Can. J. Math., Vol. XXV, No. 6, 1973, pp. 1269–1273

ON THE NUMBER OF STRUCTURES OF REFLEXIVE AND TRANSITIVE RELATIONS

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If for each permutation the number of partial orderings fixed by that permutation is known, it is possible to count the number of non-isomorphic partial orderings on a finite set using a lemma of Burnside. In this paper it is shown that knowledge of the numbers of partial orderings fixed by permutations will enable the number of non-isomorphic pre-orderings to be counted also.

1. Introduction. In the first section the basic objects and actions which will be needed are defined. The reader is directed to [**3**] where a more detailed development is set out.

Let n and j be positive integers with $n \geqq j$. Let $N = \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, j\}$. Let $\mathrm{Sur}(N, J)$ be the set of surjective functions with domain N and range J. If $x \in N$ we will write $(x)f$ for the value of $f \in \mathrm{Sur}(N, J)$ at the point x.

Let A_n be the set of $n \times n$ binary matrices representing pre-orderings on N, and B_j the set of $j \times j$ binary matrices representing partial orderings on J.

We will represent the elements of A_n and B_j by capital letters T, S, etc.

For each $f \in \mathrm{Sur}(N, J)$ define a function $\hat{f}$: $B_j \rightarrow A_n$ by sending $T \rightarrow S$, where $s_{ab} = t_{(a)f(b)f} \forall\ (a, b)$ in $N \times N$. Clearly, $(T)\hat{f}$ is well defined and is an element of A_n.

Let S_p be the group of permutations on p letters, p being a positive integer. Let $x \in N$ and $T \in A_n$.

The interrelationships between the following actions were developed in [**3**].

(a) $\quad S_n \times \mathrm{Sur}(N, J) \rightarrow \mathrm{Sur}(N, J)$

$\qquad (\phi, f) \rightarrow \phi \cdot f \qquad$ where $(x)(\phi \cdot f) = ((x)\phi)f$.

(b) $\quad \mathrm{Sur}(N, J) \times S_j \rightarrow \mathrm{Sur}(N, J)$

$\qquad (f, \sigma) \rightarrow f \cdot \sigma \qquad$ where $(x)(f \cdot \sigma) = ((x)f)\sigma$.

(c) $\quad A_n \times S_n \rightarrow A_n$

$\qquad (T, \sigma) \rightarrow S \qquad s_{ij} = t_{(i)\sigma(j)\sigma} \forall\ (i, j) \in N^2$,

(d) $\quad B_j \times S_j \rightarrow B_j \qquad$ in the same way as in (c).

If $f \in \mathrm{Sur}(N, J)$ let $||f|| = (|f^{-1}(1)|, |f^{-1}(2)|, \ldots, |f^{-1}(j)|)$ where $|Q|$ stands for the cardinality of the set Q.

Received July 18, 1972 and in revised form, April 11, 1973.

Let $\text{Norm}(N, J)$ be the set $\{||f|| \mid f \in \text{Hom}(N, J)\}$.

Define an action $\text{Norm}(N, J) \times S_j \rightarrow \text{Norm}(N, J)$ by sending

$$(||f||, \sigma) \rightarrow ||f \cdot \sigma||.$$

If $\Delta \in S_n$ and $f \in \text{Sur}(N, J)$ we say Δ *is special for* f if and only if there exists a $\sigma_\Delta \in S_j$ such that $\Delta \cdot f = f \cdot \sigma_\Delta$.

If X is an element of A_n, B_j, $\text{Sur}(N, J)$ etc. let $S_p(X)$ be the subgroup of elements of S_p fixing X under one of the given actions.

