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LINEAR TRANSFORMATIONS ON MATRICES: THE INVARIANCE OF A CLASS OF GENERAL MATRIX FUNCTIONS

HOCK ONG

1. Introduction. Let F be a field, F^* be its multiplicative group and $M_n(F)$ be the vector space of all n-square matrices over F. Let S_n be the symmetric group acting on the set $\{1, 2, \ldots, n\}$. If G is a subgroup of S_n and λ is a function on G with values in F, then the matrix function associated with G and λ, denoted by G^λ, is defined by

$$G^\lambda(X) = \sum_{\sigma \in G} \lambda(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}, \quad X = (x_{ij}) \in M_n(F)$$

and let

$$\mathscr{T}(G, \lambda) = \{T : T \text{ is a linear transformation of } M_n(F) \text{ to itself and } G^\lambda(T(X)) = G^\lambda(X) \text{ for all } X\}.$$

It is of interest to characterize all linear maps in $\mathscr{T}(G, \lambda)$. For example, if $G = S_n$ and λ is a linear character on S_n, i.e., S_n^λ is either the determinant or permanent, then $\mathscr{T}(S_n, \lambda)$ has been obtained [**3**; **4**]. If G is transitive and cyclic and λ is a function on G[**1**] or G is regular or doubly transitive and λ is a linear character on G[**2**] then $\mathscr{T}(G, \lambda)$ has also been characterized. In [**2**], it was mentioned that if G is singly transitive but not regular or doubly transitive, then the techniques in [**2**] fail and the dihedral group of degree four was given as a counter example. In this paper we show that D_4 is, in fact, an exception, i.e., we apply the techniques in [**2**] and the results in [**5**] to characterize all linear maps in $\mathscr{T}(D_n, \lambda)$ where D_n is the dihedral group of degree n, $n \geqq 5$ and λ is a function on D_n with values in F^*.

2. Definitions and statements of the main results. Recall that the dihedral group of degree n is the subgroup of S_n generated by the two permutations g and h where $g(i) = i + 1, i = 1, 2, \ldots, n - 1; g(n) = 1$ and $h(1) = 1$, $h(i) = n - i + 2$, $i = 2, 3, \ldots, n$. If we write $g_i = g^{i-1}$, $i = 1, 2, \ldots, n$, then $D_n = \{g_i, g_ih: i = 1, 2, \ldots, n\}$ and the diagonals g_i, g_ih are illustrated by the following diagram when $n = 6$, the solid lines denote the diagonals g_i,

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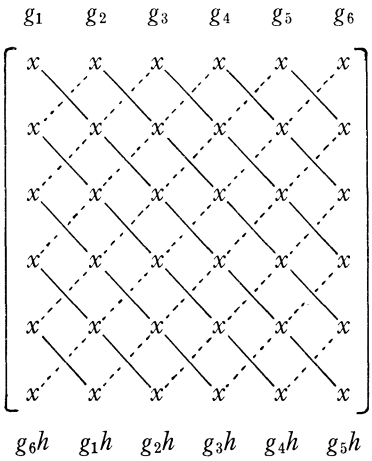
the dotted lines denote the diagonals $g_i h$.



If n is a positive integer let $K_n = \{\sigma \in S_n: \sigma$ maps even integers onto even integers$\}$. Clearly K_n is a subgroup of S_n and the permutations in $K_n g$ map even integers onto odd integers.

A subspace Z of $M_n(F)$ is a 0-subspace for D_n^λ if $\dim Z = n^2 - n$ and $X \in Z$ implies $D_n^\lambda(X) = 0$. Then we have

PROPOSITION 1. *Let n be a positive integer, $n \geqq 5$ and*

$$D_n = \{g_i, g_i h : i = 1, 2, \ldots, n\}$$

be the dihedral group of degree n. A subspace Z is a 0-subspace for D_n^λ if and only if there exist n distinct pairs of integers $(i_1, j_1), \ldots (i_n, j_n)$, $1 \leqq i_t, j_t \leqq n$ and a permutation $\alpha \in S_n$ if n is odd and $\alpha \in K_n$ if n is even such that

$$g_t(i_t) = g_{\alpha(t)}h(i_t) = j_t$$

and if $X \in Z$, $x_{i_t j_t} = 0$, $t = 1, 2, \ldots, n$.

