NOTE ON FINITE TOPOLOGICAL SPACES

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Introduction

In a recent paper [4], H. Sharp, Jr., has discussed the problem of finding formulae for the following naturally defined integers: the numbers $t(n)$, $tc(n)$, $t_0(n)$, $tc_0(n)$, and $ts(n)$ of all homeomorphism classes of finite topological spaces with n elements, which are respectively (i) arbitrary, (ii) connected, (iii) T_0, (iv) connected and T_0, (v) symmetric. Here, a finite topological space X is called *symmetric* provided the following relation $\leqq$ is symmetric: $x \leqq y$ if and only if $x \in U_y$, the intersection of all open sets containing y.

In this context, consider also the following integers: the numbers $Ps(n)$, $r(n)$, $m(n)$ and $u(n)$ of all homeomorphism classes of finite topological spaces with n elements, which are respectively (i) pseudo-metrizable, (ii) regular, (iii) measurable, (iv) uniformizable. Here, a topological space X will be called *regular* provided *only* that every closed subset can be separated in the usual way from any point in its complement, and X will be called *measurable* provided every open set is also closed.

THEOREM 1. (i) *$Ps(n) = r(n) = m(n) = u(n) = ts(n) = p(n)$, the total number of partitions of n into a sum of positive integers.*

(ii) *The formal generating functions for $t(n)$ and $t_0(n)$ satisfy the relations*

$$1+\sum_{n=1}^{\infty} t(n)x^n = \prod_{r=1}^{\infty} (1-x^r)^{-tc(r)},$$

$$1+\sum_{n=1}^{\infty} t_0(n)x^n = \prod_{r=1}^{\infty} (1-x^r)^{-tc_0(r)}.$$

REMARK. In [4], Sharp provides a table of values for $t(n)$, $tc(n)$, $t_0(n)$, $tc_0(n)$ and $ts(n)$ when $n \leqq 5$; this table is consistent with the present theorem.

Theorem 1 will be deduced from Theorem 2 and Lemma 1 in Section 1 below. I am grateful to the referee for pointing out the incorrectness of a formula originally put forward for part (ii) of Theorem 1, and also for suggesting a simplified proof of Lemma 1.

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An analogue of Theorem 1 for *finite algebraic systems* is mentioned in Section 2. Finally, an alternative but far less elementary proof of Theorem 1, based on a graph-theoretical formula of F. Harary [1], is briefly discussed in Section 3.

1. Finite spaces

Let α denote a topological property of finite topological spaces such that a space X has property α if and only if each connected component of X has property α. Let $t_\alpha(n)$ and $tc_\alpha(n)$ respectively denote the total number of homeomorphism classes of finite spaces with n elements which (i) have property α, (ii) are connected and have property α. (The vagueness concerning the property α can always be avoided by listing some particular properties of interest.)

THEOREM 2. *The formal generating function for $t_\alpha(n)$ satisfies the relation*

$$1+\sum_{n=1}^{\infty} t_\alpha(n)x^n = \prod_{r=1}^{\infty} (1-x^r)^{-tc_\alpha(r)}.$$

PROOF. Any finite topological space X is the disjoint union of its (closed) connected components. Since there are only a finite number of components, these are also open. Hence X is the *topological sum* of its components.

Now consider a partition π of a set Y with n elements into disjoint subsets $C_j(n_i)$ $[i = 1, \cdots, k; j = 1, \cdots, r_i]$ such that $C_1(n_i), \cdots, C_{r_i}(n_i)$ all have n_i elements, and $n_1, \cdots, n_k$ are distinct. Let $N(\pi)$ denote the total number of non-homeomorphic α-topologies for Y such that the $C_j(n_i)$ become the components of Y. Then, by the first remark,

$$N(\pi) = N_1 \cdot N_2 \cdot \cdots \cdot N_k$$

where N_i is the total number of non-homeomorphic α-topologies for

$$Y_i = C_1(n_i) \cup \cdots \cup C_{r_i}(n_i)$$

such that $C_1(n_i), \cdots, C_{r_i}(n_i)$ become the components of Y_i.

Next, if $t_i = tc_\alpha(n_i)$ then, by the first remark again, N_i may be regarded as the total number of un-ordered selections of r_i objects from t_i distinct objects, each of which may appear from 0 to r_i times in a selection. Hence

$$N_i = \binom{t_i+r_i-1}{r_i}$$

(cf. [2] say).

Finally, $t_\alpha(n) = \sum N(\pi)$, summing over all partitions π as above. Using the identity

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$$(1-x^n)^{-t} = \sum_{r=0}^{\infty} \binom{t+r-1}{r} x^{rn},$$

the theorem now follows.

