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# Galois level and congruence ideal for $p$-adic families of finite slope Siegel modular forms

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Abstract

We consider families of Siegel eigenforms of genus 2 and finite slope, defined as local pieces of an eigenvariety and equipped with a suitable integral structure. Under some assumptions on the residual image, we show that the image of the Galois representation associated with a family is big, in the sense that a Lie algebra attached to it contains a congruence subalgebra of non-zero level. We call the Galois level of the family the largest such level. We show that it is trivial when the residual representation has full image. When the residual representation is a symmetric cube, the zero locus defined by the Galois level of the family admits an automorphic description: it is the locus of points that arise from overconvergent eigenforms for $\mathrm{GL}_2$, via a $p$-adic Langlands lift attached to the symmetric cube representation. Our proof goes via the comparison of the Galois level with a 'fortuitous' congruence ideal. Some of the $p$-adic lifts are interpolated by a morphism of rigid analytic spaces from an eigencurve for $\mathrm{GL}_2$ to an eigenvariety for $\mathrm{GSp}_4$, while the remainder appear as isolated points on the eigenvariety.

## 1. Introduction

Drawing inspiration from earlier work of Hida and Lang, the paper [CIT16] studied the image of the Galois representations associated with $p$-adic families of modular forms, more precisely eigenforms of finite slope for the action of a Hecke algebra unramified outside of a fixed tame level. Such a family is defined by equipping a local piece of the eigencurve of the given tame level with an integral structure. A result of [CIT16] states that the Galois representation attached to a family has big image in the following sense: there is a ring $\mathbb{B}$ and a Lie subalgebra $\mathfrak{G}$ of $\mathfrak{gl}_2(\mathbb{B})$ attached to $\mathrm{Im}\,\rho$, in a meaningful way, such that $\mathfrak{G}$ contains $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B})$ for a non-zero ideal $\mathfrak{l}$ of $\mathbb{B}$. This can be seen as an analogue, for a $p$-adic family, of a classical result of Ribet and Momose on the image of the $p$-adic Galois representation attached to a classical eigenform [Rib75, Mom81]. We call the Galois level of the family the largest ideal $\mathfrak{l}$ with the above property. The arguments in [CIT16] rely heavily on the work of Hida and Lang for ordinary families [Hid15, Lan16], in particular on the study by Lang of the conjugate self-twists of the Galois representations attached to families. A new ingredient in the positive slope case is relative Sen theory, that replaces ordinarity in some crucial steps. Another result of [CIT16] is an automorphic description of the Galois level of a family: the geometric points of its zero locus are the $p$-adic CM points of the family. This is also a generalization of a theorem of Hida in the ordinary case. The proof

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goes via the comparison of the Galois level with a fortuitous congruence ideal, that encodes the information on the CM specializations of the family. We call this ideal 'fortuitous' because, in contrast to what happens in the ordinary case, the CM specializations of a non-CM family do not correspond to congruences with CM families, that do not exist when the slope is positive.

In this paper we find analogous results for $p$-adic families of Siegel modular forms of genus 2 and finite slope. We think that our work in this setting shows that the big image properties of Galois representations and their relations to congruences are part of a picture that can be extended to more general reductive groups. We remark that Hida and Tilouine already have some results for ordinary $p$-adic families of $\mathrm{GSp}_4$-eigenforms that are residually of 'twisted Yoshida type' [HT15]. Their arguments rely on the Galois ordinarity of the families and on $R = T$ results, both of which are not available when the slope is positive. They obtain congruences between families that are lifts from $\mathrm{GL}_{2/F}$, for a quadratic field $F$, and families that are not; their congruence ideals are then traditional and not fortuitous. In light of the results of the present paper, we think that fortuitous congruences should be regarded as general phenomena, that appear whenever we consider families of eigenforms for a reductive group that arise as $p$-adic Langlands lifts from a group of smaller rank.

The paper can be divided in two parts. In the first part (§§ 1–8), we define two-parameter families of $\mathrm{GSp}_4$-eigenforms of finite slope and we attach Galois representations to them; we then prove that the image of these representations is big in a Lie theoretic sense, assuming that the residual representation is either of full image or a symmetric cube. In the second part (§§ 9–11), we prove that the size of the Galois representation attached to a two-parameter family is related to the congruences of the family with lifts of eigenforms for a smaller group, constructed via a $p$-adic Langlands transfer. In the first half, we need to solve many technical problems when passing from genus 1 to genus 2, whereas the second half is substantially different from its genus 1 counterpart. We present our results and arguments in more detail in the following.

Fix a prime $p$ and an integer $M$ not divisible by $p$. Let $\mathcal{H}_2^M$ be an abstract Hecke algebra unramified outside $Mp$ and of Iwahoric level at $p$. In their paper [AIP15], Andreatta, Iovita and Pilloni constructed a rigid analytic object $\mathcal{D}_2$, that we call the $\mathrm{GSp}_4$-eigenvariety, and a map from $\mathcal{H}_2^M$ to the ring of analytic functions on $\mathcal{D}_2$, interpolating the systems of Hecke eigenvalues associated with the $p$-stabilized Siegel modular forms of genus 2 and tame level $M$. The eigenvariety $\mathcal{D}_2$ is equipped with a map to the two-dimensional weight space $\mathcal{W}_2$, that is the rigid analytic space associated with the formal scheme $\mathrm{Spf}\, \mathbb{Z}_p[[(\mathbb{Z}_p^\times)^2]]$ by Berthelot's construction [deJ95, § 7]. For our purposes, it is important that families be defined integrally, so we cannot work globally on irreducible components of the eigenvariety. We consider instead an admissible domain $D_h$ on $\mathcal{D}_2$ consisting of the points of slope bounded by a rational number $h$ and of weight in a wide open disc in the weight space. If the radius of this disc is sufficiently small with respect to $h$, the restriction of the weight map to $D_h$ is a finite map thanks to a result of Bellaïche (Proposition 4.1). A suitably chosen integral structure on the weight disc induces an integral structure on $D_h$. This means that we can define a local profinite ring $\mathbb{I}^\circ$ and a map $\mathcal{H}_2^M \to \mathbb{I}^\circ$ that interpolates the systems of Hecke eigenvalues of the classical eigenforms appearing in $D_h$. An argument by Chenevier gives a Galois pseudocharacter on $D_h$, that we lift to a representation $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ)$ (Lemma 4.7). We define the 'conjugate self-twists' of $\rho$ as automorphisms of $\mathbb{I}^\circ$ that induce an isomorphism of $\rho$ with one of its twists by a Dirichlet character (Definition 5.1). We write $\mathbb{I}_0^\circ$ for the subring of elements of $\mathbb{I}^\circ$ fixed by all the conjugate self-twists. We define a certain completion $\mathbb{B}$ of $\mathbb{I}_0^\circ[1/p]$ and a Lie subalgebra $\mathrm{Lie}(\mathrm{Im}\,\rho)$ of $\mathfrak{gsp}_4(\mathbb{B})$ attached to $\mathrm{Im}\,\rho$ (see § 7.1). We assume that $\rho$ is $\mathbb{Z}_p$-regular (Definition 3.10) and that the residual representation $\overline{\rho}$ is either full or of symmetric cube type (Definition 3.11). Our first main result is the following.

Theorem 1.1 (Theorem 8.1). *There exists a non-zero ideal $\mathfrak{l}$ of $\mathbb{B}$ such that $\mathfrak{l}\cdot\mathfrak{sp}_4(\mathbb{B}) \subset \mathrm{Lie}(\mathrm{Im}\,\rho)$.*

We call the *Galois level* of the family the largest ideal $\mathfrak{l}$ satisfying the inclusion of Theorem 1.1. We give here a summary of the proof of Theorem 1.1. We first show that, under our assumptions on $\overline{\rho}$, there exists a classical weight such that $\rho$ specializes to a representation with big image at all points of this weight appearing on the family (Theorem 3.12). Here we need the recent classicality result contained in [BPS16, Theorem 5.3.1]. Another essential ingredient is a result of Pink (Theorem 3.13) that we use to show that the representation associated with a $\mathrm{GSp}_4$-eigenform that is not a lift from a smaller group has big image with respect to the ring fixed by its conjugate self-twists. This is an analogue of the result of Ribet and Momose for $\mathrm{GL}_2$-eigenforms. We prove some results that are needed in the second part of the paper (in the proof of Theorem 1.2) and that yield as a corollary the fact that a form which is not a lift satisfies the assumptions of Pink's theorem (Corollary 3.9).

Once a classical weight with the desired properties is chosen, we follow a strategy of Lang to obtain some information on the image of $\rho$. As a first step we need to show that a big image result holds for the product of the specializations of $\rho$ of a given weight, rather than just for a single one (Proposition 6.10). The argument here relies on Goursat's lemma and on the classification of subnormal subgroups of symplectic groups by Tazhetdinov. Afterwards, we use the result of the first step to construct some non-trivial unipotent elements in the image of $\rho$. To do this, we need to prove an analogue of [Lan16, Theorem 3.1] that allows us to lift the conjugate self-twists of the specializations of $\rho$ at our chosen weight to conjugate self-twists of $\rho$ itself. The arguments of Lang about the lifting of the conjugate self-twists to automorphisms of a suitable deformation ring can be translated into the genus 2 case with little effort, but descending to a conjugate self-twist of the family requires some specific ingredients. Precisely, we prove that we can twist a family of $\mathrm{GSp}_4$-eigenforms by a Dirichlet character to obtain a new family (Lemma 5.8) and we rely on the étaleness of the eigenvariety above our chosen weight.

In § 7, we show how the relative Sen theory of [CIT16, § 5] can be extended to the group $\mathrm{GSp}_4$, to associate a Sen operator with $\rho$. The eigenvalues of this operator are given explicitly by the interpolation of the Hodge–Tate weights of the classical specializations of the family (Proposition 7.12). The exponential of the Sen operator induces by conjugation a structure of $\mathbb{Z}_p[[T_1,T_2]]$-Lie algebra on $\mathrm{Lie}(\mathrm{Im}\,\rho)$, so that the special elements we constructed generate a non-trivial congruence subalgebra. This proves Theorem 1.1.

When $\overline{\rho}$ has full image, the Galois level of the family is trivial (Corollary 11.5), so the main focus of the rest of the paper is the case where $\overline{\rho}$ is a symmetric cube. We can give two definitions of a *symmetric cube locus* on the eigenvariety: an automorphic definition, as the locus of points whose system of Hecke eigenvalues is obtained from that of an overconvergent $\mathrm{GL}_2$-eigenform via a symmetric cube morphism of Hecke algebras; and a Galois definition, as the locus of points whose Galois representation is the symmetric cube of that associated with an overconvergent $\mathrm{GL}_2$-eigenform. An important result is the following.

Theorem 1.2 (Theorem 10.1). *The automorphic and Galois definitions of the symmetric cube locus are equivalent.*

Theorem 1.2 plays an essential role in describing the Galois level of the family by automorphic means. Note that this result and its role in our work are completely new with respect to the genus 1 case: there the only possible congruences are of CM type and it is trivial to see that a point of small Galois image, contained in the normalizer of a torus, is a $p$-adic CM point (see [CIT16, Remark 3.11]).