The group $D_n' = \{g_i: i = 1, 2, \ldots, n\}$ and the set $D_n'h$ are regular and $D_n = D_n' \cup D_n'h$. Hence for each pair of integers (i, j), $1 \leqq i, j \leqq n$ there exist exactly one k and one l, $1 \leqq k, l \leqq n$ such that $g_k(i) = j$, $g_l h(i) = j$ or $g_k(i) = g_l h(i)$. We define

$$\varphi_k(i) = l.$$

If we work modulo n using $\{1, 2, \ldots, n\}$ as a system of distinct representatives, then it is known [**5**] that

$$\varphi_k(i) \equiv k + 2(i - 1) \pmod{n}, \; i, k = 1, 2, \ldots, n$$

and for n odd, φ_k are in S_n and for n even $\varphi_k(i) = \varphi_k(i + n/2)$, $k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, n/2$. For n even, since $\varphi_j(i)$, $i = 1, 2, \ldots, n$ are even if and only if j is even, we define $\varphi_{\sigma(j)}^{-1}$ such that

$$1 \leqq \varphi_{\sigma(j)}^{-1}\mu\varphi_j(i) \leqq n/2 \quad \text{if and only if} \quad 1 \leqq i \leqq n/2$$

where σ, μ are both in K_n or $K_n g$ and hence $\varphi_{\sigma(j)}^{-1}\mu\varphi_j$ are in S_n.

If $\sigma \in S_n$ then the permutation matrix corresponding to σ, $P(\sigma)$, is the n-square matrix whose (i, j) entry is 1 if $\sigma(j) = i$ and 0 elsewhere. If $A = (a_{ij})$ and $B = b(_{ij})$ are n-square matrices then the Hadamard product of A and B, $A * B$, is the n-square matrix whose (i, j) entry is $a_{ij}b_{ij}$ for all i and j. If $A = (a_{ij}) \in M_n(F)$ and $\sigma \in S_n$ then the σ-diagonal of A, A_σ, is the n-square matrix whose (i, j) entry is a_{ij} if $\sigma(i) = j$ and 0 elsewhere. If $A = (a_{ij}) \in M_n(F)$, let X^r be the n-square matrix whose (i, j) entry is $a_{i,n-j+1}$ for all i, $j = 1, 2, \ldots, n$. Let R be the linear transformation of $M_n(F)$ to itself such that $R(X) = X^r$ for all X. If T: $M_n(F) \rightarrow M_n(F)$ is a linear transformation which transforms the entries in σ-diagonal of $X \in M_n(F)$ onto the μ-diagonal where $\sigma, \mu \in S_n$ then we write $T(\sigma) = \mu$. It can be easily shown that $R(g_i) = g_{n-i+1}h$, $R(g_ih) = g_{n-i+1}$, $i = 1, 2, \ldots, n$. Now if n is odd, $(\sigma, \mu) \in S_n \times S_n$, the direct product of S_n by S_n, and $X \in M_n(F)$ we define

$$(2.1) \quad (\sigma, \mu)(X) = \sum_{i=1}^{n} P(\varphi_{\sigma(i)}{}^{-1}\mu\varphi_i)X_{g_i}P(g_i(\varphi_{\sigma(i)}{}^{-1}\mu\varphi_i)^{-1}g_{\mu(i)}{}^{-1}),$$

i.e., for $i = 1, 2, \ldots, n$, (σ, μ) permutes the entries within the g_i-diagonal of X by $\varphi_{\sigma(i)}^{-1}\mu\varphi_i$ and then transforms the entries in g_i-diagonal to $g_{\mu(i)}$-diagonal or equivalently, (σ, μ) permutes the diagonals $g_1, g_2, \ldots, g_n$ by σ and permutes the diagonals $g_1j, g_2h, \ldots, g_nh$ by μ. Then we have

THEOREM 1. *Let n be an odd positive integer, $n \geqq 5$,*

$$D_n = \{g_i, g_ih: i = 1, 2, \ldots, n\}$$

be the dihedral group of degree n and λ be a function on D_n with values in F^*. Then $T \in \mathscr{T}(D_n, \lambda)$ if and only if there exist a matrix $A = (a_{ij})$ in $M_n(F)$ and a linear transformation T' in the group $S_n \times S_n \circ \{I, R\}$ such that

$$T(X) = A * T'(X) \quad \textit{for all } X$$

with

$$\prod_{i=1}^{n} a_{i(T'(g_k))(i)} = \lambda(g_k)(\lambda(T'(g_k)))^{-1},$$

$$\prod_{i=1}^{n} a_{i(T'(g_kh))(i)} = \lambda(g_kh)(\lambda(T'(g_kh)))^{-1}, \quad k = 1, 2, \ldots, n,$$

where $\circ$ is the usual function composition and I is the identity transformation of $M_n(F)$.

