_i) + \Psi'(u-\underline{u})_k| \\ \leq A_2 \sqrt{K} F^{k\bar{k}} (|u_{ik}| + |u_{\bar{i}k}|) + |\Psi'| F^{k\bar{k}} |(u-\underline{u})_k|.$$

Using the elementary inequality $ax^2 - bx \geq -\frac{b^2}{4a}$ for $a > 0$,

$$\Psi'' F^{i\bar{i}} |(u-\underline{u})_i|^2 - C|\Psi'| F^{i\bar{i}} |(u-\underline{u})_i| \geq -\frac{C^2 \Psi'^2}{4\Psi''} \sum F^{i\bar{i}}.$$

Choosing $0 < \epsilon \ll 1$ and λ_1 large, we have

$$0 \geq \frac{1}{8} A_2 F^{k\bar{k}} (|u_{ik}|^2 + |u_{i\bar{k}}|^2) - C_I \frac{|u_{\alpha 1}|}{\lambda_1} \sum F^{i\bar{i}} \\ + \Psi' \mathcal{L}(u-\underline{u}) - \frac{C^2 \Psi'^2}{4\Psi''} \sum F^{i\bar{i}} - C_2\big(1 + \sum F^{i\bar{i}}\big). \tag{4.11}$$

Let $\Psi\colon [\inf_{\overline{M}}(u-\underline{u}), +\infty) \to \mathbb{R}$,

$$\Psi(x) = \frac{A_1}{(1 + x - \inf_{\overline{M}}(u-\underline{u}))^N}, \tag{4.12}$$

where $A_1 \geq 1$, $N \in \mathbb{N}$ to be chosen later.

We first choose N in (4.12) large such that $0 < \frac{C^2|\Psi'|}{4\Psi''} \ll \frac{1}{2}\frac{\varepsilon\kappa_0}{1+\kappa_0}$. Then by (1.3), one gets

$$0 \geq \frac{\sum |u_{\alpha 1}|}{\lambda_1} \Big(\frac{A_2}{16n^2} \frac{\rho_0}{\lambda_1} \sum |u_{\alpha 1}| - C_I \Big) \sum F^{i\bar{i}} \\ + \Big(\frac{\varepsilon\kappa_0}{2(1+\kappa_0)} |\Psi'| - C_2 \Big) \big(1 + \sum F^{i\bar{i}}\big). \tag{4.13}$$

Note that in this computation $\widetilde{\lambda}_1$ denotes the largest eigenvalue of the perturbed endomorphism $\widetilde{A} = A + B$. At the origin p_0, where we carry out the computation, $\widetilde{\lambda}_1$ coincides with the largest eigenvalue of A. However, at nearby points, it is a small perturbation. We would take $B \to 0$, and obtain the above differential inequality (4.13) as well. This only holds in a viscosity sense because, if some eigenvalues coincide, the largest eigenvalue of A may not be C^2 at the origin p_0.

Case 1. Suppose $\sum |u_{\alpha 1}| > \frac{16n^2 C_I}{\rho_0 A_2} \lambda_1$, where C_I is the constant in (4.11). Then if we choose $A_1 \gg 1$ such that

$$\frac{\varepsilon\kappa_0}{4(1+\kappa_0)} |\Psi'| - 100 C_2 - \frac{160 n^2 C_I^2}{\rho_0 A_2} > 0, \tag{4.14}$$

the right-hand side of (4.13) is positive, which is a contradiction.

Case 2. $\sum|u_{\alpha 1}| \le \frac{16n^2 C_I}{\rho_0 A_2}\lambda_1$. Fix the constants A_1, A_2, N in Case 1. Then by (4.13) and (4.14) it follows that

$$0 \ge \frac{\varepsilon\kappa_0}{2(1+\kappa_0)}|\Psi'|\Big(1+\sum F^{i\bar{i}}\Big) - \frac{16n^2 C_I^2}{\rho_0 A_2}\sum F^{i\bar{i}} - C_2\Big(1+\sum F^{i\bar{i}}\Big) > 0,$$

which is a contradiction. So $\widetilde{H}$ cannot achieve its maximal value at interior point.

Thus the proof of Theorem 1.3 is complete. ∎

5 The Dirichlet Problem on an Annulus of the Kähler Cone

In this section we consider the Dirichlet problem (1.9) on an annulus in the Kähler cone.