2. Counting relations. In this section the preliminary lemmas and the theorems are proved.

LEMMA 1. *Let T and V be in B_j, f and g in* $\text{Sur}(N, J)$. *Then*

$$[(T)\hat{f} = (V)\hat{g}] \Leftrightarrow (\exists\, \sigma \in S_j)[f = g \cdot \sigma \textit{ and } V = (T)\sigma].$$

Proof. The necessity is trivial and we need only prove sufficiency. If $(T)\hat{f} = (V)\hat{g}$ then $t_{(i)f(j)f} = v_{(i)g(j)g} \,\forall\, (i, j) \in N^2$. Because T and V are anti-symmetric, $(i)f = j(f) \Leftrightarrow (i)g = j(g)$. Because of this if $g_1: J \rightarrow N$ is any map satisfying $g_1 g = id_J$, $g_1 f$ is a permutation of J, σ say, with the property $f = g \cdot \sigma$. Also

$$t_{(i)g\sigma(j)g\sigma} = t_{(i)f(j)f} = v_{(i)g(j)g} \,\forall\, (i, j) \in N^2.$$

But g is surjective. Thus $t_{(l)\sigma(m)\sigma} = v_{lm} \,\forall\, (l, m) \in J^2$. This means $V = (T)\sigma$, completing the proof of the Lemma.

H. Gupta in [**2**] wrote down a summation for the number of preorderings on a finite set in terms of the numbers of partial orderings. He took isomorphic relations to be distinct. In the following lemma a set theoretic form of Gupta's result is derived.

LEMMA 2. *Let $[f]$ be the equivalence class to which f belongs under the relation $f \sim g \Leftrightarrow (\exists \sigma \in S_j)$ such that $f = g \cdot \sigma$. Then*

$$A_n = \bigcup_{j=1}^{n} \bigcup_{[f] \mid f \in \text{Sur}(N, J)} (B_j)\hat{f}.$$

The sets in the unions form a partition of A_n into disjoint subsets.

Proof. Clearly

$$A_n \supseteq \bigcup_{j=1}^{n} \bigcup_{f \mid f \in \text{Sur}(N, J)} (B_j)\hat{f}.$$

If $M \in A_n$ suppose that the rows $r_{11}, r_{21}, \ldots, r_{k_1 1}$ are identical but distinct from the rest; rows $r_{12}, r_{22}, \ldots, r_{k_2 2}$ are identical but distinct from the rest; and so on until finally rows $r_{1j}, \ldots, r_{k_j j}$ are identical but distinct from the others; $k_1, k_2, \ldots, k_j \geqq 1$. Then the r's are the numbers $1, 2, \ldots, n$ in some order. Let T be the $j \times j$ matrix formed by deleting the rows and columns of M apart from rows and columns numbered $r_{11}, r_{12}, \ldots, r_{1j}$.

Then, if $f \in \text{Sur}(N, J)$ is defined by the rule $f(r_{pq}) = q$, we have $M = (T)\hat{f}$ with $T \in B_j$. Thus

$$A_n = \bigcup_{j=1}^{n} \bigcup_{f \mid f \in \text{Sur}(N,J)} (B_j)\hat{f}.$$

It is easy to see that the union over j is disjoint. Suppose now that j is fixed, that f and g are in $\text{Sur}(N, J)$ and that $(B_j)\hat{f} \cap (B_j)\hat{g} \neq \emptyset$. Then there exist T and V in B_j such that $(T)\hat{f} = (V)\hat{g}$. By Lemma 1, this implies $f \sim g$. On the other hand if $f \sim g$ then $(B_j)\hat{f} = (B_j)\hat{g}$: If $\sigma \in S_j$ is such that $f = g \cdot \sigma$ then

$$(B_j)\hat{f} = (B_j)\widehat{g \cdot \sigma} = ((B_j)\sigma)\hat{g} = (B_j)\hat{g}.$$

These remarks show that the sets in the unions are disjoint as claimed.

Finally, we will show that this is equivalent to the formula of Gupta. The number of ways in which n distinct objects may be placed in exactly j like boxes is represented by $u(n, j)$ and is equal to the size of the set $\{[f] \mid f \, \text{Sur}(N, J)\}$. Because $\hat{f}$ is injective [**3**, Lemma 1(i)], $|(B_j)\hat{f}| = |B_j|$. Thus

$$|A_n| = \sum_{j=1}^{n} u(n, j)|B_j|$$

which is Gupta's result.