Next suppose $n = 2m$ is a positive even integer. Let H_n be the subgroup of S_n generated by the transpositions $(i\ m + i)$, $i = 1, 2, \ldots, m$. For $\lambda_1, \lambda_2, \ldots,$

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$\lambda_n \in H_n$, let $\Lambda_{(\lambda_1,\ldots,\lambda_n)}$ be the linear transformation of $M_n(F)$ into itself defined by

$$\Lambda_{(\lambda_1,\ldots,\lambda_n)}(X) = \sum_{i=1}^{n} P(\lambda_i)X_{g_i}P(g_i\lambda_i g_i^{-1}) \quad \text{for all } X$$

and let

$$\Lambda = \{\Lambda_{(\lambda_1,\ldots,\lambda_n)} : (\lambda_1, \ldots, \lambda_n) \in H_n \times \ldots \times H_n\}.$$

Note that Λ is a group and $\Lambda_{(\lambda_1,\ldots,\lambda_n)}$ is the linear transformation which permutes the entries in g_i-diagonal by λ_i, $i = 1, 2, \ldots, n$, i.e., $\Lambda_{(\lambda_1,\ldots,\lambda_n)}$ either interchanges the entries at positions $(k, g_i(k))$, $(k + m, g_i(k + m))$ or fixes them, $k = 1, 2, \ldots, m$; $i = 1, 2, \ldots, n$. Clearly $\Lambda_{(\lambda_1,\ldots,\lambda_n)}(\sigma) = \sigma$ for all $\sigma \in D_n$. If $A = (a_{ij})$ is an n-square matrix we denote by A_0 the n-square matrix whose (i, j) entry is a_{ij} if $i + j$ is even and 0 elsewhere and $A_e = A - A_0$. Let U, V be the linear transformations of $M_n(F)$ into itself defined by

$$U(X) = XP(g^{-1}) \quad \text{for all } X,$$
$$V(X) = X_0 + R(U(X_e)) \quad \text{for all } X.$$

Then it can be shown that

$$U(g_i) = g_{g(i)}, \quad U(g_ih) = g_{g(i)}h, \quad i = 1, 2, \ldots, n,$$
$$V(g_i) = g_i, \quad V(g_ih) = g_ih \quad \text{if } i \text{ is odd},$$
$$V(g_i) = g_{n-i}h, \quad V(g_ih) = g_{n-i} \quad \text{if } i \text{ is even [5]}.$$

If for $(\sigma, \mu) \in K_n \times K_n$ and $X \in M_n(F)$, we define $(\sigma, \mu)(X)$ by (2.1), then we can state our

THEOREM 2. *Let n be an even positive integer, $n \geqq 6$,*

$$D_n = \{g_i, g_ih : i = 1, 2, \ldots, n\}$$

be the dihedral group of degree n and λ be a function on D_n with values in F^*. Then $T \in \mathscr{T}(D_n, \lambda)$ if and only if there exist a matrix $A = (a_{ij})$ in $M_n(F)$ and a linear transformation T' in the group $\Lambda \circ K_n \times K_n \circ \{I, U\} \circ \{I, R\} \circ \{I, V\}$ such that

$$T(X) = A * T'(X) \quad \textit{for all } X$$

with

$$\prod_{t=1}^{n} a_{t(T'(g_k))(t)} = \lambda(g_k)(\lambda(T'(g_k)))^{-1},$$
$$\prod_{t=1}^{n} a_{t(T'(g_kh))(t)} = \lambda(g_kh)(\lambda(T'(g_kh)))^{-1}, \quad k = 1, 2, \ldots, n.$$

3. Proofs. For $\sigma \in S_n$, let $D(\sigma) = \{(i, \sigma(i)) : i = 1, 2, \ldots, n\}$. If S is a finite set let $|S|$ denote the number of elements in S. Then since D_n' and $D_n'g$ are regular, $|D(g_i) \cap D(g_j)| = |D(g_ih) \cap D(g_jh)| = 0$ if $i \neq j$. Furthermore we have the following properties of D_n [5].