We first construct an admissible subsolution of Dirichlet problem (1.9) under the assumption that condition (1.8) holds. Set $\underline{u} = \underline{v} + 2A(r-a)(r-b)$, $A \gg 1$, where $\underline{v}$ is the function satisfying condition (1.8).

By a simple computation, we have

$$\sqrt{-1}\Big(\partial\bar{\partial} - \partial\bar{\partial}r\frac{\partial}{\partial r}\Big)\big((r-a)(r-b)\big) = \frac{\sqrt{-1}}{2}\theta^n \wedge \overline{\theta}^n$$

and

$$\mathfrak{g}[\underline{u}] = \mathfrak{g}[\underline{v}] + A\phi(r)\sqrt{-1}\theta^n \wedge \overline{\theta}^n,$$

where $\theta^n = dr + \sqrt{-1}r\eta$ and $\overline{\theta}^n = dr - \sqrt{-1}r\eta$. Hence for $A \gg 1$, one derives

$$\begin{cases} \psi(\mathfrak{g}[\underline{u}])^n \ge b_1(\mathfrak{g}[\underline{u}])^{n-1}\wedge\overline{\omega} + b_2(\mathfrak{g}[\underline{u}])^{n-2}\wedge\overline{\omega}^2 \text{ in } M, \\ \underline{u}|_{r=a} = \varphi_a, \underline{u}|_{r=b} = \varphi_b, \end{cases}$$

provided that condition (1.8) holds.

Next we derive *a priori* C^0-estimates and gradient estimates on the boundary. Let w be a C^2 solution to

$$\begin{cases} \frac{1}{2}\phi(r)(\Delta_{\overline{g}}w - \frac{\partial w}{\partial r}\Delta_{\overline{g}}r) + \overline{g}^{i\bar{j}}\widetilde{\chi}_{i\bar{j}} \le 0 & \text{in } \overline{M}, \\ w = \varphi & \text{on } \partial M, \end{cases} \tag{5.1}$$

where $\overline{g}(X,Y) = \overline{\omega}(X,JY)$ for $X, Y \in T_{\mathbb{R}}\overline{M}$, $\Delta_{\overline{g}}$ is the standard Laplacian with respect to the Levi–Civita connection. The solvability of (5.1) can be found in [42].

Let $u \in C^2(\overline{M})$ be any solution of Dirichet problem (1.9) with $\mathfrak{g}[u] > 0$. Then

$$\frac{1}{2}\phi(r)\Big(\Delta_{\overline{g}}u - \frac{\partial u}{\partial r}\Delta_{\overline{g}}r\Big) + \overline{g}^{i\bar{j}}\widetilde{\chi}_{i\bar{j}} > 0.$$

Therefore the maximum principle yields that

$$\underline{u} \le u \le w \text{ in } \overline{M}, \quad \underline{u} = u = w = \varphi \text{ on } \partial M.$$

Hence there is a positive constant C^* depending only on $\underline{u}$ and w, such that

$$\sup_{\overline{M}}|u| + \sup_{\partial M}|\nabla u| \le C^*. \tag{5.2}$$

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Once one derives boundary estimates for second derivatives $\sup_{\partial M}|\nabla^2 u| \le C$, equation (1.9) becomes uniform elliptic. Then the existence and uniqueness of the admissible solution of equation (1.9) can be proved by the standard continuity method and the maximum principle.

Theorem 5.1 *Suppose that (1.8) holds. Then for any admissible solution*

$$u \in C^3(M) \cap C^2(\overline{M})$$

to Dirichlet problem (1.9), we have $\sup_{\partial M}|\nabla^2 u| \le C$*, where the constant* $C > 0$ *depends on* $|u|_{C^1(\overline{M})}$ *and other known data.*

5.1 Background on Sasakian Geometry

Sasakian manifolds, which can be viewed as the odd dimensional counterparts of Kähler manifolds, are odd-dimensional Riemannian manifolds whose metric cone $(C(M), \widetilde{g}) = (M \times \mathbb{R}^+, r^2 g + dr^2)$ admits a Kähler structure. These manifolds can be used to construct new Einstein manifolds in geometry. Moreover, they play an important role in the AdS/CFT correspondence in mathematical physics. We refer the reader to [5,16,17,30,31] and the references therein for relevant work on Sasakian geometry from mathematicians and physicists.