If $T \in A_n$ let $[T]$ be the equivalence class to which T belongs in the relation $T \sim S \Leftrightarrow (\exists \sigma \in S_n)[T = (S)\sigma]$. Let $\bar{A}_n = \{[T] \mid T \in A_n\}$ and $\bar{B}_j = \{[T] \mid T \in B_j\}$. The following theorem uses the result of Lemma 1 to form a partition of $\bar{A}_n$ into disjoint subsets.

THEOREM 1. *Let $f^\#: B_j \to \bar{A}_n$ be the map $T \to [(T)\hat{f}]$, and let $[[f]]$ be the equivalence class to which f belongs in the relation $f \simeq g \Leftrightarrow f = \Delta \cdot g \cdot \sigma$ for some pair (Δ, σ) in $S_n \times S_j$. Then*

$$\bar{A}_n = \bigcup_{j=1}^{n} \bigcup_{[[f]] \mid f \in \text{Sur}(N,J)} (B_j)f^\#.$$

The sets in the unions are pairwise disjoint.

Proof. From Lemma 2 we obtain the equation

$$\bar{A}_n = \bigcup_{j=1}^{n} \bigcup_{[f] \mid f \in \text{Sur}(N,J)} (B_j)f^\#.$$

Firstly, if $f \simeq g$ then $(B_j)f^\# = (B_j)g^\#$, as

$$(B_j)\hat{f} = (B_j)\widehat{\Delta \cdot g \cdot \sigma} = (B_j)\sigma\widehat{\Delta \cdot g} = (B_j)\hat{g}\Delta$$

for some pair (Δ, σ) in $S_n \times S_j$.

Also, if $(B_j)f^\# \cap (B_j)g^\# \neq \emptyset$, there is an $M \in A_n$ and a (Δ, ϕ) in S_n^2 such that $(M)\Delta = (T)\hat{f}$ and $(M)\phi = (V)\hat{g}$ for some (T, V) in B_j^2. Thus $(T)\hat{f}\Delta^{-1} = (V)\hat{g}\phi^{-1}$ and hence $(T)\widehat{\Delta^{-1} \cdot f} = (V)\widehat{\phi^{-1} \cdot g}$. By Lemma 1 there exists

https://doi.org/10.4153/CJM-1973-133-9 Published online by Cambridge University Press

a σ in S_j such that $\phi \cdot \Delta^{-1} \cdot f = g \cdot \sigma$. Therefore $f \simeq g$. The theorem now follows from these remarks.

In [**1**], Davis observed that the problem of counting the number of non-isomorphic relations of a special type could be reduced to the problem of counting the number of relations fixed by each permutation, by using Burnside's lemma. That is if a group G acts on a set S, the number of orbits, ξ, satisfies the equation

$$\xi|G| = \sum_{\pi \in G} f(\pi)$$

where $f(\pi)$ is the number of elements in S fixed by π in G. We will use this formula in the proof of Theorem 2.

If $\Delta \in S_n$ let $A_n(\Delta) = \{T \in A_n \mid T = (T)\Delta\}$. Define $B_j(\sigma)$ similarly. Theorem 2 derives a formula for $|\bar{A}_n|$ in terms of the numbers $|B_j(\sigma)|$. Three lemmas precede this derivation.

If Δ is special for $f \in \operatorname{Sur}(N, J)$ there is a relationship between $A_n(\Delta)$ and $B_j(\sigma_\Delta)$. This relationship is described in the next two Lemmas.

LEMMA 3. *If $f \in \operatorname{Sur}(N, J)$ then $(B_j)\hat{f} \cap A_n(\Delta) \neq \emptyset \Leftrightarrow \Delta$ is special for f.*

Proof. ($\Rightarrow$) If $M \in (B_j)\hat{f} \cap A_n(\Delta)$ there is a T in B_j with $M = (T)\hat{f}$ and $(T)\hat{f} = (T)\hat{f}\Delta$. Thus $(T)\hat{f} = (T)\widehat{\Delta \cdot f}$ and (by Lemma 1) there is a σ in S_j, such that $f = \Delta \cdot f \cdot \sigma$. Therefore $\Delta \cdot f = f \cdot \sigma^{-1}$ and Δ is special for f.

