Canad. Math. Bull. Vol. **29** (1), 1986

AN ENDPOINT ESTIMATE FOR CERTAIN k-PLANE TRANSFORMS

BY

S. W. DRURY

ABSTRACT. In this paper we extend a result of Oberlin and Stein on Radon Transforms to k-plane transforms for $k > \frac{1}{2}n$. Specifically let

$$F(\pi) = \sup_{\pi} |T_k f(\Pi)|$$

where the supremum is taken over all affine k-planes Π parallel to the vector k-plane π. We show that F is in L^n of the Grassmann manifold $G_{n,k}$ whenever f is in the Lorentz space $L(n/k, 1)$ of $\mathbb{R}^n$. The proof relies very heavily on the ideas of M. Christ.

Introduction. For f a suitable function defined on $\mathbb{R}^n$ the k-plane transform $T_k f$ is defined by

$$T_k f(\Pi) = \int f(x)\, d\lambda_\Pi(x)$$

where Π is an affine k-plane in $\mathbb{R}^n$ and λ_Π is Lebesgue measure on Π. Thus $T_k f$ is a function on the manifold $M_{n,k}$ of affine k-planes in $\mathbb{R}^n$. The space $M_{n,k}$ has a natural bundle structure. Each element Π of $M_{n,k}$ may be written uniquely as

$$\Pi = \pi + x$$

where π is a vector k-plane and $x \in \pi^\perp$. The space of vector k-planes, that is, the Grassmann manifold is denoted $G_{n,k}$. Clearly $M_{n,k}$ is a bundle over $G_{n,k}$ with an $(n - k)$-dimensional fibre.

There is a natural measure γ on $G_{n,k}$ (the unique rotationally invariant probability measure) and a natural measure

$$d\mu(\Pi) = d\lambda_{\pi^\perp}(x)\, d\gamma(\pi)$$

on $M_{n,k}$ which is invariant under Euclidean motions. One may define mixed norms spaces

$$L^q(d\gamma, L^r(d\lambda_{\pi^\perp}))$$

Received by the editors September 24, 1984 and, in final revised form, January 24, 1985.
Research supported by a grant from NSERC.
AMS Subject Classification (1980): 44A15.

https://doi.org/10.4153/CMB-1986-018-3 Published online by Cambridge University Press

on $M_{n,k}$ in this context and ask for estimates of the type

$$\|T_k f\|_{L^q(L^r)} \leq C\|f\|_{L^p}.$$

In a recent article M. Christ [1] makes substantial inroads into this question. In an earlier article Stein and Oberlin [6] prove a fascinating result namely

$$\|T_{n-1} f\|_{L^n(L^\infty)} \leq C_N\|f\|_{\frac{n}{n-1},1} \ (n \geq 3)$$

For the Radon Transform T_{n-1}. The norm on the right is the Lorentz space norm. The reader is referred to Strichartz [7] for this formulation of the result. The result is sharp in two separate ways. Firstly, the Lorentz space is really needed. For $a > 1$ there exists $f \in L(n/n - 1, a)$ with $T_{n-1}f$ everywhere infinite. Secondly, the condition $n \geq 3$ is essential. If $n = 2$ then taking f to be the indicator function of a Kakeya set yields a counterexample.

The purpose of this note is to rework Christ's ideas to prove an endpoint result for T_k where $k > \frac{1}{2}n$.

THEOREM 1. *For $k > \frac{1}{2}n$ we have*

$$\|T_k f\|_{L^n(L^\infty)} \leq C_n\|f\|_{n/k,1}.$$

It would be interesting to know if for $k \leq \frac{1}{2}n$ one can always find "Kakeya sets". That is, can we always find for each $\epsilon > 0$ a subset $A \subseteq \mathbb{R}^n$ with measure $< \epsilon$ but for every $\pi \in G_{n,k}$ there is a translate Π of π such that $\lambda_\Pi(A) \geq 1$. Indeed this question has already been asked by K. J. Falconer [4]. In [4] and [5], Falconer proves qualitative variants of Theorem 1.