Part (ii) of Theorem 1 is now proved. Also if α denotes the topological property of being *measurable*, it is clear that $tc_\alpha(n) = 1$ for all n. Hence

$$1+\sum_{n=1}^{\infty} m(n)x^n = \prod_{r=1}^{\infty} (1-x^r)^{-1},$$

which is the generating function for $p(n)$ (cf. [2] say). Part (i) of Theorem 1 therefore follows from the lemma:

LEMMA 1. *The following conditions on a finite topological space X are equivalent*:

(i) *X is pseudo-metrizable*;
(ii) *X is uniformizable*;
(iii) *X is regular*;
(iv) *X is measurable*;
(v) *X is symmetric.*

PROOF. The fact that, for general spaces, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is well known.

(iii) $\Rightarrow$ (iv): Regularity implies that, for any $x \in X$, the smallest neighbourhood U_x of x must contain a closed neighbourhood of x. Therefore, U_x must be closed. Since every open set is a finite union of such sets U_x, it must also be closed.

(iv) $\Rightarrow$ (v): Suppose that X is measurable, and let $x \in U_y$ in X. Since U_x and U_y have non-empty intersection, the intersection O of U_y with the complement of U_x is a proper subset of U_y. Since O is open, $y \notin O$. Hence $y \in U_x$.

(v) $\Rightarrow$ (i): If X is symmetric, it may for example be pseudometrized by defining

$$d(x, y) = \begin{cases} 0 & (\text{if } x \leqq y), \\ 1 & (\text{otherwise}). \end{cases}$$

It may be mentioned that part (i) of Theorem 1 could also be deduced directly from Lemma 1 with the aid of a result of R. E. Stong ([5], Theorem 1).

2. Finite algebraic systems

In this section, we observe that a proof, similar to that for Theorem 2, leads to the following analogue for finite *algebraic systems*:

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Consider some category of finite algebraic systems for which the *Krull-Schmidt* theorem is valid. Let $g(n)$ denote the total number of non-isomorphic systems of this type which have n elements, and let $gI(n)$ denote the total number amongst these which are indecomposable. (The vagueness here can again be avoided by listing some specific categories of interest.)

THEOREM 3. *The formal Dirichlet series for $g(n)$ satisfies the relation*

$$\sum_{n=1}^{\infty} g(n)n^{-x} = \prod_{r=2}^{\infty} (1-r^{-x})^{-gI(r)}.$$

EXAMPLES. (i) For *finite p-groups*, this gives the relation

$$\sum_{n=0}^{\infty} g(p^n)p^{-nx} = \prod_{r=1}^{\infty} (1-p^{-rx})^{-gI(p^r)}$$

(cf. the classification of groups of order p^n $(n < 5)$ in [3] for example).

(ii) For *finite abelian groups*, the only indecomposable ones are the cyclic groups of prime power order. Therefore in this case one obtains a relation of the form

$$\sum_{n=1}^{\infty} a(n)n^{-x} = \prod_{\text{primes } p} \prod_{r=1}^{\infty} (1-p^{-rx})^{-1},$$

which gives back the well known formula for $a(n)$.

(iii) For *semi-simple finite rings*, the indecomposable ones are the complete matrix rings over finite fields. Hence there is an equation of the form

$$\sum_{n=1}^{\infty} s(n)n^{-x} = \prod_{\text{primes } p} \prod_{r=1}^{\infty} \prod_{m=1}^{\infty} (1-p^{-rm^2x})^{-1}.$$

(iv) By looking at the category of finite *cyclic groups*, it may be noted that Theorem 3 gives back Euler's identity for the *Riemann zeta-function.*

3. A formula of Harary

F. Harary [1] has given a formula relating the total number of non-isomorphic *graphs*, of arbitrary given type, which have n vertices, with the total number amongst these which are connected. This result, formula (33) of [1], is there derived from a powerful 'Enumeration Theorem' due to Pólya. It may be used as follows to give an alternative, although far less elementary, proof of Theorem 2:

Firstly, by appealing to Propositions 5, 7 and 8 of Stong [5], it may be noted that connectedness and homeomorphism of finite topological spaces

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are equivalent to connectedness and isomorphism for finite transitive and directed graphs without multiple edges.

Next, Harary's formula (33) may be applied to transitive digraphs without multiple edges, which have property α; this is given in terms of the total number of edges in any such graph. Summing over the number of edges and transforming by the usual exponential-logarithmic power series identity, as in another formula (45) of Harary [1], Theorem 2 follows.

References

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