($\Leftarrow$) Let T be such that $(T)\sigma_\Delta = T$. Then

$$(T)\hat{f}\Delta = (T)\widehat{f \cdot \sigma_\Delta} = (T)\sigma_\Delta\hat{f} = (T)\hat{f}.$$

Thus $(T)\hat{f} \in (B_j)\hat{f} \cap A_n(\Delta)$, completing the proof.

LEMMA 4. *If Δ is special for f then $(B_j)\hat{f} \cap A_n(\Delta)$ and $B_j(\sigma_\Delta)$ are of the same size.*

Proof. It is easy to check that $\hat{f}\,|B_j(\sigma_\Delta)$ induces a bijection.

LEMMA 5. *Let $S_n^f = \{\Delta \in S_n |\ \Delta$ is special for $f\}$. Then*

$$S_n^f / S_n(f) \cong S_j(||f||)$$

as groups.

Proof. The map $\phi\colon S_n^f \to S_j(||f||)$ sending $\Delta \to \sigma_\Delta$ is a surjective homomorphism with kernel $S_n(f)$.

THEOREM 2.

$$n!|\bar{A}_n| = \sum_{j=1}^{n} \sum_{[f] \mid f \in \operatorname{Sur}(N,J)} \left\{ |S_n(f)| \sum_{\sigma \in S_j(||f||)} |B_j(\sigma)| \right\}.$$

Proof. From the formula given in Lemma 2 we may write

$$A_n(\Delta) = \bigcup_{j=1}^{n} \bigcup_{[f] \mid f \in \operatorname{Sur}(N,J)} \{A_n(\Delta) \cap (B_j)_f\}.$$

https://doi.org/10.4153/CJM-1973-133-9 Published online by Cambridge University Press

Because the sets in the unions are disjoint it follows that

$$|A_n(\Delta)| = \sum_{j=1}^{n} \sum_{[f]|f\in \mathrm{Sur}(N,J)} |A_n(\Delta) \cap (B_j)\hat{f}|.$$

When Δ is special for f we will write σ_Δ as $\sigma_{\Delta,f}$. Then

$$|A_n(\Delta)| = \sum_{j=1}^{n} \sum_{\substack{[f]|f\in \mathrm{Sur}(N,J) \text{ and} \\ \Delta \text{ special for } f}} |B_j(\sigma_{\Delta,f})|$$

by Lemma 4. From the formula for the number of orbits generated when S_n acts on A_n:

$$n!|\bar{A}_n| = \sum_{\Delta\in S_n} |A_n(\Delta)| = \sum_{\Delta\in S_n} \sum_{j=1}^{n} \sum_{\substack{[f]|f\in \mathrm{Sur}(N,J) \text{ and} \\ \Delta \text{ special for } f}} |B_j(\sigma_{\Delta,f})|.$$

Now let

$$\chi(j, \Delta, f) = \begin{cases} |B_j(\sigma_{\Delta,f})|, & \text{if } \Delta \in S_n^{\,f} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} n!|\bar{A}_n| &= \sum_{\Delta\in S_n} \sum_{j=1}^{n} \sum_{[f]|f\in \mathrm{Sur}(N,J)} \chi(j, \Delta, f), \\ &= \sum_{j=1}^{n} \sum_{[f]|f\in \mathrm{Sur}(N,J)} \left\{ \sum_{\Delta\in S_n} \chi(j, \Delta, f) \right\}, \\ &= \sum_{j=1}^{n} \sum_{[f]|f\in \mathrm{Sur}(N,J)} \left\{ \sum_{\Delta\in S_n^{\,f}} |B_j(\sigma_{\Delta,f})| \right\}, \\ &= \sum_{j=1}^{n} \sum_{[f]|f\in \mathrm{Sur}(N,J)} \left\{ |S_n(f)| \sum_{\sigma\in S_j(||f||)} |B_j(\sigma)| \right\} \end{aligned}$$

using the isomorphism in Lemma 5. This completes the proof.

Conclusion. From Burnside's formula,

$$j!|\bar{B}_j| = \sum_{\sigma\in S_j} |B_j(\sigma)|.$$

If one could calculate the numbers $|B_j(\sigma)|$ one could not only count the number of non-isomorphic partial orderings but also, by the result of Theorem 2, count the number of non-isomorphic preorderings.

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