A natural generalization of T_k is $T_{l,k}$ the "l-plane to k-plane" transform mentioned by Strichartz [7, page 701]. To define it we consider $\Pi \in M_{n,k}$ as fixed and let μ_Π be the natural measure on $M_{k,l}(\Pi)$ the space of l-planes Θ contained in Π. For f a function on $M_{n,l}$ we set

$$T_{l,k}f(\Pi) = \int f(\Theta)\,d\mu_\Pi(\Theta).$$

The function $T_{l,k}f$ is defined on $M_{n,k}$. When $l = 0$, 0-planes are just points and $T_{0,k} = T_k$. These transforms compose nicely

$$T_{l,k} \circ T_{m,l} = T_{m,k}.$$

We may now restate Lemma 3.5 of Christ [1] as follows.

THEOREM 2. (Christ). *For $0 \leq k \leq n - 1$*

$$\|T_{k,n-1}\|_{L^B(L^R)(M_{n,n-1})} \leq C\|f\|_{L^A(L^P)(M_{n,k})}$$

where $A' = nP = (n - k)B'$, $P' = (n - k)R'$, $P \geq 1$, $R < \infty$.

The operator which Christ calls S is, suitably interpreted, $T^*_{k,n-1}$. Christ's condition $p < n/k$ corresponds to our $R < \infty$.

One idea of this paper is to use $T_{k,n-1}$ as a link between T_k and T_{n-1} according to Lemma 1 below. We give some motivation for this in the remarks following the proof of theorem 1.

LEMMA 1.

$$T_k = T^*_{k,n-1} R_{-(n-1-k)} T_{n-1}. \tag{0}$$

Here $R_{-(n-1-k)}$ is a suitably normalized Riesz potential (of negative order) taken on each fibre of $M_{n,n-1}$.

PROOF OF LEMMA 1. It will be easier to verify the equivalent adjoint statement

$$T^*_k = T^*_{n-1} R_{-(n-1-k)} T_{k,n-1}. \tag{1}$$

First we tackle $T_{k,n-1}$.

$$T_{k,n-1} f(\Pi) = \int f(\Theta) d\mu_{\Pi}(\Theta)$$

for $\Pi \in M_{n,n-1}$, $\Theta \in M_{n,k}$. If we write $\Pi = (\pi, x)$, $\pi \in G_{n,n-1}$, $x \in \pi^\perp$ and $\Theta = (\theta, y)$, $\theta \in G_{n,k}$, $y \in \theta^\perp$ then the condition $\Theta \subseteq \Pi$ amounts to $\theta \subseteq \pi$ and $y \in \pi + x$. The latter can be written $y = x + y'$ with $y' \in \pi \cap \theta^\perp$. The measure $d\mu_\Pi(\Theta) = d\lambda_{\pi\cap\theta^\perp}(y') d\gamma_\pi(\theta)$ where γ_π is the invariant measure on the Grassmann of k-planes contained in Π. Thus

$$T_{k,n-1} f(\pi, x) = \int f(\theta, x + y') d\lambda_{\pi\cap\theta^\perp}(y') d\gamma_\pi(\theta).$$

Next we take the Fourier transform on the fibre $\pi^\perp$

$$T_{k,n-1} f^\wedge(\pi, u) = \int f(\theta, x + y')\, e^{-iux} d\lambda_{\pi\cap\theta^\perp}(y') d\gamma_\pi(\theta) d\lambda_{\pi^\perp}(x)$$

which is defined for $u \in \pi^\perp$. Now $y' \in \pi \cap \theta^\perp \subseteq \pi$ is orthogonal to u so that $e^{-iux} = e^{-iu(x+y')}$. Thus we find

$$T_{k,n-1} f^\wedge(\pi, u) = \int f(\theta, y) e^{-iuy} d\lambda_{\theta^\perp}(y) d\gamma_\pi(\theta)$$

or

$$T_{k,n-1} f^\wedge(\pi, u) = \int f^\wedge(\theta, u) d\gamma_\pi(\theta) \tag{2}$$

where the "$\wedge$" on the right is the Fourier transform along the fibre in $M_{n,k}$.