The proof of Theorem 1.2 goes via the theory of $(\varphi,\Gamma)$-modules. It is known from the work of Kisin and Emerton that a two-dimensional $p$-adic representation of $G_{\mathbb{Q}}$ is associated with an overconvergent $\mathrm{GL}_2$-eigenform, up to a twist, if and only if it is trianguline. Thanks to the recent work of Kedlaya, Pottharst and Xiao on triangulations over eigenvarieties, we know that the 'only if' part also holds for overconvergent $\mathrm{GSp}_4$-eigenforms (Theorem 3.2). We combine their results with some arguments of Di Matteo [DiM13b] that relate the triangulinity of a representation to that of its symmetric cube, giving the desired result. As a corollary we deduce that if a $p$-old point of symmetric cube type of $\mathcal{D}_2^M$ is classical, then it is obtained from a classical point of an eigencurve for $\mathrm{GL}_2$, via the classical Langlands lift attached to the symmetric cube representation by Kim and Shahidi [KS02].

We study further the symmetric cube locus and show that it is Zariski-closed with zero- and one-dimensional irreducible components. The one-dimensional part of the locus can be constructed as the image of a morphism from an eigencurve for $\mathrm{GL}_2$, of a suitable tame level, to $\mathcal{D}_2^M$. This morphism is obtained by interpolating $p$-adically the classical symmetric cube Langlands lift, via an argument that goes back to Chenevier's work on the $p$-adic Jacquet–Langlands correspondence [Che05] and is now quite standard. We assume the existence of this morphism (Proposition 9.11) and refer to [Con16, §§ 3.6–3.9] for a construction of the morphism that relies on the results of [BC09, § 7.2.3].

The zero-dimensional components of the symmetric cube locus are given by isolated $p$-adic Langlands lifts, that cannot be interpolated owing to the fact that their slopes do not vary analytically. The appearance of such points is related to the existence of more than one crystalline period for the corresponding Galois representation (Proposition 3.4).

Restricting once again our attention to a local piece of the eigencurve describing a family, we define a *symmetric cube congruence ideal* that measures the locus of symmetric cube specializations of the family (Definition 11.1). We call it a *fortuitous* congruence ideal: since there are no two-parameter families of symmetric cube type, the congruences detected by this ideal are symmetric cube specializations of a family that is not globally a symmetric cube. Thanks to Theorem 1.2, that serves as a bridge between the automorphic and Galois sides, we can relate the congruence ideal with the Galois level of the family.

THEOREM 1.3 (Theorem 11.4). *The sets of prime divisors of the Galois level and of the symmetric cube congruence ideal coincide outside of a finite and explicit bad locus.*

We think that the results of this paper can be generalized by allowing for different residual representations, hence different types of congruences, or by replacing $\mathrm{GSp}_4$ by other reductive groups for which an eigenvariety has been constructed. We hope to come back to this problem in a later work.

*Notation.* We fix some notation and conventions. In the text, $p$ will always denote a prime number strictly larger than three. Most arguments work for every odd $p$; we specify when this is not sufficient. We choose algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively. If $K$ is a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$, we denote by $G_K$ its absolute Galois group. We equip $G_K$ with its profinite topology. We denote by $\mathcal{O}_K$ the ring of integers of $K$. If $K$ is local, we denote by $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$. For every prime $p$, we fix an embedding $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, identifying $G_{\mathbb{Q}_p}$ with a decomposition group of $G_{\mathbb{Q}}$. We fix a valuation $v_p$ on $\overline{\mathbb{Q}}_p$ normalized so that $v_p(p) = 1$. It defines a norm given by $|\cdot| = p^{-v_p(\cdot)}$. We denote by $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to this norm.

All rigid analytic spaces will be considered in the sense of Tate (see [BGR84, Part C]). Let $K/\mathbb{Q}_p$ be a field extension and let $X$ be a rigid analytic space over $K$. We denote by $\mathcal{O}(X)$ the $K$-algebra of rigid analytic functions on $X$, and by $\mathcal{O}(X)^\circ$ the $\mathcal{O}_K$-subalgebra of functions of norm bounded by 1 (we often say 'functions bounded by 1' meaning that they are bounded in norm). When $f\colon X \to Y$ is a map of rigid analytic spaces, we denote by $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ the map induced by $f$. There is a Grothendieck topology on $X$, called the Tate topology; we refer to [BGR84, Proposition 9.1.4/2] for the definition of its admissible open sets and admissible coverings. There is a notion of irreducible components for $X$; see [Con99] for the details. We say that $X$ is equidimensional of dimension $d$ if all its irreducible components have dimension $d$.

We say that $X$ is *wide open* if there exists an admissible covering $\{X_i\}_{i\in\mathbb{N}}$ of $X$ by affinoid domains $X_i$ such that, for every $i$, $X_i \subset X_{i+1}$ and the map $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ induced by the previous inclusion is compact.

Let $S$ be any subset of $X(\mathbb{C}_p)$. We say that $S$ is:

(i) a discrete subset of $X(\mathbb{C}_p)$ if $S \cap A$ is a finite set for any open affinoid $A \subset X(\mathbb{C}_p)$;

(ii) a Zariski-dense subset of $X(\mathbb{C}_p)$ if, for every $f \in \mathcal{O}(X)$ vanishing at every point of $S$, $f$ is identically zero;

(iii) an accumulation subset of $X(\mathbb{C}_p)$ if for every $x \in S$ there exists a basis $\mathcal{B}$ of affinoid neighborhoods of $x$ in $X$ such that for every $A \in \mathcal{B}$ the set $S \cap A(\mathbb{C}_p)$ is Zariski-dense in $A$ (this term is borrowed from [BC09, § 3.3.1]).

We denote by $\mathbb{A}^d$ the $d$-dimensional rigid analytic affine space over $\mathbb{Q}_p$. Given a point $x \in \mathbb{A}^d(\mathbb{C}_p)$ and $r \in p^{\mathbb{Q}}$, we denote by $B_d(x,r)$ the $d$-dimensional closed disc of centre $x$ and radius $r$. It is an affinoid domain defined over $\mathbb{C}_p$. We denote by $B_d(x,r^-)$ the $d$-dimensional wide open disc of centre $x$ and radius $r$, defined as the rigid analytic space over $\mathbb{C}_p$ given by the increasing union of the $d$-dimensional affinoid discs of centre $x$ and radii $\{r_i\}_{i\in\mathbb{N}}$ with $r_i < r$ and $\lim_{i\mapsto+\infty} r_i = r$.

For every $n \geqslant 1$ we denote by $\mathbb{1}_n$ the $n \times n$ unit matrix. Let $g \geqslant 1$ be an integer and let $s$ be the $g \times g$ antidiagonal unit matrix $(\delta_{i,n-i}(i,j))_{1\leqslant i,j\leqslant g}$. Let $J_g$ be the $2g \times 2g$ matrix $\left(\begin{smallmatrix} 0 & s \\ -s & 0 \end{smallmatrix}\right)$. We denote by $\mathrm{GSp}_{2g}$ the algebraic group of symplectic similitudes for $J_g$, defined over $\mathbb{Z}$; for every ring $R$ the $R$-points of this group are given by

$$\mathrm{GSp}_{2g}(R) = \{A \in \mathrm{GL}_4(R) \,|\, \exists\, \nu(A) \in R^\times \text{ s.t. } {}^tAJA = \nu(A)J\}.$$

The map $A \to \nu(A)$ defines a character $\nu\colon\ \mathrm{GSp}_4(R) \to R^\times$. We refer to $\nu$ as the similitude factor and we set $\mathrm{Sp}_{2g}(R) = \{A \in \mathrm{GSp}_{2g}(R) \,|\, \nu(A) = 1\}$.

We denote by $B_g$ the Borel subgroup of $\mathrm{GSp}_{2g}$ such that for every ring $R$ the $R$-points of $B_g$ are the upper triangular matrices in $\mathrm{GSp}_{2g}(R)$. We let $T_g$ be the maximal torus such that for every ring $R$ the $R$-points of $T_g$ are the diagonal matrices in $\mathrm{GSp}_{2g}(R)$. We write $U_g$ for the unipotent radical of $B_g$. We have $B_g = T_gU_g$. We will always speak of weights and roots for $\mathrm{GSp}_{2g}$ with respect to the previous choice of Borel subgroup and torus. For every root $\alpha$, we denote by $U^\alpha$ the corresponding one-parameter unipotent subgroup of $\mathrm{GSp}_{2g}$. For every prime $\ell$, we write $I_{g,\ell}$ for the Iwahori subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ corresponding to our choice of Borel subgroup, and we define some compact open subgroups of $\mathrm{GSp}_{2g}(\mathbb{A}_{\mathbb{Q}})$ by:

(i) $\Gamma^{(g)}(\ell^n) = \{h \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) \,|\, h_\ell \cong \mathbb{1}_{2g} \pmod{\ell^n}\}$;

(ii) $\Gamma_1^{(g)}(\ell^n) = \{h \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) \,|\, h_\ell \pmod{\ell^n} \in U_g(\mathbb{Z}/\ell^n\mathbb{Z})\}$;

(iii) $\Gamma_0^{(g)}(\ell^n) = \{h \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) \,|\, h_\ell \pmod{\ell^n} \in B_g(\mathbb{Z}/\ell^n\mathbb{Z})\}$.



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Let $N$ be an arbitrary positive integer. Write $N = \prod_i \ell_i^{n_i}$ for some distinct primes $\ell_i$ and some $n_i \in \mathbb{N}$. For a positive integer $N$ factoring as $\prod_i \ell_i^{n_i}$, we set $\Gamma_?^{(g)}(N) = \bigcap_i \Gamma_?^{(g)}(\ell_i^{n_i})$ for $? = \emptyset, 0, 1$. For $g = 1$, we will omit the upper index (1).

**List of notation**

In addition to the notation described above, we give the references to the pages where the following notation is introduced.