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LEMMA 1. *For each pair g_j, $g_k h$ in D_n, $1 \leqq j, k \leqq n$, if n is odd then*

$$|D(g_j) \cap D(g_k h)| = 1$$

and if n is even then

$$|D(g_j) \cap D(g_k h)| = 0 \quad \text{if } 2 \nmid (j - k),$$
$$|D(g_j) \cap D(g_k h)| = 2 \quad \text{if } 2 | (j - k).$$

Suppose Z is a subspace of $M_n(F)$ and $\dim Z = n^2 - n$. By using the reduction of a basis for Z to Hermite normal form we can assume that there exist n distinct pairs of integers $\{(i_1, j_1), \ldots, (i_n, j_n)\} = M$ such that the matrices

$$A_{ij} = E_{ij} + \sum_{t=1}^{n} c_t^{ij} E_{i_t j_t}, \quad c_t^{ij} \in F, \ (i, j) \notin M$$

form a basis for Z. Here E_{ij} is the matrix whose (i, j) entry is 1 and 0 elsewhere.

If G is a subgroup of S_n let $G(i, j) = \{\sigma \in G: \sigma(i) = j\}$. If G is transitive then $|G| = np$ and $|G(i, j)| = p$ for all $1 \leqq i, j \leqq n$ where p is an integer and $p \geqq 1$.

LEMMA 2. *If G is a transitive subgroup of S_n and for some $\sigma \in G$, $|D(\sigma) \cap M| = k > 1$, then there exist at least $k - 1$ elements $\mu_1, \ldots, \mu_{k-1}$ in G such that $|D(\mu_i) \cap M| = 0$, $i = 1, 2, \ldots, k - 1$.*

Proof. If $|D(\sigma) \cap M| = k > 1$ say $D(\sigma) \cap M = \{(i_1, j_1), \ldots, (i_k, j_k)\}$, then for $t = 2, 3, \ldots, k$, $|G(i_1, j_1) \cap G(i_t, j_t)| \geqq 1$. Hence

$$\left| \bigcup_{r=1}^{n} G(i_r, j_r) \right| \leqq \sum_{r=1}^{n} |G(i_r, j_r)| - \sum_{t=2}^{k} |G(i_1, j_1) \cap G(i_t, j_t)|$$
$$= np - (k - 1) = |G| - (k - 1).$$

Therefore there exist $\mu_1, \ldots, \mu_{k-1} \in G$ such that $\mu_i \notin \bigcup_{t=1}^{n} G(i_t, j_t)$, i.e., $|D(\mu_i) \cap M| = 0$, $i = 1, 2, \ldots, k - 1$.

LEMMA 3. *If $n \geqq 5$ and $|D(\sigma) \cap M| = 0$ for some $\sigma \in D_n$ then there exists a matrix B in Z such that $D_n^{\lambda}(B) \neq 0$.*

Proof. Consider the matrix

$$(b_{rs}) = B = \sum_{i=1}^{n} A_{i\sigma(i)} = P(\sigma) + \sum_{t=1}^{n} c_t E_{i_t j_t}$$

where $c_t = \sum_{t=1}^{n} c_t^{i\sigma(i)}$, $t = 1, 2, \ldots, n$. Clearly $B \in Z$, $b_{rs} = 0$ if $(r, s) \notin D(\sigma) \cup M$ and

$$(3.1) \quad D_n^{\lambda}(B) = \lambda(\sigma) + \sum_{\tau \neq \sigma} \lambda(\tau) \prod_{i=1}^{n} b_{i\tau(i)}.$$

If for all $\tau \neq \sigma$, $\prod_{i=1}^{n} b_{i\tau(i)} = 0$ then $D_n^{\lambda}(B) = \lambda(\sigma) \neq 0$. Hence assume that for some $\tau \neq \sigma$, $\prod_{i=1}^{n} b_{i\tau(i)} \neq 0$. Then $D(\tau) \subset D(\sigma) \cup M$. By Lemma 1,

$|D(\sigma) \cap D(\tau)| = 0$ or 1 if n is odd and 0 or 2 if n is even. Hence $|D(\tau) \cap M| = n$ or $n - 1$ if n is odd and n or $n - 2$ if n is even. Since D_n is transitive and $|D(\tau) \cap M| \geqq n - 2 \geqq 3$, by Lemma 2 there exists $\mu \in D_n$, $\mu \neq \sigma$ such that $|D(\mu) \cap M| = 0$. Let

$$(b_{rs}') = B' = \sum_{i=1}^{n} A_{i\mu(i)} = P(\mu) + \sum_{t=1}^{n} c_t' E_{i_t j_t}$$

where $c_t' = \sum_{i=1}^{n} c_t^{i\mu(i)}$. Then $B' \in Z$, $b_{rs}' = 0$ if $(r, s) \notin D(\mu) \cup M$ and

$$(3.2) \quad D_n^{\lambda}(B') = \lambda(\mu) + \sum_{\nu \neq \mu} \lambda(\nu) \prod_{i=1}^{n} b_{i\nu(i)}'.$$

We consider the cases $|D(\tau) \cap M| = n, n - 1, n - 2$ separately.

