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A METRISATION THEOREM FOR PSEUDOCOMPACT SPACES

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In this paper we prove that a completely regular pseudocompact space with a quasi-regular–G_δ-diagonal is metrisable.

1. Introduction

Recently, we have considered the question of what topological properties imply metrisability in the presence of a weak diagonal property. For example, it is well-known that the existence of a quasi–G_δ–diagonal is sufficient for metrisability in countably compact spaces [7]. In [3] we proved that a manifold with a quasi–regular–G_δ–diagonal is metrisable. In this present paper, we give a diagonal condition on pseudocompact spaces to get metrisability.

A countable family $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets of a space X is called a *quasi-G_δ-diagonal* (*quasi-G_δ^*-diagonal*), if for each $x \in X$ we have $\bigcap\limits_{n\in c(x)} st(x,\mathcal{G}_n) = \{x\}$ $\Big(\bigcap\limits_{n\in c(x)} \overline{st(x,\mathcal{G}_n)} = \{x\}\Big)$ where $c(x) = \{n : x \in G \text{ for some } G \in \mathcal{G}_n\}$ and $st(x,\mathcal{G}_n)$ is the union of all sets in $\mathcal{G}_n$ which contain x.

A space X has a *quasi–regular–G_δ–diagonal* [3] if and only if there is a countable sequence $\langle U_n : n \in \mathbb{N}\rangle$ of open subsets in X^2, such that for all $(x,y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x,x) \in U_n$ but $(x,y) \notin \overline{U_n}$.

A space X is called *quasi–developable* if there is a countable family $\{\,\mathcal{G}_n : n \in \mathbb{N}\,\}$ of collections of open subsets of X such that for all $x \in X$ the nonempty sets of the form $st(x,\mathcal{G}_n)$ form a local base at x.

In this paper all spaces will be completely regular, unless we state otherwise.

2. The main results

Pseudocompact spaces were first defined and investigated by Hewitt in [4].

Definition 2.1. *A space X is pseudocompact if every real–valued continuous function on X is bounded.*

The following characterisation of pseudocompactness may be found in [2].

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Lemma 2.2. *A space X is pseudocompact if and only if for every decreasing sequence $\langle U_n : n \in \mathbb{N}\rangle$ of nonvoid open subsets of X, $\bigcap_{n\in\mathbb{N}} \overline{U_n} \neq \emptyset$.*

McArthur in [6] proved the following lemma.

Lemma 2.3. *Let X be a pseudocompact space. Suppose $\langle U_n : n \in \mathbb{N}\rangle$ is a decreasing sequence of open sets such that $\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n} = \{x\}$ for a point $x \in X$. Then the sets U_n form a local neighbourhood base at x.*

The proof of our main result relies on a metrisation theorem.

Theorem 2.4. [3] *Let X be a space with a sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ of open families such that, for each $x \in X$, $\{st^2(x, \mathcal{G}_n)\}_{n\in\mathbb{N}} - \{\emptyset\}$ (that is, the union of all sets $st(y, \mathcal{G}_n)$ with $y \in st(x, \mathcal{G}_n)$) is a local base at x. Then X is metrisable.*

Lemma 2.5. *Let X be a pseudocompact space with a quasi–G_δ^*–diagonal. Then X is quasi–developable.*

Proof: Let $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ be a quasi–G_δ^*–diagonal sequence for X. Without loss of generality we may assume that $\mathcal{V}_1 = \{X\}$. Set $c_{\mathcal{V}}(x) = \{n : st(x, \mathcal{V}_n) \neq \emptyset\}$. Then $\bigcap_{n\in c_{\mathcal{V}}(x)} \overline{st(x, \mathcal{V}_n)} = \{x\}$. Let $\mathcal{F}$ denote the set of non–empty finite subsets of $\mathbb{N}$. For each $F \in \mathcal{F}$ set

$$\mathcal{G}_F = \Big\{\bigcap_{i\in F} V_i : V_i \in \mathcal{V}_i\Big\}.$$

We show that $\{\mathcal{G}_F : F \in \mathcal{F}\}$ is a quasi–development of X. For each $n \in \mathbb{N}$, $x \in X$ put $F_n(x) = c_{\mathcal{V}}(x) \cap \{1, 2, \ldots, n\}$. Then $F_n(x) \neq \emptyset$. Note that $st\big(x, \mathcal{G}_{F_n(x)}\big) \subseteq st(x, \mathcal{V}_m)$ for each $n \in \mathbb{N}$, each $x \in X$ and each $m \in F_n(x)$. Note also that

$$\bigcap_{n\in\mathbb{N}} \overline{st\big(x, \mathcal{G}_{F_n(x)}\big)} = \bigcap_{n\in\mathbb{N}} st\big(x, \mathcal{G}_{F_n(x)}\big) = \{x\}.$$

By Lemma 2.3, $\big\{st(x, \mathcal{G}_{F_n(x)}) : n \in \mathbb{N}\big\}$ forms a local neighbourhood base at x. Hence, $\big\{st(x, \mathcal{G}_F) : F \in \mathcal{F}\big\} - \emptyset$ forms a local neighborhood base at x. ∎

Theorem 2.6. *Let X be a pseudocompact space with a quasi–regular–G_δ–diagonal. Then X is metrisable.*

Proof: By Theorem 2.4, we only need to show that X has a quasi–development $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ such that, for each $x \in X$, $\{st^2(x, \mathcal{G}_n)\}_{n\in\mathbb{N}} - \{\emptyset\}$ is a local base at x.