$\mathcal{W}_g, \mathcal{W}_g^\circ$ ... 781
$\Lambda_g$ ... 782
$\mathcal{H}_g^{Np}, \mathcal{H}_g^N$ ... 782
$\mathcal{D}_g^N$ ... 783
$\mathcal{D}_g^{N,h}$ ... 791
$A_{r_i}^\circ, A_{r_i} = A_{r_i}^\circ[p^{-1}]$ ... 792
$B_{g,h}$ ... 792
$\Lambda_{g,h}, \Lambda_h \coloneqq \Lambda_{2,h}$ ... 792
$\mathcal{D}_{g,h}^N$ ... 793
$\mathbb{T}_{g,h}, \mathbb{T}_h \coloneqq \mathbb{T}_{2,h}$ ... 793
$\mathbb{I}^\circ$ ... 793
$\mathbb{I}_{\mathrm{Tr}}^\circ$ ... 794
$\mathbb{I}_0^\circ, \mathbb{I}_0 = \mathbb{I}_0^\circ[p^{-1}]$ ... 795
$\mathbb{I}_{r_i,0}^\circ, \mathbb{I}_{r_i,0} = \mathbb{I}_{r_i,0}^\circ[p^{-1}]$ ... 806
$H_{r_i}, H_{r_i,p}, K_{H_{r_i},p}$ ... 807
$\mathbb{B}_r$ ... 808
$\mathfrak{G}_r, \mathfrak{G}_r^{\mathrm{loc}}$ ... 808
$\phi_r, \phi_r'$ ... 811
$\Phi_{\mathbb{B}_r}$ ... 811
$\mathfrak{l}_\theta$ ... 814
$\lambda^{Np} \colon \mathcal{H}_2^{Np} \to \mathcal{H}_1^{Np}, \lambda_1 \colon \mathcal{H}_2^N \to \mathcal{H}_1^N$ ... 817
$\iota \colon \mathcal{W}_1^\circ \to \mathcal{W}_2^\circ$ ... 820
$\mathcal{I}_{\mathrm{Sym}^3}, \mathcal{D}_{2,\mathrm{Sym}^3}^M$ ... 824
$\mathfrak{c}_\theta$ ... 825


## 2. Preliminaries on eigenvarieties

In this section, we define the basic objects we are going to work with: weight spaces, Hecke algebras and eigenvarieties. We recall some of their properties.

### 2.1 The weight spaces

We choose once and for all $u = 1+p$ as a generator of $\mathbb{Z}_p^\times$. This choice determines an isomorphism $\mathbb{Z}_p^\times \cong (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$. Let $g$ be a positive integer. Consider the Iwasawa algebra $\mathbb{Z}_p[[(\mathbb{Z}_p^\times)^g]]$. A construction by Berthelot [deJ95, § 7] attaches to the formal scheme $\operatorname{Spf} \mathbb{Z}_p[[(\mathbb{Z}_p^\times)^g]]$ a rigid analytic space that we denote by $\mathcal{W}_g$ .If $A$ is a $\mathbb{Q}_p$-algebra, the $A$-points of $\mathcal{W}_g$ are the continuous characters $(\mathbb{Z}_p^\times)^g \to A^\times$. Denote by $(\mathbb{Z}/(p-1)\mathbb{Z})^{\wedge,g}$ the group of characters of $(\mathbb{Z}/(p-1)\mathbb{Z})^g$. The following map gives an isomorphism from $\mathcal{W}_g$ to a disjoint union of $g$-dimensional open discs $B_g(0, 1^-)$ indexed by $(\mathbb{Z}/(p-1)\mathbb{Z})^{\wedge,g}$:



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$$\eta_g\colon \mathcal{W}_g \to (\mathbb{Z}/(p-1)\mathbb{Z})^{\wedge,g} \times B_g(0,1^-),$$
$$\kappa \mapsto (\kappa|_{(\mathbb{Z}/(p-1)\mathbb{Z})^g}, (\kappa(u,1,\dots,1)-1, \kappa(1,u,1,\dots,1)-1,\dots,\kappa(1,\dots,1,u)-1)).$$

We denote by $\mathcal{W}_g^\circ$ the connected component of $\mathcal{W}_g$ that maps to $0 \in (\mathbb{Z}/(p-1)\mathbb{Z})^g$. We write $\Lambda_g$ for the algebra $\mathbb{Z}_p[[T_1,T_2,\dots,T_g]]$ of formal series in $g$ variables over $\mathbb{Z}_p$. It is the ring of rigid analytic functions bounded by 1 on a connected component of the weight space. The weight space $\mathcal{W}_g$ carries a universal character $\kappa_{\mathcal{W}_g}\colon \mathbb{Z}_p^\times \to \mathbb{Z}_p[[(\mathbb{Z}_p^\times)^g]]^\times$.

We call *arithmetic primes* the primes of $\Lambda_g$ of the form $P_{\underline{k}} = (1+T_1-u^{k_1}, 1+T_2-u^{k_2},\dots, 1+T_g-u^{k_g})$ for a $g$-tuple of integers $\underline{k}=(k_1,k_2,\dots,k_g)$ (in the usual definition an auxiliary character can appear, but we will never need it). We say that a $\mathbb{Q}_p$-point $\kappa\colon \mathbb{Z}_p^\times \to \mathbb{Q}_p^\times$ of $\mathcal{W}_g^\circ$ is *classical* if it is the specialization of the universal character of $\mathcal{W}_g$ at $P_{\underline{k}}$, for some $\underline{k}\in\mathbb{Z}^g$.

### 2.2 The abstract Hecke algebras

Let $\ell$ be a prime. Let $G$ be a $\mathbb{Z}$-subgroup scheme of $\mathrm{GSp}_{2g}$ and let $K \subset G(\mathbb{Q}_\ell)$ be a compact open subgroup. For $\gamma \in G(\mathbb{Q}_\ell)$ we denote by $\mathbb{1}([K\gamma K])$ the characteristic function of the double coset $[K\gamma K]$. Let $\mathcal{H}(G(\mathbb{Q}_\ell),K)$ be the $\mathbb{Q}$-algebra generated by the functions $\mathbb{1}([K\gamma K])$ for $\gamma \in G(\mathbb{Q}_\ell)$, equipped with the convolution product. We call *spherical (or unramified) Hecke algebra of* $\mathrm{GSp}_{2g}$ *at* $\ell$ the $\mathbb{Q}$-algebra $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),\mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$. It is generated by the elements $T_{\ell,i}^{(g)} = \mathbb{1}([\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)\,\mathrm{diag}(\mathbb{1}_i, \ell\mathbb{1}_{2g-2i}, \ell^2\mathbb{1}_i)\,\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)])$, for $i=0,1,\dots g$, and $(T_{\ell,0}^{(g)})^{-1}$.

The Hecke algebra $\mathcal{H}(T_g(\mathbb{Q}_\ell),T_g(\mathbb{Z}_\ell))$ carries a natural action of the Weyl group $W_g = \mathscr{S}_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$ of $\mathrm{GSp}_{2g}$, where $\mathscr{S}_g$ is the group of permutations of $\{1,2,\dots,g\}$: on an element $\mathrm{diag}(\nu t_1,\dots,\nu t_g, t_g^{-1},\dots,t_1^{-1})$ of the torus, $\mathscr{S}_g$ acts by permuting the $t_i$ and the non-trivial element in each $\mathbb{Z}/2\mathbb{Z}$ sends $t_i$ to $t_i^{-1}$. We denote the action of $w\in W_g$ on $t\in T(\mathbb{Q}_\ell)$ by $t\mapsto w.t$. Via the twisted Satake transform, the algebra $\mathcal{H}(T_g(\mathbb{Q}_\ell),T_g(\mathbb{Z}_\ell))$ obtains a structure of Galois extension of $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),\mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$. Its Galois group is $W_g$.

For $i=0,1,\dots,g$, let $t_{\ell,i}^{(g)} = \mathbb{1}([\mathrm{diag}(\mathbb{1}_i,\ell\mathbb{1}_{2g-2i},\ell^2\mathbb{1}_i)\mathbb{T}_g(\mathbb{Z}_\ell)])$. Note that $t_{\ell,0}^{(g)} = S_{\mathrm{GSp}_{2g}}^{T_g}(T_{\ell,0}^g)$. The set $(t_{\ell,i}^{(g)})_{i=1,\dots,g}$ generates the extension $\mathcal{H}(T_g(\mathbb{Q}_\ell),T_g(\mathbb{Z}_\ell))$ over $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),\mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$.

We call an element $\gamma\in T_g(\mathbb{Z}_\ell)$ dilating if $v_p(\alpha(\gamma))\leqslant 0$ for every positive root $\alpha$. Let $T_g(\mathbb{Z}_\ell)^-$ be the subset of $T_g(\mathbb{Z}_\ell)$ consisting of dilating elements and let $\mathcal{H}(T_g(\mathbb{Q}_\ell),T_g(\mathbb{Z}_\ell))^-$ be the $\mathbb{Q}$-subalgebra of $\mathcal{H}(T_g(\mathbb{Q}_\ell),T_g(\mathbb{Z}_\ell))$ generated by the functions $\mathbb{1}([\gamma T_g(\mathbb{Z}_\ell)])$ with $\gamma\in T_g(\mathbb{Q}_\ell)^-$. The functions $\mathbb{1}([\gamma T_g(\mathbb{Z}_\ell)])$ with $\gamma\in T_g(\mathbb{Q}_\ell)^-$ also form a basis of $\mathcal{H}(T_g(\mathbb{Q}_\ell),T_g(\mathbb{Z}_\ell))^-$ as a $\mathbb{Q}$-vector space.

Let $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),I_{g,\ell})^-$ be the subalgebra of $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),I_{g,\ell})$ generated by the functions $\mathbb{1}([I_{g,\ell}\gamma I_{g,\ell}])$ with $\gamma\in T(\mathbb{Z}_\ell)^-$. We call $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),I_{g,\ell})^-$ the *dilating Iwahori–Hecke algebra* at $\ell$. It is generated by the elements $U_{\ell,i}^{(g)} = \mathbb{1}([I_{g,\ell}\,\mathrm{diag}(\mathbb{1}_i,\ell\mathbb{1}_{2g-2i},\ell^2\mathbb{1}_i)I_{g,\ell}])$, for $i=0,1,\dots,g$, and $(U_{\ell,0}^{(g)})^{-1}$.

We define a morphism of $\mathbb{Q}$-algebras $\iota_{I_{g,\ell}}^{T_g}\colon \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),I_{g,\ell})^- \to \mathcal{H}(T_g(\mathbb{Q}_\ell),T_g(\mathbb{Z}_\ell))^-$ by sending $\mathbb{1}(I_{g,\ell}\gamma I_{g,\ell})$ to $\mathbb{1}(T_g(\mathbb{Z}_\ell)\gamma T_g(\mathbb{Z}_\ell))$ for every $\gamma\in T(\mathbb{Z}_\ell)^-$. The map $\iota_{I_{g,\ell}}^{T_g}$ is an isomorphism; this can be proved as [BC09, Proposition 6.4.1].

Let $p$ be a prime and $N$ be a positive integer such that $(N,p)=1$. Set

$$\mathcal{H}_g^{Np} = \bigotimes_{\mathbb{Q},\ell\nmid Np} \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)) \tag{1}$$
$$\mathcal{H}_g^{N} = \mathcal{H}_g^{Np} \otimes_{\mathbb{Q}} \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_p),I_{g,p})^-. \tag{2}$$



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We call $\mathcal{H}_g^N$ the *abstract Hecke algebra spherical outside $N$ and Iwahoric dilating at $p$*. With an abuse of notation, we will consider the elements of one of the local algebras as elements of $\mathcal{H}_g^N$ by tensoring with 1 at all the other primes.

