Path decompositions of digraphs

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A *path decomposition* of a digraph G (having no loops or multiple arcs) is a family of simple paths such that every arc of G lies on precisely one of the paths of the family. The *path number*, $pn(G)$ is the minimal number of paths necessary to form a path decomposition of G .

We show that $pn(G) \geq \sum_{v} \max\{0, \text{od}(v)\text{-id}(v)\}$ the sum taken over all vertices v of G , with equality holding if G is acyclic. If G is a subgraph of a tournament on n vertices we show that $pn(G) \leq [\![n^2/4]\!]$ with equality holding if G is transitive.

We conjecture that $pn(G) \leq [\![n^2/4]\!]$ for any digraph G on n vertices if n is sufficiently large, perhaps for all $n \geq 4$.

In this paper the word "digraph" will be reserved for directed graphs having neither loops nor multiple arcs. A *path decomposition* of a digraph G is a family of simple paths such that every arc of G lies on precisely one of the paths of the family. The *path number* of G is the minimal number of paths necessary to form a path decomposition of G ; it is denoted by $pn(G)$.

Counterparts of these ideas for undirected graphs were studied by Lovász [3] in response to a problem posed by Erdös (see also Erdös [1], Harary, Schwenk [2] and Stanton, Cowan and James [4]).

We obtain a lower bound for the path number of a digraph involving only the in- and out-degrees of its vertices (Theorem 1). In Theorem 3 we show that $[\![n^2/4]\!]$ is an upper bound for $pn(G)$ when G is a tournament or a subgraph of a tournament and in Corollary 4.2 we show that this bound is sharp. A formula is given in Theorem 4 expressing the path number of any acyclic digraph entirely in terms of its in- and out-degrees.

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Notation and preliminaries

The set of arcs of a digraph G will be denoted by $A(G)$, the set of vertices by $V(G)$ and vw will be the notation for the arc from the vertex v to the vertex w . For every vertex v of G , if $od(v)$ arcs begin at v and $id(v)$ arcs terminate at v we define:

$x(v) = \max\{0, od(v)\text{-}id(v)\}$ called the *excess* at v ,

$d(v) = |od(v)\text{-}id(v)|$ called the *net degree* at v , and

$\mu(v) = \max\{od(v), id(v)\}$ called the *maximum degree* at v .

The graph obtained from G by deleting a vertex v and all its adjacent arcs will be denoted by G/v . If a vertex on a path is neither the initial nor the terminal vertex of the path we will call it an *intermediate* vertex of the path. Hereafter "path" will mean *simple* path.

The following inequality is useful for estimating path numbers.

THEOREM 1. *For every digraph G* ,

(1) $$pn(G) \geq \sum_{v \in V(G)} x(v) .$$

Proof. Let P be any path decomposition of G and $P(v)$ be the set of paths belonging to P which begin at v . Suppose $x(v) > 0$. The vertex v is an intermediate vertex for at most $id(v)$ paths of P so at least $od(v) - id(v)$ paths begin at v . Therefore $|P(v)| \geq x(v)$ but $|P| = \sum_{v \in V(G)} |P(v)|$ and hence a minimal path decomposition has at least

$$\sum_{v \in V(G)} x(v) \quad \text{members.}$$

EXAMPLE 1. The digon below is an example of a digraph in which strict inequality holds in (1):
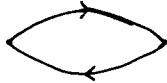
Another proposition useful in estimating the path number of a digraph is:

THEOREM 2. *If v is any vertex of an arbitrary digraph G then $pn(G) \le pn(G/v) + \mu(v)$.*

Proof. Suppose there are $t \ge 0$ digons adjacent to v . If $t \ne 1$ then the digons' arcs can be partitioned into t paths of length 2 . The remaining arcs adjacent at v (if any) can be partitioned into $\max\{od(v)-t, id(v)-t\}$ paths and hence

$$pn(G) \le pn(G/v) + t + \max\{od(v)-t, id(v)-t\}$$
$$\le pn(G/v) + \mu(v) .$$

If $t = 1$ and $od(v)$ or $id(v)$ isn't 1 it is easy to verify that $\mu(v)$ paths suffice to form a decomposition of the arcs adjacent at v . If $od(v) = id(v) = t = 1$ let w be the vertex of G/v adjacent to v and P be a minimal path decomposition of G/v . If w is an initial (or terminal) vertex of some path of P then we can augment that path by vw (or wv) to obtain a path decomposition for G having $pn(G/v) + 1$ members. If w is an intermediate vertex of some path p of P augment the part of p from its beginning to w by wv (call the resulting path r), augment the part of p from w to the end of p by vw (call the resulting path s), then $(P \sim \{p\}) \cup \{r, s\}$ is a path decomposition of G having $pn(G/v) + 1$ members.

Asymmetric digraphs

A digraph G is *asymmetric* iff wv is not an arc of G whenever vw is an arc of G for every arc vw of G . Alternatively, G is asymmetric iff there exists a tournament T such that G is a subgraph of T .

THEOREM 3. *If G is an asymmetric digraph with n vertices then*
$pn(G) \leq [\![n^2/4]\!]$.

Proof (by induction on n). It is easy to check the proposition is
true for $1 \leq n \leq 3$.