(i) $M = D(\tau)$. Consider (3.1). Since for $\nu \neq \sigma, \tau$, $|D(\nu) \cap D(\sigma)| \leqq 2$, $|D(\nu) \cap D(\tau)| \leqq 2$, it follows that $|D(\nu) \cap (D(\sigma) \cup D(\tau))| \leqq 4$ and $\prod_{i=1}^{n} b_{i\nu(i)} = 0$ since $n \geqq 5$. Hence

$$D_n^{\lambda}(B) = \lambda(\sigma) + \lambda(\tau) \prod_{t=1}^{n} c_t$$

with $\prod_{t=1}^{n} c_t \neq 0$. Similarly in (3.2) we have $\prod_{i=1}^{n} b_{i\nu(i)}' = 0$ for $\nu \neq \mu, \tau$. If $\prod_{i=1}^{n} b_{i\tau(i)}' = 0$ then $D_n^{\lambda}(B') = \lambda(\mu) \neq 0$. Hence suppose

$$D_n^{\lambda}(B') = \lambda(\mu) + \lambda(\tau) \prod_{t=1}^{n} c_t'$$

with $\prod_{i=1}^{n} c_t' \neq 0$. Since $c_1' \neq 0$ there exists i' such that $c_1^{i'\mu(i')} \neq 0$. Consider the matrix

$$A(x) = \sum_{i=1}^{n} A_{i\sigma(i)} + xA_{i'\mu(i')}$$

where x is an indeterminate over F. Then the (i_1, j_1) entry of $A(x)$ is a nonzero polynomial of degree one; hence we may choose $c \in F$ so that this entry is zero. Let

$$(a_{rs}) = A(c) = \sum_{i=1}^{n} A_{i\sigma(i)} + cA_{i'\mu(i')}.$$

Then $a_{rs} = 0$ if $(r, s) \notin D(\sigma) \cup \{(i', \mu(i'))\} \cup (M - \{(i_1, j_1)\}) = \Omega_A$. Clearly for $\nu \in D_n$, $\nu \neq \sigma, \tau$, $|D(\nu) \cap \Omega_A| \leqq 3$ if n is odd and $|D(\nu) \cap \Omega_A| \leqq 5$ if n is even. Since $n \geqq 5$, $\prod_{i=1}^{n} b_{i\nu(i)} = 0$ for all $\nu \neq \sigma$. Hence $D_n^{\lambda}(A(c)) = \lambda(\sigma) \neq 0$.

(ii) $|D(\tau) \cap M| = n - 1$. Then n is odd and $|D(\sigma) \cap D(\tau)| = 1$. Consider (3.1). For $\nu \neq \sigma, \tau$, since $|D(\nu) \cap D(\sigma)| \leqq 1$ and $|D(\nu) \cap M| \leqq 2$ it follows that $|D(\nu) \cap (D(\sigma) \cup M)| \leqq 3$ and $\prod_{i=1}^{n} b_{i\tau(i)} = 0$ since $n \geqq 5$. Hence

$$D_n^{\lambda}(B) = \lambda(\sigma) + \lambda(\tau) \prod_{i=1}^{n} b_{i\tau(i)}.$$

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Applying the same argument to (3.2) we have $\prod_{i=1}^{n} b_{i\nu(i)} = 0$ if $\nu \neq \mu, \tau$. Furthermore $|D(\mu) \cap D(\tau)| = 0$ for otherwise $|D(\mu) \cap M| \neq 0$ or $\mu = \sigma$. Hence $(i, \tau(i)) \notin D(\mu) \cup M$ for some i, i.e., $b_{i\tau(i)}' = 0$ and $D_n^{\lambda}(B') = \lambda(\mu) \neq 0$.