2.2.1 *The Hecke polynomials.* We record here some explicit formulas for the minimal polynomials $P_{\min}(t_{\ell,i}^{(g)};X)$ of the elements $t_{\ell,i}^{(g)}$ over $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell),\mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$ when $g$ is 1 or 2.

For $g=1$, the element $t_{\ell,1}^{(1)}=\mathbb{1}([\mathrm{diag}(1,\ell)T_1(\mathbb{Z}_\ell)])$ generates the degree two extension $\mathcal{H}(T_1(\mathbb{Q}_\ell),T_1(\mathbb{Z}_\ell))$ of $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell),\mathrm{GL}_2(\mathbb{Z}_\ell))$. Let $w$ be the only non-trivial element of the Weyl group of $\mathrm{GL}_2$. The minimal polynomial of $t_{\ell,1}^{(1)}$ is $P_{\min}(t_{\ell,1}^{(1)})(X)=(X-t_{\ell,1}^{(1)})(X-(t_{\ell,1}^{(1)})^w)$. An explicit calculation gives

$$P_{\min}(t_{\ell,1}^{(1)};X)=(X-t_{\ell,1}^{(1)})(X-(t_{\ell,1}^{(1)})^w)=X^2-T_\ell^{(1)}X+\ell T_{\ell,0}^{(1)}. \tag{3}$$

For $g=2$, the degree eight extension $\mathcal{H}(T_2(\mathbb{Q}_\ell),T_2(\mathbb{Z}_\ell))$ over $\mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_\ell),\mathrm{GSp}_4(\mathbb{Z}_\ell))$ is generated by $t_{\ell,1}^{(2)}=\mathbb{1}([\mathrm{diag}(1,\ell,\ell,\ell^2)T_2(\mathbb{Z}_\ell)])$ and $t_{\ell,2}^{(2)}=\mathbb{1}([\mathrm{diag}(1,1,\ell,\ell)T_2(\mathbb{Z}_\ell)])$. Each of them has an orbit of order four under the action of the Weyl group. If $t=\mathrm{diag}(\nu t_1,\nu t_2,t_1^{-1},t_2^{-1})$ is an element of the torus we denote by $w_0$, $w_1$, $w_2$ the generators of the Weyl group satisfying $t^{w_0}=\mathrm{diag}(\nu t_2,\nu t_1,t_2^{-1},t_1^{-1})$, $t^{w_1}=\mathrm{diag}(\nu t_1^{-1},\nu t_2,t_1,t_2^{-1})$, $t^{w_2}=\mathrm{diag}(\nu t_1,\nu t_2^{-1},t_1^{-1},t_2)$. Note that $t_{\ell,2}^{(2)}$ is invariant under $w_0$. The calculation in the proof of [And87, Lemma 3.3.35] gives

$$\begin{aligned}P_{\min}(t_{\ell,2}^{(2)};X)&=(X-t_{\ell,2}^{(2)})(X-(t_{\ell,2}^{(2)})^{w_1})(X-(t_{\ell,2}^{(2)})^{w_2})(X-(t_{\ell,2}^{(2)})^{w_1w_2})\\&=X^4-T_{\ell,2}^{(2)}X^3+((T_{\ell,2}^{(2)})^2-T_{\ell,1}^{(2)}-\ell^2T_{\ell,0}^{(2)})X^2-\ell^3T_{\ell,2}^{(2)}T_{\ell,0}^{(2)}X+\ell^6(T_{\ell,0}^{(2)})^2.\end{aligned} \tag{4}$$

2.2.2 *Normalized systems of Hecke eigenvalues.* Let $f$ be a classical $\mathrm{GSp}_{2g}$-eigenform of level $\Gamma_1(N)\cap\Gamma_0(p)$ and weight $\underline{k}=(k_1,k_2,\ldots,k_g)$. Let $\chi\colon\mathcal{H}_g^N\to\overline{\mathbb{Q}}_p$ be the system of Hecke eigenvalues associated with $f$.

Definition 2.1. For $g\in\{1,2\}$, let $\chi^{\mathrm{norm}}\colon\mathcal{H}_g^N\to\overline{\mathbb{Q}}_p$ be the character defined by:

- $\chi^{\mathrm{norm}}|_{\mathcal{H}_g^{Np}}=\chi|_{\mathcal{H}_g^{Np}}$;
- $\chi^{\mathrm{norm}}(U_{p,i}^{(g)})=p^{-\sum_{j=1}^{g-i}(k_j-j)}$ for $i=1,2,\ldots,g$ (where the exponent of $p$ is 0 for $i=g$).

We call $\chi^{\mathrm{norm}}$ the normalized system of Hecke eigenvalues associated with $f$.

### 2.3 The cuspidal $\mathrm{GSp}_{2g}$-eigenvariety

Let $g$ be a positive integer. Let $p$ be an odd prime and $N$ a positive integer such that $(N,p)=1$. Let $\mathcal{H}_g^N$ be the abstract Hecke algebra for $\mathrm{GSp}_{2g}$, spherical outside $N$ and Iwahoric dilating at $p$. Let $\mathcal{W}_g$ be the $g$-dimensional weight space. For every affinoid $A=\mathrm{Spm}\,R\subset\mathcal{W}_g$ and every sufficiently large rational number $w$, Andreatta, Iovita and Pilloni [AIP15, § 8.2] defined a Banach $R$-module $M_g(A,w)$ of $w$-overconvergent cuspidal $\mathrm{GSp}_{2g}$-modular forms of weight $\kappa_A$ and tame level $\Gamma_1(N)$. For each pair $(A,w)$ there is an action $\phi_{A,w}^g\colon\mathcal{H}_g^N\to\mathrm{End}_{R,\mathrm{cont}}\,M_g(A,w)$. Set $U_p^{(g)}=\prod_{i=1}^g U_{p,g}^{(g)}$. It is shown in [AIP15, § 8.1] that $(\mathcal{W}_g,\mathcal{H}_g^N,(M_g(A,w))_{A,w},(\phi^g)_{A,w},U_p^{(g)})$ is an eigenvariety datum in the sense of [Buz07, § 5]. Buzzard's 'eigenvariety machine' [Buz07, Construction 5.7] produces from this datum a rigid analytic variety over $\mathbb{Q}_p$, equidimensional of dimension $g$. We call it the $\mathrm{GSp}_{2g}$-*eigenvariety* of tame level $N$ and we denote it by $\mathcal{D}_g^N$. It is equipped with a weight morphism $w_g\colon\mathcal{D}_g^N\to\mathcal{W}_g$ and a homomorphism $\psi_g\colon\mathcal{H}_g^N\to\mathcal{O}(\mathcal{D}_g^N)$, that



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interpolates the normalized systems of Hecke eigenvalues of classical cuspidal $\mathrm{GSp}_{2g}$-eigenforms of tame level $\Gamma_1(N)$. The images of the elements $T_{i,\ell}^{(g)}$ and $U_{i,p}^{(g)}$, $1 \leqslant i \leqslant g$, belong to $\mathcal{O}(\mathcal{D}_g^N)^\circ$.

When $g = 1$ we call $\mathcal{D}_1^N$ the *eigencurve*. It was constructed by Coleman and Mazur in [CM98] for $N = 1$ and $p > 2$, building on earlier ideas of Coleman. Their construction was extended to all $N$ and $p$ by Buzzard in [Buz07].

We call a point $x \in \mathcal{D}_g^N(\mathbb{C}_p)$ *classical* if the evaluation of $\psi_g$ at $x$ is the Hecke eigensystem a classical $\mathrm{GSp}_{2g}$-eigenform $f$ of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight $w_g(x)$. In this case, $w_g(x)$ is clearly a classical weight.

There is a slope function $\mathrm{sl}\colon \mathcal{D}_g^N(\mathbb{C}_p) \to \mathbb{R}^+$ defined as the $p$-adic valuation of $\psi_g(U_{p,g}^{(g)})$. Let $x$ be a $\overline{\mathbb{Q}}_p$-point of $\mathcal{D}_g^N$ of weight $\underline{k} = (k_1, k_2, \ldots, k_g) \in \mathbb{Z}^g$, so that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_g$. We recall the following result.

Proposition 2.2 (Coleman [Col96, Theorem 6.1] when $g = 1$; Bijakowski, Pilloni and Stroh [BPS16, Theorem 5.3.1, see also Remark 1 in the Introduction] when $g > 1$). *If* $\mathrm{sl}(x) < k_g - g(g+1)/2$, *then the point $x$ is classical.*

2.3.1 *The non-CM eigencurve.* We say that a classical point of $\mathcal{D}_1^N$ is a *CM point* if it corresponds to a classical CM modular form. We say that an irreducible component of $\mathcal{D}_1^N$ is a CM component if all its classical specializations are CM points.

*Remark* 2.3. By [Hid15, Proposition 5.1], if an ordinary irreducible component of the eigencurve contains a classical CM eigenform of weight $k \geqslant 2$, then the component is CM. In contrast, the CM classical points of the positive slope eigencurve form a discrete set. This is a consequence of [CIT16, Corollary 3.6], where it is shown that the eigencurve $\mathcal{D}^{+,\leqslant h}$ contains a finite number of CM classical points.

Let $\mathcal{D}_1^{N,\mathcal{G}}$ be the Zariski-closure in $\mathcal{D}_1^N$ of the set of non-CM classical points. We call $\mathcal{D}_1^{N,\mathcal{G}}$ the *non-CM eigencurve*. The upper index $\mathcal{G}$ stands for 'general', because CM components are exceptional among the irreducible components of $\mathcal{D}_1^N$.

*Remark* 2.4. It follows from Remark 2.3 that $\mathcal{D}_1^{N,\mathcal{G}}$ is the union of all the non-CM irreducible components of $\mathcal{D}_1^N$. In particular, $\mathcal{D}_1^{N,\mathcal{G}}$ is equidimensional of dimension 1 and it contains the positive slope eigencurve. Moreover the set of non-CM classical points is a Zariski-dense and accumulation subset of $\mathcal{D}_1^{N,\mathcal{G}}$.