As in the Kähler setting, the Sasakian metric can be locally generated by a free real function of $2(n-1)$ variables. More precisely, for any $p \in S$, there is a local *basic* function h and a local coordinate chart $(z^1, \dots, z^{n-1}, x) \in \mathbb{C}^{n-1} \times \mathbb{R}$ on a small neighborhood U around p such that

$$\begin{cases} \xi = \dfrac{\partial}{\partial x}, \quad g = \eta \otimes \eta + 2h_{i\bar{j}} dz^i d\bar{z}^j, \quad \eta = dx - \sqrt{-1}(h_j dz^j - h_{\bar{j}} d\bar{z}^j), \\ \Phi = \displaystyle\sum_{i=1}^{n-1} \Big\{ \sqrt{-1}\Big(\frac{\partial}{\partial z^i} + \sqrt{-1} h_i \frac{\partial}{\partial x}\Big) \otimes dz^i - \sqrt{-1}\Big(\frac{\partial}{\partial \bar{z}^i} - \sqrt{-1} h_{\bar{i}} \frac{\partial}{\partial x}\Big) \otimes d\bar{z}^i \Big\}, \end{cases}$$

and $\mathcal{D}^{\mathbb{C}}$ is spanned by

$$X_i = \frac{\partial}{\partial z^i} + \sqrt{-1} h_i \frac{\partial}{\partial x}, \quad \overline{X}_i = \frac{\partial}{\partial \bar{z}^i} - \sqrt{-1} h_{\bar{i}} \frac{\partial}{\partial x}, \quad 1 \le i \le n-1,$$

where $2dz^i d\bar{z}^j = dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i$, $h_i = \frac{\partial h}{\partial z^i}$, $h_{i\bar{j}} = \frac{\partial^2 h}{\partial z^i \partial \bar{z}^j}$. Moreover, one can change the local coordinates to normal coordinates such that $h_i(p) = 0$, $h_{i\bar{j}}(p) = \delta_{ij}$, and $d(h_{i\bar{j}})|_p = 0$. The proof can be found in [19].

For the normal local coordinate chart $(z^1, \dots, z^{n-1}, x)$ on a Sasakian manifold (S, η, ξ, Φ, g), set $(z^1, \cdots, z^{n-1}, \widetilde{z})$ on $U \times \mathbb{R}^+ \subset C(S)$, where $\widetilde{z} = r + \sqrt{-1}x$, and

$$X_n = \frac{1}{2}\Big(\frac{\partial}{\partial r} - \sqrt{-1}\frac{1}{r}\frac{\partial}{\partial x}\Big), \quad \overline{X}_n = \frac{1}{2}\Big(\frac{\partial}{\partial r} + \sqrt{-1}\frac{1}{r}\frac{\partial}{\partial x}\Big),$$

where h is the Sasakian potential function (which is *basic*). Then $JX_i = \sqrt{-1}X_i$, $J\overline{X}_i = -\sqrt{-1}\overline{X}_i$ for $i = 1, \dots, n$. The background information on Sasakian manifolds can be found in [4].

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5.2 Proof of Theorem 5.1

Given any point $p = (q, a) \in S \times \{a\}$ (or $p = (q, b) \in S \times \{b\}$), set

$$\rho(z) := \mathrm{dist}_{\overline{M}}(z, p) \quad \text{and} \quad \Omega_\delta = \{z \in M : \rho(z) < \delta\}, \quad 0 < \delta \ll 1,$$

and

$$\sigma(\hat{p}) = (r - a) \text{ or } \sigma(\hat{p}) = (b - r), \quad \text{where } \hat{p} = (\hat{q}, r) \in \overline{M}, \hat{q} \in S.$$

To construct a barrier function, we we will follow Guan [20] and employ a barrier function of the form

$$v = u - \underline{u} + \sigma - \sigma^2 \text{ in } \Omega_{\delta_0} \text{ for } 0 < \delta_0 \ll 1, \tag{5.3}$$

where $\underline{u} \in C^2(\overline{M})$ is the admissible subsolution of Dirichlet problem (1.9). The following lemma was first proved in [20] for domains in $\mathbb{C}^n$.