Let $\langle U_n : n \in \mathbb{N}\rangle$ be as in the definition of quasi–regular–G_δ–diagonal. So, the sets U_n are open in X^2 and for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U}_n$. Put $\mathcal{H}_n = \{H : H \text{ is open }, H \times H \subseteq U_n\}$. As in the proof of Lemma 2.5, let $\mathcal{F}$ denote the set of non–empty finite subsets of $\mathbb{N}$, and for $F \in \mathcal{F}$ put

$$\mathcal{G}'_F = \Big\{\bigcap_{i\in F} H_i : H_i \in \mathcal{H}_i\Big\}.$$

We show that for each $x \in X$, $\{st^2(x, \mathcal{G}'_F)\}_{F\in\mathcal{F}} - \{\emptyset\}$ is a local base at x. Take any $x \in X$. For each $n \in \mathbb{N}$ put $F_n(x) = \{i : st(x, \mathcal{H}_i) \neq \emptyset\} \cap \{1, 2, \dots, n\}$. Without loss, $\mathcal{H}_i = \{X\}$, so $F_n(x) \neq \emptyset$. We prove that $\bigcap_{n\in\mathbb{N}} \overline{st^2(x, \mathcal{G}'_{F_n(x)})} = \{x\}$.

Suppose, for a contradiction, for all $n \in \mathbb{N}$, $y \in \overline{st^2(x, \mathcal{G}'_{F_n(x)})}$ and $x \neq y$. So by the definition of quasi–regular–G_δ–diagonal, there is k such that $(x, x) \in U_k$ but $(x, y) \notin \overline{U}_k$.

By the same argument as in Lemma 2.5, we know that $\{\mathcal{G}'_F : F \in \mathcal{F}\}$ is a quasi–development of X. Therefore there exist I and $J \in \mathcal{F}$ such that

$$(x, y) \in st(x, \mathcal{G}'_I) \times st(y, \mathcal{G}'_J) \subseteq X^2 - \overline{U}_n.$$

Choose $m \geqslant \max\{I, k\}$, so that $I \subseteq F_m(x)$. It follows that $y \in \overline{st^2(x, \mathcal{G}'_{F_m(x)})}$, so $st^2(x, \mathcal{G}'_{F_m(x)}) \cap st(y, \mathcal{G}'_J) \neq \emptyset$. Then there exists $G_1, G_2 \in \mathcal{G}'_{F_m(x)}$ and $G_3 \in \mathcal{G}'_J$ such that $y \in G_3$, $x \in G_1$, $G_1 \cap G_2 \neq \emptyset$ and $G_2 \cap G_3 \neq \emptyset$. Let $z_1 \in G_1 \cap G_2$ and $z_2 \in G_2 \cap G_3$. Then $(z_1, z_2) \in (G_1 \times G_3) \cap (G_2 \times G_2)$. Now, $G_1 \in \mathcal{G}'_{F_m(x)}, G_3 \in \mathcal{G}'_J$, so $G_1 \times G_3 \subseteq st(x, \mathcal{G}'_{F_m(x)}) \times st(y, \mathcal{G}'_J)$. Also, $G_2 \in \mathcal{G}'_{F_m(x)}$ and $k \in F_m(x)$, so $G_2 \subseteq H$ for some $H \in \mathcal{H}_k$. Therefore $G_2 \times G_2 \subseteq H \times H \subseteq U_k$, so $(z_1, z_2) \in U_k$.

In other words, $(z_1, z_2) \in (G_2 \times G_3) \cap U_k \subseteq \big(st(x, \mathcal{G}'_{F_m(x)}) \times st(y, \mathcal{G}'_J)\big) \cap U_k$, and this is a contradiction. Therefore, $\bigcap_{n\in c_{\mathcal{G}'}(x)} \overline{st^2(x, \mathcal{G}'_{F_n(x)})} = \{x\}$. We conclude by Lemma 2.3 that for each $x \in X$, $\{st^2(x, \mathcal{G}'_F)\}_{F\in\mathcal{F}} - \{\emptyset\}$ is a local base at x. Hence, X is metrisable. ∎

Example 2.7. *The space $E \cap [0, 1]$ of [2, Problem 3J] is submetrisable (that is, it is a space with a coarser metric topology) pseudocompact and Hausdorff. Since the space is not completely regular, it is not metrisable.*

Example 2.8. *The Mrowka space Ψ (see [2, 1, 5]) is completely regular, pseudocompact and developable but does not have a quasi–regular–G_δ–diagonal, and hence is not metrisable.*

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