(iii) $|D(\tau) \cap M| = n - 2$. Then n is even and $|D(\sigma) \cap D(\tau)| = 2$. We may assume $\sigma = g_k$, $\tau = g_l h$ for some k and l and $2|(k - l)$. Consider (3.1). For $\nu \neq \sigma, \tau$, since either $|D(\nu) \cap D(\sigma)| = 0$ or $|D(\nu) \cap D(\tau)| = 0$, it follows that $|D(\nu) \cap (D(\sigma) \cup M)| \leqq 4$ and $\prod_{i=1}^{n} b_{i\nu(i)} = 0$. Hence

$$D_n^{\lambda}(B) = \lambda(\sigma) + \lambda(\tau) \prod_{i=1}^{n} b_{i\tau(i)}.$$

Since there are $n/2$ ($\geqq 3$) g_i-diagonals with $2 \nmid (i - l)$ which do not intersect with the diagonal $\tau = g_l h$ and since there are only two positions in M which do not lie in $D(\tau)$, we may choose $\mu = g_q$ with $2 \nmid (l - q)$. Applying the above argument to (3.2) we have $\prod_{i=1}^{n} b_{i\nu(i)}' = 0$ if $\nu \neq \mu, \tau$. Since $2 \nmid (l - q)$, $|D(\mu) \cap D(\tau)| = 0$ and $\prod_{i=1}^{n} b_{i\tau(i)}' = 0$. Hence $D_n^{\lambda}(B') = \lambda(\mu) \neq 0$.

By Lemmas 2 and 3, we have

LEMMA 4. *If Z is a 0-subspace for D_n^{λ}, $n \geqq 5$, then for every $\sigma \in D_n$, $|D(\sigma) \cap M| = 1$.*

LEMMA 5. *Suppose $n \geqq 5$ and Z is a 0-subspace for D_n^{λ}. Then Z consists of all matrices with n fixed positions $\{(i_1, j_1), \ldots, (i_n, j_n)\} = M$ equal to zero.*

Proof. We need only to show that $c_t^{ij} = 0$ for all $(i, j) \notin M$ and $1 \leqq t \leqq n$. Suppose the contrary, i.e., $c_t^{ij} \neq 0$ for some (i, j) and some t. Since $|D_n(i_t, j_t)| = 2$ let $D_n(i_t, j_t) = \{\sigma, \nu\}$. Let x be an indeterminate over F.

(i) $\sigma(i) \neq j$. Let

$$B(x) = \sum_{k \neq i_t} A_{k\sigma(k)} + xA_{ij}.$$

Then the (i_t, j_t) entry of $B(x)$ is a nonzero polynomial of degree 1 in x so we may choose $c \in F$ so that the entry is nonzero. Let $B(c) = (b_{rs})$. Then $b_{rs} = 0$ if $(r, s) \notin M \cup D(\sigma) \cup \{(i, j)\}$ and $b_{i\sigma(i)} \neq 0$, $i = 1, 2, \ldots, n$. Now

$$D_n^{\lambda}(B(c)) = \lambda(\sigma) \prod_{k=1}^{n} b_{k\sigma(k)} + \sum_{\mu \neq \sigma} \lambda(\mu) \prod_{k=1}^{n} b_{k\mu(k)}.$$

If $\mu \neq \sigma$, then there exist $p \neq q$ such that $\mu(p) \neq \sigma(p)$, $\mu(q) \neq \sigma(q)$ and hence $|D(\sigma) \cap D(\mu)| \leqq n - 2$. If there exists $\mu \neq \sigma$ and $\prod_{k=1}^{n} b_{k\mu(k)} \neq 0$ then $D(\mu) \subseteq M \cup D(\sigma) \cup \{(i, j)\}$. Since by Lemma 4, $|D(\mu) \cap M| = 1$, it follows that $|D(\sigma) \cap D(\mu)| = n - 2$. If $n \geqq 5$, this is impossible since by Lemma 1, $|D(\sigma) \cap D(\mu)| = 0$, 1 or 2. Hence $D_n^{\lambda}(B(c)) = \lambda(\sigma) \prod_{k=1}^{n} b_{k\sigma(k)} \neq 0$ and $B(c) \in Z$, a contradiction.