Lemma 5.2 *There are positive constants N, t, and c_1 such that for small δ_0 we have*

$$v \geq 0 \text{ and } \mathcal{L}v \leq -c_1(1 + F^{i\bar{j}}\overline{g}_{i\bar{j}}) \text{ in } V_{\delta_0} := \{z \in M : \sigma < \delta\}. \tag{5.4}$$

Proof By the maximum principle, $u \geq \underline{u}$. Thus $v \geq 0$ in Ω_{δ_0}, $0 < \delta_0 \ll 1$. By Lemma 2.2, we divide the proof into two cases as follows.

Case 1: $|\lambda| \geq R_0$. If $\mathcal{L}(u - \underline{u}) \leq -\varepsilon F^{i\bar{j}}\overline{g}_{i\bar{j}}$ for the positive constant ε in Lemma 2.2, then

$$\mathcal{L}(u - \underline{u}) \leq -\frac{\varepsilon\kappa_0}{2(1 + \kappa_0)}\big(1 + F^{i\bar{j}}\overline{g}_{i\bar{j}}\big).$$

Then we have (5.4). On the other hand, it follows from Lemma 2.2 and Lemma 2.1 that

$$F^{i\bar{j}} \geq \varepsilon F^{p\bar{q}}\overline{g}_{p\bar{q}}\overline{g}^{i\bar{j}} \geq \varepsilon\kappa_0\overline{g}^{i\bar{j}}.$$

Note that $|\nabla r| = \frac{1}{2}$ and $\mathcal{L}(u - \underline{u}) \leq 0$. Then

$$\mathcal{L}v = \mathcal{L}(u - \underline{u}) - 2F^{i\bar{j}}r_i r_{\bar{j}} \leq -c_1\Big(1 + \sum F^{i\bar{j}}\overline{g}_{i\bar{j}}\Big).$$

Case 2: $|\lambda| < R_0$, $F^{i\bar{j}} \geq \theta\overline{g}^{i\bar{j}}$ and $\sum F^{i\bar{j}}\overline{g}_{i\bar{j}} \leq C$. The proof is similar, and we omit it here. ∎

Tangential-Normal derivatives The tangential-normal case will be proved by constructing barrier functions. This type of construction of barrier functions follows from[20, 24, 27].

Given $p = (q, a) \in S \times \{a\}$ (or $p = (q, b) \in S \times \{b\}$), we can pick the local coordinates $z' = (z'_1, \ldots, z'_n)$, $z'_j = x'_j + \sqrt{-1}y'_j$, where $\frac{\partial}{\partial x'_n}$ is the interior normal direction to ∂M at p. Here we identify p with $z' = 0$ and assume that $\overline{g}_{i\bar{j}}(0) = \delta_{ij}$ and $\{\mathfrak{g}_{\alpha\bar{\beta}}(0)\}$ is diagonal for $1 \leq \alpha, \beta \leq n - 1$. We also use the notations $t_{2k} = x'_k$, $t_{2k-1} = y'_k$ and $v_i = \frac{\partial}{\partial z'_i}v = \partial_i v$, $v_{t_i} = \frac{\partial v}{\partial t_i}$, $v_{i\bar{j}} = \partial_i\overline{\partial}_j v, \ldots$, etc. Let $D = \pm(\frac{\partial}{\partial t_\alpha} - r_{t_\alpha}\frac{\partial}{\partial r})$, $\alpha \leq 2n - 1$.

The proof of the following lemma can be found in [20, 22]. We omit the proof here.

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Lemma 5.3 *There is a positive constant C_0 depending on*

$$\sup_{\overline{M}}|\nabla u|,\quad |\varphi|_{C^{2,1}(\overline{M})},\quad |\overline{\chi}|_{C^{0,1}(\overline{M})},$$

and other known data such that

$$\mathcal{L}(D(u-\varphi)) \le C_0(\sup_{\overline{M}}|\nabla\psi| + F^{i\bar{j}}\overline{g}_{i\bar{j}}) + F^{i\bar{j}}u_{y'_n i}u_{y'_n\bar{j}}.$$

The barrier function will be given as $\Psi = A_1 v + A_2\rho^2 + D(u-\underline{u}) - (u_{y'_n} - \varphi_{y'_n})^2$, where $A_1 \gg A_2 \gg 1$ and v is the function in (5.3), *i.e.*, $v = u - \underline{u} - \sigma^2 + \sigma$.