Suppose $n = 2m > 2$. If $d(\overset{\bullet}{v}) \leq 1$ for some $v \in V(G)$ then
$\mu(v) \leq m$ because G is asymmetric. Assuming inductively that

$$pn(G/v) \leq [\![(n-1)^2/4]\!]$$

we have $pn(G) \leq m^2$ by Theorem 2. If on the other hand, $d(w) \geq 2$ for
all $w \in V(G)$ we may assume that $x(w) > 0$ for at least m vertices w
of G . Let v be one of these vertices. If $od(v) \leq m$ then $\mu(v) \leq m$
and we have $pn(G) \leq m^2$ by Theorem 2 as before. Therefore we may assume
that $od(v) > m$. Notice that the vertices of G other than v having
positive excess as vertices of G have positive excess as vertices of G/v
because their net degrees (relative to G) exceed 1 . If P is a
minimal path decomposition of G/v then these vertices of positive excess
are initial vertices of paths of P . Let $k = od(v) - m$ and
$W = \{w \in V(G) : vw \in A(G)$ and $x(w) > 0\}$. W has at least k members.
Let p_w denote a path of P beginning at w and $\hat{p}_w$ be the path
obtained by augmenting p_w by vw . The arcs of G which are adjacent at
v but do not end in W can be decomposed into $id(v)$ paths of length 2
and at most $m - id(v)$ paths of length 1 . Calling this path
decomposition $P*$ we see that

$$\left(P \sim \{p_w : w \in W\}\right) \cup \{\hat{p}_w : w \in W\} \cup P*$$

is a path decomposition of G of cardinality not exceeding

$$\left(pn(G/v) - |W|\right) + |W| + m$$

and hence

$$pn(G) \leq pn(G/v) + m$$
$$\leq m^2 \text{ by our inductive assumption.}$$

A similar argument shows that $pn(G) \leq m^2 + m$ when $n = 2m + 1 > 1$
and hence

$$pn(G) \leq [\![n^2/4]\!] \ .$$

COROLLARY 3.1. *For every tournament* T ,

$$pn(T) \leq [\![n^2/4]\!] \ .$$

Later in Corollary 4.2 we will show that the path number of the transitive tournament on n vertices is $[\![n^2/4]\!]$ so the bound given by Corollary 3.1 is best possible with respect to the class of tournaments as well as the class of asymmetric digraphs. In Example 3 below we present an arc-minimal family of digraphs realizing this upper bound for each n .

Acyclic digraphs

LEMMA. *If* P *is a path decomposition of an acyclic digraph* G *and there exist distinct paths* p *and* r *of* P *such that* p *ends at the vertex where* r *begins then* P *is not minimal.*

Proof. Augment p by r to obtain a path decomposition smaller than P .

THEOREM 4. *If* G *is an acyclic digraph then* $pn(G) = \sum\limits_{v \in V(G)} x(v)$.

Proof. Suppose P is a minimal path decomposition of G . If $x(v) = 0$ then $\mathrm{id}(v) \geq \mathrm{od}(v)$ and hence if some path of P began at v then some other path ends at v . This would imply by the Lemma that P is not minimal. Therefore $P(v) = \emptyset$ whenever $x(v) = 0$. Consequently $\{P(v) : x(v) > 0\}$ partitions P and we have
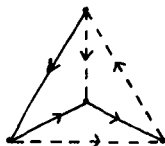
$$(2) \qquad\qquad pn(G) = \sum\limits_{v \in V(G)} |P(v)| \ .$$

Now suppose $x(v) > 0$. No path of P entering v ends at v otherwise as $\mathrm{od}(v) > \mathrm{id}(v)$, some other path of P would begin at v which is impossible by the Lemma. Therefore v is an intermediate vertex for $\mathrm{id}(v)$ paths and hence $\mathrm{od}(v) - \mathrm{id}(v)$ paths begin at v . Therefore $|P(v)| = x(v)$ for all $v \in V(G)$. Applying (2) we have

$$(3) \qquad\qquad pn(G) = \sum\limits_{v \in V(G)} x(v) \ .$$

EXAMPLE 2. That the converse to Theorem 4 is false is shown by the

digraph below:

COROLLARY 4.1. *If G is an acyclic digraph on n vertices, precisely k of which have positive excess then $pn(G) \leq k(n-k)$.*

Proof. Let P denote the vertices of G of positive excess. If an arc of G joins two vertices of P then it may be removed from G without altering $\sum_{v \in V(G)} x(v)$ which is $pn(G)$ by Theorem 4. Therefore we may assume, without loss of generality, that any arc of G beginning at a vertex of P must end in a vertex which is not in P . Thus $x(v) \leq n - k$ for all $v \in P$ and hence $pn(G) \leq k(n-k)$.

This gives us an alternate proof of

COROLLARY 4.2. *If G is an acyclic digraph on n vertices then $pn(G) \leq [\![n^2/4]\!]$.*

COROLLARY 4.3. $pn(T) = [\![n^2/4]\!]$ *if T is a transitive tournament.*

EXAMPLE 3. Define a bipartite digraph B_k on n vertices as follows: let $V(B_k) = \{1, 2, \ldots, n\}$ and let $vw \in A(B_k)$ iff $1 \leq v \leq k$ and $k < w \leq n$. It follows that $x(v) = n - k$ if $1 \leq v \leq k$ and $x(v) = 0$ if $k < v \leq n$. Theorem 4 implies that $pn(B_k) = k(n-k)$.

If we take $q = [\![(n+1)/2]\!]$ then B_q is an acyclic digraph whose path number is $[\![n^2/4]\!]$. This shows that the inequalities of Corollary 4.2 and Theorem 3 are best possible. No digraph G can have fewer than $pn(G)$ arcs. Therefore B_q is an arc-minimal digraph with path number $[\![n^2/4]\!]$. In fact, for each $1 \leq k \leq [\![n^2/4]\!]$ we can exhibit an arc-minimal connected digraph on n vertices having path number k , because if $k \geq n - 1$ a suitable bipartite digraph with k arcs can always be chosen, and if $1 \leq k < n - 1$ one can orient the arcs on a hamiltonian path suitably to obtain the required path number.

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