(ii) $\sigma(i) = j$ and $|F| > 2$. Let

$$B(x) = \sum_{k \neq i_t, i} A_{k\sigma(k)} + xA_{i\sigma(i)}.$$

Then we may choose $c \in F^*$ so that the (i_t, j_t) entry of $B(c)$ is nonzero. Again set $B(c) = (b_{rs})$. Then $b_{rs} = 0$ if $(r, s) \notin M \cup D(\sigma)$ and $b_{i\sigma(i)} \neq 0$, $i = 1, 2, \ldots, n$. Since $|D(\mu) \cap M| = 1$ for $\mu \in D_n$ it follows that $D_n^\lambda(B(c)) = \lambda(\sigma) \prod_{k=1}^n b_{k\sigma(k)} \neq 0$ and $B(c) \in Z$, a contradiction.

(iii) $\sigma(i) = j$ and $F = \{0, 1\}$. Let $(b_{rs}) = B = \sum_{k \neq i_t} A_{k\sigma(k)}$. If $b_{i_t j_t} = 1$ then $D_n^\lambda(B) = \lambda(\sigma) \neq 0$ and $B \in Z$, a contradiction. Hence assume $b_{i_t j_t} = 0$. Then $c_t^{i'j'} = 1$ for some $(i', j') \notin M$, $i' \neq i$ and $\sigma(i') = j'$. Since $|D(\sigma) \cap D(\nu)| \leqq 2$, it follows that at least one of (i, j), (i', j') is not in $D(\nu)$. This reduces to case (i) with σ replaced by ν.

Proof of Proposition 1. By Lemma 4, $|D(g_i) \cap M| = 1$ for $i = 1, 2, \ldots, n$. Hence the pairs (i_t, j_t), $t = 1, 2, \ldots, n$ may be arranged so that $g_t(i_t) = j_t$, $t = 1, 2, \ldots, n$. Since $|D(g_ih) \cap M| = 1$ for $i = 1, 2, \ldots, n$ there exists a permutation α such that $g_{\alpha(t)}h(i_t) = j_t$, $t = 1, 2, \ldots, n$. If n is odd, it follows from $|D(g_j) \cap D(g_kh)| = 1$ that $\alpha \in S_n$. Suppose n is even. By Lemma 1, $|D(g_j) \cap D(g_kh)| \neq 0$ only if $2|(j - k)$. Hence $2|(t - \alpha(t))$ and $\alpha \in K_n$. By Lemma 5 the result follows.

LEMMA 6. *If $T \in \mathscr{T}(D_n, \lambda)$, $n \geqq 3$, then T is nonsingular.*

Proof. Suppose T is singular. Then $T(A) = 0$ for some $A \neq 0$. Hence

$$D_n^\lambda(X - A) = D_n^\lambda(T(X - A)) = D_n^\lambda(T(X) - T(A)) = D_n^\lambda(T(X)) = D_n^\lambda(X)$$

for all X. If $A = (a_{ij})$ then $a_{ij} \neq 0$ for some i, j. We know that $|D_n(i, j)| = 2$ and let $\sigma \in D_n(i, j)$. Set

$$c_1 = \sum_{\nu \in D_n(i,j)} \lambda(\nu) \prod_{t=1}^n a_{t\nu(t)}, \quad c_2 = \sum_{\nu \notin D_n(i,j)} \lambda(\nu) \prod_{t=1}^n a_{t\nu(t)}.$$

Then $D_n^\lambda(A) = c_1 + c_2 = 0$ since $D_n^\lambda(A) = D_n^\lambda(T(A)) = D_n^\lambda(0) = 0$. We consider two cases:

(i) $c_1 = -c_2 \neq 0$. Let $X = a_{ij}E_{ij}$. Then $D_n^\lambda(X) = 0$ and

$$D_n^\lambda(X - A) = \sum_{\nu \in D_n(i,j)} \lambda(\nu) \prod_{t=1}^n (\delta_{j\nu(t)}a_{ij} - a_{t\nu(t)}) + c_2 = 0 + c_2 \neq 0$$

since $\delta_{j\nu(i)}a_{ij} - a_{i\nu(i)} = 0$. Hence we have $D_n^\lambda(X - A) \neq D_n^\lambda(X)$, a contradiction.