### 2.4 The Galois pseudocharacters on the eigenvarieties

In this section, $p$ is a fixed prime, $M$ is a positive integer prime to $p$ and $g$ is 1 or 2. For a point $x \in \mathcal{D}_g^M(\mathbb{C}_p)$, we denote by $\mathrm{ev}_x\colon \mathcal{O}(\mathcal{D}_g^M) \to \mathbb{C}_p$ both the evaluation at $x$ and the map $\mathrm{GSp}_{2g}(\mathcal{O}(\mathcal{D}_g^M)) \to \mathrm{GSp}_{2g}(\mathbb{C}_p)$ induced by $\mathrm{ev}_x$. Recall that the $\mathrm{GSp}_{2g}$-eigenvariety $\mathcal{D}_g^M$ is endowed with a morphism $\psi_g\colon \mathcal{H}_g^M \to \mathcal{O}(\mathcal{D}_g^M)$ that interpolates the normalized systems of Hecke eigenvalues associated with the cuspidal $\mathrm{GSp}_{2g}$-eigenforms of level $\Gamma_1(N) \cap \Gamma_0(p)$. Also recall that the images of $T_{i,\ell}^{(g)}$ and $U_{i,p}^{(g)}$, $1 \leqslant i \leqslant g$, are elements of $\mathcal{O}(\mathcal{D}_g^M)^\circ$. Let $S^{\mathrm{cl}}$ denote the set of classical $\overline{\mathbb{Q}}_p$-points of $\mathcal{D}_g^M$. For $x \in S^{\mathrm{cl}}$ let $\psi_x = \mathrm{ev}_x \circ \psi_g$. Let $f_x$ be the classical $\mathrm{GSp}_{2g}$-eigenform having system of Hecke eigenvalues $\psi_x$. Let $\rho_x\colon G_{\mathbb{Q}} \to \mathrm{GSp}_{2g}(\overline{\mathbb{Q}}_p)$ be the $p$-adic Galois representation attached to $f_x$ and let $T_x$ be the pseudocharacter defined as the trace of $\rho_x$. When $x$ varies, the traces $T_x$ can be interpolated by a pseudocharacter with values in $\mathcal{O}(\mathcal{D}_g^M)^\circ$. This is stated precisely in the following proposition.

For every ring $R$, we implicitly extend a character of the Hecke algebra $\mathcal{H}_g^M \to R^\times$ to a morphism of polynomial algebras $\mathcal{H}_g^M[X] \to R[X]$ by applying it to the coefficients. Recall that we fixed an embedding $G_{\mathbb{Q}_\ell} \hookrightarrow G_{\mathbb{Q}}$ for every prime $\ell$, hence an embedding of the inertia subgroup $I_\ell$ in $G_{\mathbb{Q}}$. As usual $\mathrm{Frob}_\ell$ denotes a lift of the Frobenius at $\ell$ to $G_{\mathbb{Q}_\ell}$.

Proposition 2.5. *There exists a pseudocharacter $T_g\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{D}_g^M)$ of dimension $2g$ with the following properties:*

(i) *for every prime $\ell$ not dividing $Np$ and every $h \in I_\ell$, we have $T_g(h) = 2$, where $2 \in \mathcal{O}(\mathcal{D}_g^M)$ denotes the function constantly equal to 2;*

(ii) *for every prime $\ell$ not dividing $Np$, we have $P_{\mathrm{char}}(T_g)(\mathrm{Frob}_\ell; X) = \psi_g(P_{\min}(t_{\ell,g}^{(g)}; X))$;*

(iii) *for every $x \in S^{\mathrm{cl}}$, we have $\mathrm{ev}_x \circ T_g = T_x$.*

*Proof.* The pseudocharacter $T_g$ is constructed via the interpolation argument of [Che04, Proposition 7.1.1]. Its properties are a consequence of those of the classical representations. See [Con16, Theorem 3.5.10] for a detailed proof of the proposition. □

*Remark* 2.6. (i) Let $x \in \mathcal{D}_g^M(\overline{\mathbb{Q}}_p)$. Consider the $2g$-dimensional pseudocharacter $T_x\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ defined by $T_x = \mathrm{ev}_x \circ T_g$. By a well-known theorem of Taylor (see [Tay91]) there exists a Galois representation $\rho_x\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_p)$ satisfying $T_x = \mathrm{Tr}(\rho_x)$. We show in § 4.2 that, when $\overline{\rho}_x$ is absolutely irreducible, $\rho_x$ is isomorphic to a representation $G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$.

(ii) When $x$ varies in a connected component of $\mathcal{D}_g^M$, the residual representation $\overline{\rho}_x\colon G_{\mathbb{Q}} \to \mathrm{GSp}_{2g}(\overline{\mathbb{Q}}_p)$ is independent of $x$. We call it the residual representation associated with the component.

## 3. Image of Galois representations attached to $\mathrm{GSp}_4$-eigenforms

Let $N$ be a positive integer and let $p$ be a prime not dividing $N$. Let $F$ be an overconvergent $\mathrm{GSp}_{2g}$-eigenform of level $\Gamma_1(N)$. Let $\rho_{F,p}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_{2g}(\overline{\mathbb{Q}}_p)$ be the $p$-adic Galois representation associated with $F$. It is defined over a $p$-adic field $K$. Under the technical condition of '$\mathbb{Z}_p$-regularity' of $\rho_{F,p}$ and an assumption on the associated residual representation, we prove that the image of $\rho_{F,p}$ is 'big' when $g = 2$ and $F$ is a classical eigenform that is not a lift of a $\mathrm{GL}_2$-eigenform (Theorem 3.12). On our way towards this result, we prove a theorem that is needed in § 10 (Theorem 3.8).

### 3.1 Trianguline parameters of overconvergent $\mathrm{GSp}_4$-eigenforms

We refer to [Ber02, Col08] for the definitions and results that we need from the theory of $(\varphi, \Gamma)$-modules and trianguline representations. Let $F$ be as in the beginning of the section. Part (i) of the following theorem is a classical result of Faltings [Fal89]. Part (ii) is a combination of [Kis03, Theorem 6.3] and [Col08, Proposition 4.3], as Berger observed in [Ber11, § 4.3].

Theorem 3.1.

(i) *If $F$ is a classical eigenform of cohomological weight, then $\rho_{F,p}|_{G_{\mathbb{Q}_p}}$ is crystalline.*

(ii) *If $g = 1$ and the slope of $F$ is finite, then $\rho_{F,p}|_{G_{\mathbb{Q}_p}}$ is trianguline.*



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If $g=2$, an analogue of Theorem 3.1(ii) for $\rho_{F,p}$ can be deduced from the work of Kedlaya, Pottharst and Xiao [KPX14]. Moreover, the results of [KPX14] allow us to write the parameters of the triangulation of $\rho_{F,p}$ in terms of a Hecke polynomial, as for classical points. Recall that we are working on an eigenvariety $\mathcal{D}_2^M$ with a fixed residual Galois representation $\overline{\rho}$. Suppose that $\overline{\rho}$ is irreducible. By lifting pseudocharacters to representations and considering the associated $(\varphi,\Gamma)$-modules we can define a family of $(\varphi,\Gamma)$-modules over $\mathcal{D}_2^M$ in the sense of [KPX14, § 2.1]. For $x\in\mathcal{D}_2^M(\mathbb{C}_p)$, let $\rho_x\colon G_{\mathbb{Q}}\to \mathrm{GL}_4(\overline{\mathbb{Q}}_p)$ and $\psi_x\colon \mathcal{H}_2^N\to\overline{\mathbb{Q}}_p$ be the Galois representation and the system of Hecke eigenvalues, respectively, attached to $x$. Let $M_x$ be the $(\varphi,\Gamma)$-module over $\overline{\mathbb{Q}}_p$ attached to $\rho_x$. Denote by $\mathrm{ev}_x$ the evaluation of rigid analytic functions on $\mathcal{D}_2^M$ at $x$. We identify the weight of $x$ with a character $(\kappa_1(x),\kappa_2(x))\colon(\mathbb{Z}_p^\times)^2\to\mathbb{C}_p^\times$. Let $\mathrm{id}\colon\mathbb{Z}_p^\times\to\mathbb{Z}_p^\times$ be the identity. Let $\delta_i$, $1\leqslant i\leqslant 4$ be the characters $\mathbb{Z}_p^\times\to\mathcal{O}(\mathcal{D}_2^M)^\times$ defined by

$$\delta_1|_{\mathbb{Z}_p^\times}=1,\quad \delta_1(p)=\psi_x(U_{p,2}^{(2)});\quad \delta_2|_{\mathbb{Z}_p^\times}=\kappa_1/\mathrm{id},\quad \delta_2(p)=\psi_x((U_{p,2}^{(2)})^{w_1});$$
$$\delta_3|_{\mathbb{Z}_p^\times}=\kappa_2/\mathrm{id}^2,\quad \delta_3(p)=\psi_x((U_{p,2}^{(2)})^{w_2});\quad \delta_4|_{\mathbb{Z}_p^\times}=\kappa_1\kappa_2(p)/\mathrm{id}^3,\quad \delta_4(p)=\psi_x((U_{p,2}^{(2)})^{w_1w_2}).$$

For $x\in\mathcal{D}_2^M(\mathbb{C}_p)$, let $\delta_{i,x}=\mathrm{ev}_x\circ\delta_i\colon\mathbb{Q}_p^\times\to\overline{\mathbb{Q}}_p$.

Proposition 3.2.

(i) *For every $x\in\mathcal{D}_2^M(\mathbb{C}_p)$, the $(\varphi,\Gamma)$-module $M_x$ is trianguline.*

(ii) *There exists a Zariski-open rigid analytic subspace $\widetilde{\mathcal{D}}_2^M$ of $\mathcal{D}_2^M$ such that for every $x\in\mathcal{D}_2^M(\mathbb{C}_p)$ the $(\varphi,\Gamma)$-module $M_x$ is triangulable with parameters $\mathrm{ev}_x\circ\delta_i\colon\mathbb{Q}_p^\times\to\overline{\mathbb{Q}}_p$.*

*Proof.* The first statement follows immediately from [KPX14, Corollary 6.3.13]. The second follows from [KPX14, Theorem 6.3.10] after checking that the functions $\delta_i$, $1\leqslant i\leqslant 4$, interpolate the parameters of the triangulations at the classical points. This is true because the parameters of the filtration correspond to the eigenvalues of the crystalline Frobenius at a crystalline point by [Col08, Proposition 1.8] (that restates a result of Berger), and these are identified with the roots of $P_{\min}(U_{p,2}^{(2)})$ by [Urb05, Théorème 1]. □

Now let $N$ be a positive integer prime to $p$ and let $M=N^3$. Let $F$ be an overconvergent $\mathrm{GSp}_4$-eigenform corresponding to a point of $\widetilde{\mathcal{D}}_2^M$. Suppose that there is a $\mathrm{GL}_2$-eigenform $f$ of level $N$ such that $\rho_{F,p}\cong\mathrm{Sym}^3\rho_{f,p}$, with the usual notation. Let $\chi_F\colon\mathcal{H}_2^N\to\overline{\mathbb{Q}}_p$ and $\chi_f\colon\mathcal{H}_1^N\to\overline{\mathbb{Q}}_p$ be the systems of Hecke eigenvalues of the two forms. For $1\leqslant i\leqslant 8$, define $\chi_2^{\mathrm{st},i}$ as in Proposition 9.7 by replacing $\chi_1^{\mathrm{st}}$ by $\chi_f$.

Proposition 3.3. *There exists $i\in\{1,2,\dots,8\}$ such that $\chi_F=\chi_2^{\mathrm{st},i}$.*

*Proof.* The proof is completely analogous to that of Proposition 9.7, once we replace $\mathbf{D}_{\mathrm{cris}}(\rho_{f,p})$ and $\mathbf{D}_{\mathrm{cris}}(\rho_{f,p})$ by the $(\varphi,\Gamma)$-modules $\mathbf{D}_{\mathrm{rig}}(\rho_{f,p})$ and $\mathbf{D}_{\mathrm{rig}}(\rho_{F,p})$, respectively, and we use the result of Proposition 3.2. □

Let $F$ be a finite slope overconvergent $\mathrm{GSp}_4$-eigenform of tame level $M$.