On one hand, by Lemma 5.2 and Lemma 5.3, we know that $\mathcal{L}\Psi \le 0$ in $\Omega_{\delta_0}(\delta \ll 1)$ if $A_1 \gg A_2 \gg 1$. On the other hand,

$$\begin{cases} D(u-\underline{u}) = 0,\ |u_{y'_n} - \varphi_{y'_n}| \le a_0\rho & \text{on } \partial M \cap \overline{\Omega}_\delta,\\ \rho = \delta,\ |D(u-\underline{u})| \le a_0 & \text{on } \partial\Omega_\delta \smallsetminus \partial M.\end{cases}$$

Therefore, $\Psi \ge 0$ on $\partial\Omega_\delta$. By the maximum principle $\Psi \ge 0$ on Ω_δ. Note that $\Psi(p) = 0$. Then

$$|\nabla_{\frac{\partial}{\partial r}} Du(p)| \le C, \tag{5.5}$$

where C is a positive constant depending on $|u|_{C^1(\overline{M})}$, $|\underline{u}|_{C^2(\overline{M})}$, $|\varphi|_{C^{2,1}(\overline{M})}$, $|\psi|_{C^{1,1}(\overline{M})}$, and other known data.

Pure normal/pure tangential derivatives: Next, we need to consider boundary estimates for pure normal derivatives and pure tangential derivatives. Since $u - \underline{u} = 0$ on ∂M, we can therefore write $u - \underline{u} = \hat{h}\sigma$, in $V_\delta := \{z \in M : \sigma(z) < \delta\}$ for some $0 < \delta \ll 1$. Given $p = (q, r_p) \in \partial M$, $r_p = a$ or b at p. It follows that

$$\begin{cases} \dfrac{\partial}{\partial r}(u-\underline{u}) = \dfrac{\partial\hat{h}}{\partial r}\sigma + \dfrac{\partial\sigma}{\partial r}\hat{h},\\ (u-\underline{u})_{\alpha\bar{\beta}} = \hat{h}_{\alpha\bar{\beta}}\sigma + \hat{h}_\alpha\sigma_{\bar{\beta}} + \hat{h}_{\bar{\beta}}\sigma_\alpha + \hat{h}\sigma_{\alpha\bar{\beta}},\\ (u-\underline{u})_{t_\alpha t_\beta} = \hat{h}_{t_\alpha t_\beta}\sigma + \hat{h}_{t_\alpha}\sigma_{t_\beta} + \hat{h}_{t_\beta}\sigma_{t_\alpha} + \hat{h}\sigma_{t_\alpha t_\beta}.\end{cases}$$

For $\alpha, \beta \le n-1$, we have at p

$$(u-\underline{u})_{\alpha\bar{\beta}} = \frac{\partial(u-\underline{u})}{\partial r}r_{\alpha\bar{\beta}} \quad\text{and}\quad (u-\underline{u})_{t_\alpha t_\beta} = \frac{\partial(u-\underline{u})}{\partial r}r_{t_\alpha t_\beta}. \tag{5.6}$$

In particular, by (5.6), we have

$$\mathfrak{g}_{\alpha\bar{\beta}}[u] = \mathfrak{g}_{\alpha\bar{\beta}}[\underline{u}] \text{ at } p,\ \text{ for } 1 \le \alpha, \beta \le n-1,\quad \sup_{\partial M}|u_{t_\alpha t_\beta}| \le C_a. \tag{5.7}$$

Here we use (5.2), and we obtain the uniform bound of second derivatives for pure tangential. Finally, we shall show that $\mathfrak{g}_{n\bar{n}}[u](p) \le C$.

We rewrite equation (1.9) as follows

$$\begin{aligned}[n\psi(\mathfrak{g}[u])^{n-1} - (n-1)b_1(\mathfrak{g}[u])^{n-2} \wedge \overline{\omega} - (n-2)b_2(\mathfrak{g}[u])^{n-3} \wedge \overline{\omega}^2]&\\ \wedge \sqrt{-1}\mathfrak{g}_{n\bar{n}}[u]dz'^n \wedge d\bar{z}'^n = (*),&\end{aligned}$$

where $(*)$ does not have any terms containing $\sqrt{-1}\mathfrak{g}_{n\bar{n}}dz'^n \wedge d\bar{z}'^n$. Then it follows from (5.5), (5.7), and (1.8) that $\mathfrak{g}_{n\bar{n}}[u](p)$ has a uniform bound $\mathfrak{g}_{n\bar{n}}[u](p) \le C$.

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