Now we turn our attention to T_k. It is well-known (see [1] or [7]) that

$$T_k f^\wedge(\theta, u) = \hat{f}(u)$$

for $u \in \theta^\perp$. A straightforward change of variables argument now leads to

$$T^*_k g^\wedge(u) = c_{n,k} |u|^{-k} \int g^\wedge(\theta, u) d\gamma_{u^\perp}(\theta) \tag{3}$$

for the adjoint. In case $k = n - 1$ this simplies further to

$$T_{n-1}^{*} g^{\wedge}(u) = c_n |u|^{-(n-1)} g^{\wedge}(u^{\perp}, u) \tag{4}$$

as there is only one linear hyperplane perpendicular to u.

Combining (2), (3) and (4) now leads immediately to (1). The dedicated reader may wish to observe that the statement

$$I = T_{k,n-1}^{*} R_{-(n-1-k)} T_{k,n-1}$$

is false. (If it were true we could deduce (0) by multiplying on the right by T_k).

We now establish an interpolation lemma which is probably well-known. The lemma is stated in one dimension: we leave the every task of generalizing this result to n dimensions to the reader. L^p_α denotes the Sobolev space defined with the Riesz potential.

LEMMA 2. *Let* $\alpha > \frac{1}{2}$ *and* $1 \leq r < \infty$. *Suppose that*

$$\|h\|_{L^2_\alpha(\mathbb{R})} \leq M_1$$

and

$$\|h\|_{L^r(\mathbb{R})} \leq M_2 .$$

Then

$$\|h\|_{L^\infty(\mathbb{R})} \leq C M_1^{\beta} M_2^{1-\beta}$$

where $\beta = (1/r)\,(\alpha - \frac{1}{2} + 1/r)^{-1}$.

PROOF OF LEMMA 2. By translation invariance it is enough to control $|h(0)|$. Let δ denote the delta function at 0, let θ be a Schwarz class function with integral one and denote $\theta_t(x) = t^{-1}\theta(t^{-1}x)$. Then

$$h(0) = \langle \delta - \theta_t, h \rangle + \langle \theta_t, h \rangle. \tag{5}$$

The second term on the right is controlled by Hölder's inequality

$$|\langle \theta_t, h \rangle| \leq \|h\|_r \|\theta_t\|_{r'} \leq t^{-1/r} \|\theta\|_{r'} M_2. \tag{6}$$

For the first term we take Fourier transforms and follow up by using Cauchy–Schwarz

$$|\langle \delta - \theta_t, h \rangle| \leq \|h\|_{L^2_\alpha} \left\{ \int |u|^{-2\alpha} |1 - \hat{\theta}(tu)|^2 \, du \right\}^{1/2}.$$

Since $\hat{\theta}$ is smooth at the origin and $\hat{\theta}$ decays at infinity the second factor is finite and yields

$$|\langle \delta - \theta_t, h \rangle| \leq C\, t^{-1/2+\alpha} M_1 . \tag{7}$$

Combining (5), (6) and (7) and choosing t appropriately now yields the conclusion of the lemma.

PROOF OF THEOREM 1. As in Oberlin and Stein [6] we have

$$\|T_{n-1} f\|_{L^2(L^2_{(n-1)/2})} \leq C\|f\|_{L^2}$$

and

$$\|T_{n-1} f\|_{L^\infty(L^1)} \leq C\|f\|_{L^1}.$$

Interpolating yields

$$\|T_{n-1} f\|_{L^{p'}(L^p_{(n-1)/p'})} \leq C\|f\|_{L^p}$$

for $1 \leq p \leq 2$. Equivalently

$$\|R_{-(n-1-k)}\, T_{n-1} f\|_{L^{p'}(L^p_{k-(n-1)/p})} \leq C\|f\|_{L^p} \tag{8}$$

for $1 \leq p \leq 2$. We now use the Hardy, Littlewood, Sobolev lemma in the one dimensional fibre of $M_{n,n-1}$ to obtain

$$\|R_{-(n-1-k)}\, T_{n-1} f\|_{L^{p'}(L^r)} \leq C\|f\|_p \tag{9}$$

where $1/r = n/p - k$ and for $(n-1)/k \leq p < n/k$. Next we apply Lemma 2 on each fibre — that is to the function

$$h = R_{-(n-1-k)}\, T_{n-1} f(\pi, \cdot)$$

for each $\pi \in G_{n,n-1}$ and where $\pi^\perp$ is identified to $\mathbb{R}$. The hypotheses of Lemma 2 are provided by (8) with $p = 2$ and (9) with some fixed p in the range $(n-1)/k \leq p < n/k$. Thus $\alpha = k - (n-1)/2$ and $\alpha > \frac{1}{2}$ by virtue of the hypothesis $k > n/2$.