Proposition 3.4. *There are at most $2\dim_{\overline{\mathbb{Q}}_p}\mathbf{D}_{\mathrm{cris}}(\rho_{F,p})$ points of $\mathcal{D}_2^M$ whose associated Galois representation is isomorphic to $\rho_{F,p}$.*



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*Proof.* The accumulation and Zariski-dense set $Z$ of classical points of $\mathcal{D}_2^M$ satisfies the assumptions (CRYS) and (HT) of [BC09, § 3.3.2]. Then [BC09, Theorem 3.3.3] implies that, for every $\mathbb{C}_p$-point $x$ of $\mathcal{D}_2^M$, $\psi_x(U_{p,2}^{(2)})$ is an eigenvalue of the crystalline Frobenius acting on $\mathbf{D}_{\mathrm{cris}}(\rho_x)$. There are exactly two characters of the Iwahori–Hecke algebra giving the same value for $\psi_x(U_{p,2}^{(2)})$, hence the desired result. $\square$

Now let $f$ be a finite slope overconvergent $\mathrm{GL}_2$-eigenform of tame level $N$.

Corollary 3.5. *There are at most* $2\dim_{\overline{\mathbb{Q}}_p}\mathbf{D}_{\mathrm{cris}}(\rho)$ *points of* $\mathcal{D}_2^M$ *whose associated Galois representation is isomorphic to* $\mathrm{Sym}^3\rho_{f,p}$.

### 3.2 Representations with symmetric cube of automorphic origin

Let $V$ be a two-dimensional representation of $G_{\mathbb{Q}}$ over a $p$-adic field. We recall a special case of a result of Di Matteo.

Proposition 3.6 [DiM13a, Theorem 2.4.2]. *If the representation* $\mathrm{Sym}^3 V$ *is de Rham, then* $V$ *is the twist by a character of a de Rham representation.*

When 'de Rham' is replaced by 'trianguline', the techniques of [DiM13b] can be modified to prove the following result.

Proposition 3.7. *If the representation* $\mathrm{Sym}^3 V$ *is trianguline, then* $V$ *is either trianguline or the twist by a character of a de Rham representation.*

We refer to [Con16, Proposition 3.10.25] for the proof. We only remark here that to obtain triangulinity, rather than potential triangulinity, we need a result of Berger and Chenevier [BC10, Théorème A].

Given a two-dimensional modulo $p$ representation $\overline{\tau}$ of $G_{\mathbb{Q}}$, we list here for future reference some assumptions that we need in applying the results of [Eme14]. Here $\chi$ denotes the modulo $p$ cyclotomic character:

$$\begin{gathered}\overline{\tau}|_{\mathbb{Q}(\zeta_p)} \text{ is absolutely irreducible}\\ \overline{\tau} \text{ is not equivalent to a twist by a character of } \left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix}1 & *\\ 0 & \chi\end{smallmatrix}\right).\end{gathered} \tag{$*_{\overline{\tau}}$}$$

For convenience we also restate the symmetric cube of the above assumptions, for a four-dimensional modulo $p$ representation $\overline{\tau}$ of $G_{\mathbb{Q}}$:

$$\begin{gathered}\overline{\tau}|_{\mathbb{Q}(\zeta_p)} \text{ is not the symmetric cube of a non-absolutely irreducible representation}\\ \overline{\tau} \text{ is not equivalent to a twist by a character of } \mathrm{Sym}^3\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right) \text{ or } \mathrm{Sym}^3\left(\begin{smallmatrix}1 & *\\ 0 & \chi\end{smallmatrix}\right).\end{gathered} \tag{$**_{\overline{\tau}}$}$$

We will always replace the subscript $\overline{\tau}$ with the representation we are making the above assumptions on.

Let $\rho_1\colon G_{\mathbb{Q}}\to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ and $\rho_2\colon G_{\mathbb{Q}}\to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ be two continuous representations. We will deduce the following theorem from the two previous propositions.

Theorem 3.8. *Suppose that:*

(1) $\rho_2\cong \mathrm{Sym}^3\rho_1$;

(2) $\rho_2$ *is odd and unramified outside a finite set of primes;*

(3) *the residual representation* $\overline{\rho}_1$ *associated with* $\rho_1$ *satisfies assumptions* $(*_{\overline{\rho}_1})$.



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*Then the following conclusions hold.*

(i) *If $\rho_2$ is associated with an overconvergent cuspidal* $\mathrm{GSp}_4$*-eigenform, then $\rho_1$ is associated with an overconvergent cuspidal* $\mathrm{GL}_2$*-eigenform.*

(ii) *If $\rho_2$ is associated with a classical cuspidal* $\mathrm{GSp}_4$*-eigenform, then $\rho_1$ is associated with a classical cuspidal* $\mathrm{GL}_2$*-eigenform.*

*Proof.* Suppose that $\rho_2$ is associated with an overconvergent cuspidal $\mathrm{GSp}_4$-eigenform $F$. Now $\rho_2$ is trianguline by Proposition 3.2(i), so Proposition 3.7 implies that the representation $\rho_1$ is either trianguline or the twist by a character of a de Rham representation. Thanks to assumptions (2) and (3) we can apply [Eme14, Theorem 1.2.4(2)] to deduce that $\rho_1$ is a twist of a representation associated with a cuspidal overconvergent $\mathrm{GL}_2$-eigenform. A study of the Hodge–Tate–Sen weights of $\rho_1$ and $\rho_2$ shows that the twist occurring here can be taken to be trivial, giving conclusion (i).

We prove conclusion (ii). Since $\rho_2$ is associated with a classical cuspidal $\mathrm{GSp}_4$-eigenform, it is a de Rham representation. Then Proposition 3.6 implies that $\rho_1$ is also a de Rham representation up to a twist by a character. As in the previous paragraph, we conclude that $\rho_1$ is attached to a classical cuspidal $\mathrm{GL}_2$-eigenform from [Eme14, Theorem 1.2.4(1)] and a study of the Hodge–Tate–Sen weights of $\rho_1$ and $\rho_2$. □

Corollary 3.9. *If $\rho_1$, $\rho_2$ satisfy the assumptions of Theorem 3.8 and $\rho_2$ is associated with a classical cuspidal* $\mathrm{GSp}_4$*-eigenform $F$, then there exists a* $\mathrm{GL}_2$*-eigenform $f$ such that $F$ is the symmetric cube lift* $\mathrm{Sym}^3 f$ *given by Corollary 9.2.*

*Proof.* The representation $\rho_1$ is attached to a classical cuspidal $\mathrm{GL}_2$-eigenform $f$ by Theorem 3.8(ii). Then $\rho_2$ is the $p$-adic Galois representation attached to the form $\mathrm{Sym}^3 f$. We conclude that $F = \mathrm{Sym}^3 f$. □

### 3.3 A big image result for classical $\mathrm{GSp}_4$-eigenforms

In the following definitions, let $E$ be a finite extension of $\mathbb{Q}_p$. Let $R$ be a local ring with maximal ideal $\mathfrak{m}_R$ and residue field $\mathbb{F}$. Let $\tau\colon G_E \to \mathrm{GSp}_4(R)$ be a representation. Let $\mathrm{PGSp}_4(R) = \mathrm{GSp}_4(R)/R^\times$, where $R^\times$ is identified with the subgroup of scalar matrices. We denote by $\overline{\tau}\colon G_E \to \mathrm{GSp}_4(\mathbb{F})$ the reduction of $\tau$ modulo $\mathfrak{m}_R$. Recall that $T_2$ is the torus consisting of diagonal matrices in $\mathrm{GSp}_4$. We give a notion of $\mathbb{Z}_p$-regularity of $\tau$, analogous to that in [HT15, Lemma 4.5(2)].

Definition 3.10. We say that $\tau$ is *$\mathbb{Z}_p$-regular* if there exists $d \in \mathrm{Im}\,\tau \cap T_2(\mathbb{Z}_p)$ with the following property: if $\alpha$ and $\alpha'$ are two distinct roots of $\mathrm{GSp}_4$, then $\alpha(d) \not\equiv \alpha'(d) \pmod{\mathfrak{m}_R}$. If $d$ has this property we call it a *$\mathbb{Z}_p$-regular element*.

From now on we focus on representations that are either 'residually full' or 'residually of symmetric cube type', in the sense of the following definition. Note that these two types of representations appear in [Pil12, § 5.8] as examples of those for which Pilloni can construct a sequence of Taylor–Wiles primes.

Definition 3.11. We say that $\tau$ is:

(i) *residually full* if there exists a non-trivial subfield $\mathbb{F}'$ of $\mathbb{F}$ and an element $g \in \mathrm{GSp}_4(\mathbb{F})$ such that

$$\mathrm{Sp}_4(\mathbb{F}') \subset g(\mathrm{Im}\,\overline{\tau})g^{-1} \subset \mathrm{GSp}_4(\mathbb{F}');$$

(ii) *residually of symmetric cube type* if there exist a non-trivial subfield $\mathbb{F}'$ of $\mathbb{F}$ and an element $g \in \mathrm{GSp}_4(\mathbb{F})$ such that

$$\mathrm{Sym}^3\,\mathrm{SL}_2(\mathbb{F}') \subset g(\mathrm{Im}\,\overline{\tau})g^{-1} \subset \mathrm{Sym}^3\,\mathrm{GL}_2(\mathbb{F}').$$

We also say that $\overline{\tau}$ is *full* in case (i) and *of symmetric cube type* in case (ii).

We write $\mathfrak{sp}_4(K)$ for the Lie algebra of $\mathrm{Sp}_4(K)$ and $\mathrm{Ad}\colon \mathrm{GSp}_4(K) \to \mathrm{End}(\mathfrak{sp}_4(K))$ for the adjoint representation. Let $F$ and $\rho_{F,p}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathcal{O}_K)$ be as in the beginning of the section. Let $E$ be the subfield of $K$ generated over $\mathbb{Q}_p$ by the set $\{\mathrm{Tr}(\mathrm{Ad}(\rho(g)))\}_{g\in G_{\mathbb{Q}}}$. Let $\mathcal{O}_E$ be the ring of integers of $E$. For a $\mathrm{GL}_2$-eigenform $f$, we denote by $\rho_{f,p}$ the associated $p$-adic Galois representation. We will prove the following result.

Theorem 3.12. *Assume that $\rho_{F,p}$ is $\mathbb{Z}_p$-regular and that one of the following two conditions is satisfied:*

(i) *$\rho_{F,p}$ is residually full;*
(ii) *$F$ is not a $p$-stabilization of the symmetric cube lift of a $\mathrm{GL}_2$-eigenform, defined by Corollary 9.2, and $\overline{\rho}_{F,p}$ satisfies the assumptions $(**_{\overline{\rho}_{F,p}})$.*

*Then the image of $\rho_{F,p}$ contains a principal congruence subgroup of $\mathrm{Sp}_4(\mathcal{O}_E)$.*

For use in the proof of Theorem 3.12 we state a result of Pink.