(ii) $c_1 = c_2 = 0$. Let X be the matrix whose (r, s) entry is a_{ij} if $\sigma(r) = s$ and zero elsewhere. Then $D_n^\lambda(X) = \lambda(\sigma)a_{ij}^n \neq 0$. Write $B = X - A = (b_{ij})$. Then

$D_n^\lambda(B) = d_1 + d_2$ where

$$d_1 = \sum_{\nu \in D_n(i,j)} \lambda(\nu) \prod_{t=1}^{n} b_{t\nu(t)}, \quad d_2 = \sum_{\nu \notin D_n(i,j)} \lambda(\nu) \prod_{t=1}^{n} b_{t\nu(t)}.$$

Since $b_{ij} = a_{ij} - a_{ij} = 0$ we have $d_1 = 0$. If $d_2 = 0$ then $D_n^\lambda(B) = 0$ and $D_n^\lambda(X - A) \neq D_n^\lambda(X)$, a contradiction. Therefore we suppose $d_2 \neq 0$. Since $c_2 \neq d_2$ there exists $\mu \notin D_n(i, j)$ such that

$$\text{(3.3)} \quad \prod_{t=1}^{n} b_{t\mu(t)} \neq \prod_{t=1}^{n} a_{t\mu(t)}.$$

Since A and B differ only at positions in $D(\sigma)$ we have $|D(\sigma) \cap D(\mu)| \neq 0$ and $|D(\sigma) \cap D(\mu)| = 1$ or 2 depending on whether n is odd or even. If n is odd let $(k, l) \in D(\sigma) \cap D(\mu)$ and $X_1 = a_{ij}(E_{ij} + E_{kl})$. If n is even let (k, l), $(k', l') \in D(\sigma) \cap D(\mu)$ and $X_1 = a_{ij}(E_{ij} + E_{kl} + E_{k'l'})$. In both cases we have $D_n^\lambda(X_1) = 0$ since $n \geqq 3$. Now let $X_1 - A = (b_{rs}')$. Then $D_n^\lambda(X_1 - A) = d_1' + d_2'$ with

$$d_1' = \sum_{\nu \in D_n(i,j)} \lambda(\nu) \prod_{t=1}^{n} b_{t\nu(t)}' = 0$$

since $b_{i\nu(i)}' = b_{ij}' = a_{ij} - a_{ij} = 0$ and

$$d_2' = \sum_{\nu \notin D_n(i,j) \cup D_n(k,l)} \lambda(\nu) \prod_{t=1}^{n} b_{t\nu(t)}' + \lambda(\mu) \prod_{t=1}^{n} b_{t\mu(t)}'$$

since $D_n(k, l) = \{\sigma, \mu\}$. Note that $b_{t\nu(t)}' = a_{t\nu(t)}$ if $\nu \notin D_n(i, j) \cup D_n(k, l)$ and $b_{t\mu(t)}' = b_{t\mu(t)}$ for all $t = 1, 2, \ldots, n$. Hence

$$d_2' = \sum_{\nu \notin D_n(i,j) \cup D_n(k,l)} \lambda(\nu) \prod_{t=1}^{n} a_{t\nu(t)} + \lambda(\mu) \prod_{t=1}^{n} b_{t\mu(t)}$$

$$= c_2 - \lambda(\mu) \prod_{t=1}^{n} a_{t\mu(t)} + \lambda(\mu) \prod_{t=1}^{n} b_{t\mu(t)}.$$

Now $c_2 = 0$ and by (3.3), $d_2' \neq 0$. Hence $D_n^\lambda(X_1 - A) \neq 0$ and $D_n^\lambda(X_1 - A) \neq D_n^\lambda(X_1)$, a contradiction.

Now suppose $T \in \mathscr{T}(D_n, \lambda)$. Then by Proposition 1 and Lemma 6, applying the same argument as in [**2**], it can be shown that for each pair $1 \leqq i, j \leqq n$ there exist $1 \leqq p, q \leqq n$ and $a_{pq} \in F^*$ such that

$$T(E_{ij}) = a_{pq}E_{pq}$$

and for distinct (i, j) we have distinct (p, q), i.e., the matrix representation of T with respect to the basis $\{E_{ij} : i, j = 1, 2, \ldots, n\}$ is a generalized permutation matrix.

For $\sigma \in D_n$, since $D_n^\lambda(P(\sigma)) = \lambda(\sigma) \neq 0$, it follows that $T(P(\sigma)) = A * P(\mu)$ for some $\mu \in D_n$, i.e., T transforms diagonals to diagonals. Furthermore since $D_n^\lambda(T(P(\sigma))) = \lambda(\mu) \prod_{i=1}^n a_{i\mu(i)}$, we have $\prod_{i=1}^n a_{i\mu(i)} = \lambda(\sigma)(\lambda(\mu))^{-1}$. By minor modifications on the proofs in [**5**], Theorems 1 and 2 follow.

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Tunku Abdul Rahman College,
Kuala Lumpur, Malaysia