Together with an application of Hölder's inequality on the base $G_{n,n-1}$ we obtain

$$\|R_{-(n-1-k)}\, T_{n-1} f\|_{L^{q'}(L^\infty)} \leq C\|f\|_2^\beta \|f\|_p^{1-\beta}$$

with β as in the lemma and $1/q = \beta/2 + (1-\beta)/p = k/n$. Now let $|f| \leq \mathbf{1}_E$ where E is a set of finite measure m in $\mathbb{R}^n$. Then

$$\|R_{-(n-1-k)}\, T_{n-1} f\|_{L^{n/(n-k)}(L^\infty)} \leq C\, m^{k/n}.$$

Normalized convex combinations of such functions generate the Lorentz space $L(n/k, 1)$ so that we now obtain

$$\|R_{-(n-1-k)}\, T_{n-1} f\|_{L^{n/(n-k)}(L^\infty)} \leq C\|f\|_{n/k,1}.$$

Now apply $T^*_{k,n-1}$ to $R_{-(n-1-k)}\, T_{n-1} f$ and use Christ's Theorem with $R = 1$, $P = 1$, $B = n/k$ and $A = n/(n-1)$ to control the result. Together with (0) this yields the Theorem.

We now offer a rather inadequate motivation for the contortions we have used in the proof of Theorem 1. It is instructive to observe that if we take the two estimates (see [7])

$$\|T_k f\|_{L^2(L^2_{k/2})} \leq C\|f\|_2 \tag{10}$$

and

$$\|T_k f\|_{L^\infty(L^1)} \leq \|f\|_1 \tag{11}$$

and apply the $(n - k)$-dimensional analogue of Lemma 2 directly on each fibre of $M_{n,k}$ we obtain

$$\|T_k f\|_{L^{n/(n-k)}(L^\infty)} \leq C\|f\|_{n/k,1} \ (k > n/2).$$

This statement is weaker than Theorem 1. Despite the fact that (10) is actually an equality, this is where the weakness lies. Equation (10) does not carry *enough* information.

A partial explanation of this comes from observing that (11) above and

$$\|T_k f\|_{L^{n+1}(M_{n,k})} \leq C\|f\|_{(n+1)/(k+1)}$$

are affinely invariant inequalities — see equation (4) of Drury [3]. By extrapolation Theorem 1 is also affinely invariant. This suggests that one seek an affinely invariant proof. The Hardy Littlewood Sobolev lemma is affinely invariant only in dimension 1. This indicates the virtue of working on the 1-dimensional fibre of $M_{n,n-1}$.

References

1. M. Christ, *Estimates for the k-plane transform*, Indiana J. Math. 33/6 (1984) pp. 891–910.
2. S. W. Drury, *L^p estimates for the X-ray Transform*, Illinois J. Math. 27/1 (1983) pp. 125–129.
3. S. W. Drury, *Generalizations of Riesz Potentials and L^p estimates for certain k-plane transforms*, Illinois J. Math. 28/3 (1984) pp. 495–512.
4. K. J. Falconer, *Continuity properties of k-plane integrals and Besicovitch sets*, Math. Proc. Cam. Phil. Soc. **87** (1980), pp. 221–226.
5. K. J. Falconer, *Sections of sets of zero Lebesgue measure*, Mathematika **27** (1980), pp. 90–96.
6. D. M. Oberlin and E. M. Stein, *Mapping properties of the Radon transform*, Indiana J. Math., **31** (1982), pp. 641–650.
7. R. Strichartz, *L^p estimates for Radon transforms in Euclidean and non-Euclidean spaces*, Duke J. Math. **48** (1981), pp. 699–727.

McGill University.
Montreal, Quebec