Theorem 3.13 [Pin98, Theorem 0.7]. *Let $L$ be a local field and let $H$ be an absolutely simple connected adjoint group over $L$. Let $\Gamma$ be a compact Zariski-dense subgroup of $H(L)$. Suppose that the adjoint representation of $\Gamma$ is irreducible. Then there exists a closed subfield $E$ of $L$ and a model $H_E$ of $H$ over $E$ such that $\Gamma$ is an open subgroup of $H_E(E)$.*

We also need the following lemma, that results from an application of Theorem 5.12. We refer to [Con16, Lemma 3.11.5] for a detailed proof.

Lemma 3.14. *Let $\mathcal{G}$ be a profinite group and let $\mathcal{G}_1$ be a normal open subgroup of $\mathcal{G}$. Let $L$ be a field. Let $\tau\colon \mathcal{G} \to \mathrm{GSp}_4(L)$ be a continuous representation. Suppose that:*

(i) *there exists a representation $\tau_1'\colon \mathcal{G}_1 \to \mathrm{GL}_2(L)$ such that $\tau|_{\mathcal{G}_1} \cong \mathrm{Sym}^3\,\tau_1'$;*
(ii) *the image of $\tau_1'$ contains a principal congruence subgroup of $\mathrm{SL}_2(L)$;*
(iii) *there exists a character $\eta\colon \mathcal{G} \to L^\times$ such that $\det\tau \cong \eta^6$.*

*Then there exists a finite extension $\iota\colon L \hookrightarrow L'$ and a representation $\tau'\colon \mathcal{G} \to \mathrm{GL}_2(L')$ such that $\iota\circ\tau \cong \mathrm{Sym}^3\,\tau'$.*

The rest of the section is devoted to the proof of Theorem 3.12. Let $(\mathrm{Im}\,\rho_{F,p})'$ be the derived subgroup of $\mathrm{Im}\,\rho_{F,p}$ and let $G = (\mathrm{Im}\,\rho_{F,p}) \cap \mathrm{Sp}_4(K)$. We denote by $\overline{G}$ the Zariski-closure of $G$ in $\mathrm{Sp}_4(K)$. As in [HT15, § 3], we will show first that under the hypotheses of Theorem 3.12 we have $\overline{G} = \mathrm{Sp}_4(K)$, and second that $G$ is $p$-adically open in $\overline{G}$. We will replace the ordinarity assumption in [HT15, § 3] by that of $\mathbb{Z}_p$-regularity. Let $\overline{G}^\circ$ denote the connected component of the identity in $\overline{G}$.

Let $H$ be any connected, Zariski-closed subgroup of $\mathrm{Sp}_4$, defined over $K$. As in [HT15, § 3.4], we have six possibilities for the isomorphism class of $H$ over $K$:



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(i) $H \cong \mathrm{Sp}_4$;
(ii) $H \cong \mathrm{SL}_2 \times \mathrm{SL}_2$;
(iii) $H \cong \mathrm{SL}_2$ embedded in a Klingen parabolic subgroup;
(iv) $H \cong \mathrm{SL}_2$ embedded in a Siegel parabolic subgroup;
(v) $H \cong \mathrm{SL}_2$ embedded via the symmetric cube representation $\mathrm{SL}_2 \to \mathrm{Sp}_4$ (in this case we write $H \cong \mathrm{Sym}^3 \mathrm{SL}_2$);
(vi) $H \cong \{1\}$.

We show that only choice (i) is possible for $H = \overline{G}^{\circ}$.

LEMMA 3.15. *If condition (i) or (ii) in Theorem 3.12 holds, then* $\overline{G}^{\circ} \cong \mathrm{Sp}_4$.

*Proof.* Let $\mathfrak{m}_K$ be the maximal ideal of $\mathcal{O}_K$ and let $\mathbb{F}_K = \mathcal{O}_K/\mathfrak{m}_K$. The group $(\mathrm{Im}\,\rho_{F,p})'$ is contained in $\overline{G}^{\circ}(\mathcal{O}_K)$. By reducing modulo $\mathfrak{m}_K$ we obtain that the derived subgroup $(\mathrm{Im}\,\overline{\rho}_{F,p})'$ of $\mathrm{Im}\,\overline{\rho}_{F,p}$ is contained in $\overline{G}^{\circ}(\mathbb{F}_K)$. If $\rho_{F,p}$ is residually full, then the only choice for the isomorphism class of $\overline{G}^{\circ}$ is $\overline{G}^{\circ} \cong \mathrm{Sp}_4$. If $\rho_{F,p}$ is residually of symmetric cube type, then either $\overline{G}^{\circ} \cong \mathrm{Sp}_4$ or $\overline{G}^{\circ} \cong \mathrm{Sym}^3 \mathrm{SL}_2$.

Suppose that $\overline{G}^{\circ} \cong \mathrm{Sym}^3 \mathrm{SL}_2$. We show that there exists a $\mathrm{GL}_2$-eigenform $f$ such that $\rho_{F,p} \cong \mathrm{Sym}^3 \rho_{f,p}$. This will contradict the second part of condition (ii) of Theorem 3.12, concluding the proof of Lemma 3.15. As $\overline{G}^{\circ}(K)$ is of finite index in $\overline{G}(K)$, Lemma 3.14 implies that $\overline{G}(K) \subset \mathrm{Sym}^3 \mathrm{SL}_2(K)$, so $\mathrm{Im}\,\rho_{F,p} \subset \mathrm{Sym}^3 \mathrm{GL}_2(K)$. Hence, there exists a representation $\rho'$ satisfying $\rho_{F,p} \cong \mathrm{Sym}^3 \rho'$. As $\rho_{F,p}$ is associated with a $\mathrm{GSp}_4$-eigenform, Corollary 3.9 implies that $\rho'$ is associated with a $\mathrm{GL}_2$-eigenform $f$. □

We equip all groups with their $p$-adic topology. The proof of Theorem 3.12 is completed by the following proposition.

PROPOSITION 3.16. *Suppose that* $\overline{G} \cong \mathrm{Sp}_4(K)$. *Then the group* $G$ *contains an open subgroup of* $\mathrm{Sp}_4(E)$.

*Proof.* Consider the image $G^{\mathrm{ad}}$ of $G$ under the projection $\mathrm{Sp}_4(K) \to \mathrm{PGSp}_4(K)$. It is a compact subgroup of $\mathrm{PGSp}_4(K)$. Since $\overline{G} \cong \mathrm{Sp}_4(K)$, the group $G^{\mathrm{ad}}$ is Zariski-dense in $\mathrm{PGSp}_4(K)$. By Theorem 3.13 there is a model $H$ of $\mathrm{PGSp}_4$ over $E$ such that $G^{\mathrm{ad}}$ is an open subgroup of $H(E)$. By the assumption of $\mathbb{Z}_p$-regularity of $\rho$, there is a diagonal element $d$ with pairwise distinct eigenvalues. The group $H(E)$ must contain the centralizer of $d$ in $\mathrm{PGSp}_4(E)$, which is a split torus in $\mathrm{PGSp}_4(E)$. As $H$ is split and $H \times_E K \cong \mathrm{PGSp}_{4/K}$, $H$ is a split form of $\mathrm{PGSp}_4$ over $E$. Then $H$ must be isomorphic to $\mathrm{PGSp}_4$ over $E$ by unicity of the quasi-split form of a reductive group. Hence, $G^{\mathrm{ad}}$ is an open subgroup of $\mathrm{PGSp}_4(E)$. As the map $\mathrm{Sp}_4(K) \to \mathrm{PGSp}_4(K)$ has degree 2 and $G \cap \mathrm{Sp}_4(E)$ surjects onto $G^{\mathrm{ad}} \cap \mathrm{PGSp}_4(E)$, $G$ must contain an open subgroup of $\mathrm{Sp}_4(E)$. In particular, $G$ contains a principal congruence subgroup of $\mathrm{Sp}_4(\mathcal{O}_E)$. □

Theorem 3.12 states that, when $\overline{\rho}_{F,p}$ is either full or of symmetric cube type, the image of $\rho_{F,p}$ is large if and only if $F$ is not a lift of an eigenform from a smaller group, the only possible such lift under these assumptions being associated with the symmetric cube representation of $\mathrm{GL}_2$. We think that a similar result should hold under more general assumptions on the residual representation, and that it would follow from Pink's theorem together with an analogue of Corollary 3.9 for the other possible Langlands lifts to $\mathrm{GSp}_4$.

## 4. Finite slope families of $\mathrm{GSp}_{2g}$-eigenforms

In this section, we define families of finite slope $\mathrm{GSp}_{2g}$-eigenforms of level $\Gamma_1(N)\cap\Gamma_0(p)$, extending the definitions given in [CIT16, § 3.1] for $g=1$. Our goal is to define such families integrally. In the following sections, we only use families of genus 1 or 2, but we can give the definitions for general genus with no extra effort.

Let $p$ be a prime number and let $N$ be a positive integer prime to $p$. For $g\geqslant 1$ let $\mathcal{D}_g^{N,h}$ be the $\mathrm{GSp}_{2g}$-eigenvariety of tame level $\Gamma_1(N)$. Let $h\in\mathbb{Q}^{+,\times}$. As the slope $\mathrm{sl}\colon \mathcal{D}_g^N(\mathbb{C}_p)\to\mathbb{R}^{\geqslant 0}$ is the valuation of a rigid analytic function on $\mathcal{D}_g^N$, the locus of $\mathbb{C}_p$-points $x\in\mathcal{D}_g^N$ satisfying $\mathrm{sl}(x)\leqslant h$ admits a structure of rigid analytic subvariety of $\mathcal{D}_g^N$. We denote it by $\mathcal{D}_g^{N,h}$. We write $w_g^h$ for the restriction of the weight map to $\mathcal{D}_g^{N,h}$. Recall that we always identify the $g$-dimensional weight space $\mathcal{W}_g$ with a disjoint union of open discs of centre 0 and radius 1. A standard way to obtain an integral structure on an admissible domain of an eigenvariety is to pull back the integral structure on the weight space via the weight map. The restriction of the weight map to $\mathcal{D}_g^{N,h}$ is not, in general, finite if $h>0$, but it becomes finite when restricted to a sufficiently small admissible domain in $\mathcal{D}_g^{N,h}$. This is assured by a result of Bellaïche that we recall in the following proposition. For every affinoid subdomain $V$ of $\mathcal{W}_g^\circ$, let $\mathcal{D}_{g,V}^{N,h}=\mathcal{D}_g^{N,h}\times_{\mathcal{W}_g^\circ}V$ and let $w_{g,V}^h=w_g^h|_{\mathcal{D}_{g,V}^{N,h}}\colon\mathcal{D}_{g,V}^{N,h}\to V$.

Proposition 4.1 (Bellaïche [Bel12]).

(i) *For every $\kappa\in\mathcal{W}_g^\circ(\overline{\mathbb{Q}}_p)$ there exists an affinoid neighborhood $V_{h,\kappa}$ of $\kappa$ in $\mathcal{W}_g^\circ$ such that the map $w_{g,V_{h,\kappa}}^h$ is finite.*

(ii) *When $h$ varies in $\mathbb{Q}^{+,\times}$ and $\kappa$ varies in $\mathcal{W}_g^\circ$, the set $\{(w_{g,V_{h,\kappa}}^h)^{-1}(V_{h,\kappa})\}_{h,\kappa}$ is an admissible affinoid covering of $\mathcal{D}_g^N$.*

In Bellaïche's terminology, a pair $(V_{h,\kappa},h)$ such that $V_{h,\kappa}$ has the property described in part (i) is called an *adapted pair*. Part (i) of Proposition 4.1 follows from the fact that the characteristic power series of $U_p^{(2)}$ acting on modules of overconvergent eigenforms is strictly convergent, in particular from the calculation in [Bel12, Proposition II.1.12] and the fact that the map from the eigenvariety to the spectral variety is finite. Part (ii) follows from part (i) together with the admissibility of Buzzard's covering of the spectral variety [Buz07, Theorem 4.6] and the construction of the eigenvariety (see [Bel12, Theorem II.3.3]).

*Remark* 4.2. (i) By Proposition 4.1 there exists a radius $r_{h,\kappa}\in p^{\mathbb{Q}}$ such that

$$w^h_{g,B_g(\kappa,r_{h,\kappa}^-)}\colon\mathcal{D}^{N,h}_{g,B_g(\kappa,r_{h,\kappa}^-)}\to B_g(\kappa,r_{h,\kappa}^-)$$

is a finite morphism. By the results of Hida theory for $\mathrm{GSp}_{2g}$, we can take $r_{0,\kappa}=1$ for every $\kappa$.

(ii) We would like to have an estimate for $r_{h,\kappa}$ independent of $\kappa$ and with the property that $r_{h,\kappa}\to 1$ for $h\to 0$, to recover the ordinary case in this limit. This is not available at the moment for the group $\mathrm{GSp}_{2g}$. An estimate of the analogue of this radius is known for the eigenvarieties associated with unitary groups compact at infinity by the work of Chenevier [Che04, Théorème 5.3.1].

### 4.1 Families defined over $\mathbb{Z}_p$

For our purpose of studying the images of Galois representations, we will need to have our finite slope families defined over $\mathbb{Z}_p$. For this reason we specialize to families over weight discs for which we can construct a $\mathbb{Z}_p$-model. For simplicity we only work on the connected component $\mathcal{W}_g^\circ$. Recall that we defined coordinates $T_1, T_2, \dots, T_g$ on $\mathcal{W}_g^\circ$. Let $\kappa$ be a point of $\mathcal{W}_g^\circ$ with coordinates $(\kappa_1, \kappa_2, \dots, \kappa_g)$ in $\mathbb{Z}_p^g$; for instance, we can take as $\kappa$ the arithmetic prime $P_{\underline{k}}$ for some $\underline{k} \in \mathbb{Z}^g$. Let $r_{h,\kappa}$ be the largest radius in $p^{\mathbb{Q}}$ such that the map $w_{\kappa, B_g(\kappa, r_{h,\kappa}^-)} \colon \mathcal{D}^{N,h}_{g, B_g(\kappa, r_{h,\kappa}^-)} \to B_g(\kappa, r_{h,\kappa}^-)$ is finite. Such a radius is non-zero thanks to Remark 4.2(i). Let $s_h$ be a rational number satisfying $r_h = p^{s_h}$. We define a model for $B_g(\kappa, r_h^-)$ over $\mathbb{Q}_p$ by adapting Berthelot's construction for the wide open unit disc (see [deJ95, § 7]). Write $s_h = b/a$ for some $a, b \in \mathbb{N}$. For $i \geqslant 1$, let $s_i = s_h + 1/2^i$ and $r_i = p^{-s_i}$. Set

$$A^\circ_{r_i} = \mathbb{Z}_p\langle t_1, t_2, \dots, t_g, X_i \rangle / (t_j^{2^i a} - p^{a + 2^i b} X_i)_{j=1,2,\dots,g}$$

and $A_{r_i} = A^\circ_{r_i}[p^{-1}]$. Set $B_i = \operatorname{Spm} A_{r_i}$. Then $B_i$ is a $\mathbb{Q}_p$-model of the disc of centre $\kappa$ and radius $r_i$. We define morphisms $A^\circ_{r_{i+1}} \to A^\circ_{r_i}$ by

$$\begin{aligned} X_{i+1} &\mapsto p^a X_i^2, \\ t_j &\mapsto t_j \quad \text{for } j = 1, 2, \dots, g. \end{aligned}$$

They induce compact maps $A_{r_{i+1}} \to A_{r_i}$ which give open immersions $B_i \hookrightarrow B_{i+1}$. We define $B_{g,h} = \varinjlim_i B_i$ where the limit is taken with respect to the above immersions. Let $\Lambda_{g,h} = \mathcal{O}(B_{g,h})^\circ$. Then $\Lambda_{g,h} = \varprojlim_i \mathcal{O}(\operatorname{Spm} B_i)^\circ = \varprojlim_i A^\circ_{r_i}$. We call $\Lambda_{g,h}$ the *genus $g$, $h$-adapted Iwasawa algebra*; we leave its dependence on $\kappa$ implicit. We define $t_1, t_2, \dots, t_g \in \Lambda_{g,h}$ as the projective limits of the variables $t_1, t_2, \dots, t_g$, respectively, of $A^\circ_{r_i}$.

There is a map of $\mathbb{Z}_p$-algebras $\iota^*_{g,h} \colon \Lambda_g \to \Lambda_{g,h}$ defined by $T_j \mapsto t_j + \kappa_j$ for $j = 1, 2, \dots, g$. The inclusion $\iota_{g,h} \colon B_{g,h} \hookrightarrow \mathcal{W}_g^\circ$ induced by $\iota^*_{g,h}$ makes $B_{g,h}$ into a $\mathbb{Q}_p$-model of $B_g(\kappa, r_h^-)$, endowed with the integral structure defined by $\Lambda_{g,h}$.

Let $\eta_h$ be an element of $\overline{\mathbb{Q}}_p$ satisfying $v_p(\eta_h) = s_h$. Let $K_h = \mathbb{Q}_p(\eta_h)$ and let $\mathcal{O}_h$ be the ring of integers of $K_h$. The algebra $\Lambda_{g,h}$ is not a ring of formal series over $\mathbb{Z}_p$, but there is an isomorphism $\Lambda_{g,h} \otimes_{\mathbb{Z}_p} \mathcal{O}_h \cong \mathcal{O}_h[[t_1, t_2, \dots, t_g]]$.

We say that a prime of $\Lambda_{g,h}$ is arithmetic if it lies over an arithmetic prime of $\Lambda_g$. By an abuse of notation we will write again $P_{\underline{k}}$ for an arithmetic prime of $\Lambda_{g,h}$ lying over the arithmetic prime $P_{\underline{k}}$ of $\Lambda_g$.

*Remark* 4.3. Let $\underline{k} = (k_1, k_2, \dots, k_g)$ be a cohomological weight for $\mathrm{GSp}_{2g}$. There exists a prime $\mathfrak{P}$ of $\Lambda_{g,h}$ lying over the prime $P_{\underline{k}}$ of $\Lambda_g$ if and only if the classical weight $\underline{k}$ belongs to the disc $B_g(0, r_h^-)$; otherwise, we have $P_{\underline{k}}\Lambda_g = \Lambda_g$. This happens if and only if $v_p(k_i) > -v_p(r_h) - 1$ for $i = 1, 2, \dots, g$, as we can see via a simple calculation.

Let $\mathcal{D}^N_{g,h}$ be the rigid analytic space fitting in the following Cartesian diagram.

$$\begin{tikzcd} \mathcal{D}^N_{g,h} \arrow[r] \arrow[d] & \mathcal{D}^{N,h}_g \arrow[d, "w_g^h"] \\ B_{g,h} \arrow[r, "\iota_{g,h}"] & \mathcal{W}_g^\circ \end{tikzcd} \tag{5}$$

The rigid analytic space $\mathcal{D}^N_{g,h}$ is a model of $\mathcal{D}^{N,h}_{g,B_g(\kappa,r^-_{h,\kappa})}$ over a $p$-adic field, but it is not necessarily defined over $\mathbb{Q}_p$ since the map $\iota_{g,h}$ may not be. We say that a $\mathbb{C}_p$-point of $\mathcal{D}^N_{g,h}$ is classical if it is a classical point of $\mathcal{D}^{N,h}_{g,B_g(\kappa,r^-_{h,\kappa})}$.

Let $\mathbb{T}_{g,h} = \mathcal{O}(\mathcal{D}^N_{g,h})^\circ$. We call $\mathbb{T}_{g,h}$ the *genus $g$, $h$-adapted Hecke algebra*; we leave its dependence on $\kappa$ implicit again. The weight map induces a morphism $w_{g,h}\colon \mathcal{D}^N_{g,h} \to B_{g,h}$, hence a morphism $w^*_{g,h}\colon \Lambda_{g,h} \to \mathbb{T}_{g,h}$. Thanks to our choice of $r_h$, $w^*_{g,h}$ gives $\mathbb{T}_{g,h}$ a structure of finite $\Lambda_{g,h}$-algebra. The $\mathcal{D}^N_{g,h} \to \mathcal{D}^{N,h}_g$ appearing in the diagram induces a map $\mathcal{O}(\mathcal{D}^{N,h}_g)^\circ \to \mathbb{T}_{g,h}$, that we compose with $\psi_g\colon \mathcal{H}^N_g \to \mathcal{O}(\mathcal{D}^{N,h}_g)^\circ$ to obtain a morphism $\psi_{g,h}\colon \mathcal{H}^N_g \to \mathbb{T}_{g,h}$.

For a prime $\mathfrak{P}$ of $\mathbb{T}_{g,h}$, we denote by $\mathrm{ev}_\mathfrak{P}\colon \mathbb{T}_{g,h} \to \overline{\mathbb{Z}}_p$ the evaluation at $\mathfrak{P}$. We say that $\mathfrak{P}$ is a classical point of $\operatorname{Spec}\mathbb{T}_{g,h}$ if $\mathrm{ev}_\mathfrak{P}\circ\psi_{g,h}\colon \mathcal{H}^N_g \to \overline{\mathbb{Z}}_p$ is the system of Hecke eigenvalues attached to a classical $\mathrm{GSp}_{2g}$-eigenform. These systems of eigenvalues also appear at classical points of $\mathcal{D}^N_{g,h}$.

Definition 4.4. We call the *family of* $\mathrm{GSp}_{2g}$*-eigenforms of slope bounded by $h$* an irreducible component $I$ of $\mathcal{D}^N_{g,h}$, equipped with the integral structure defined by $\mathbb{T}_{g,h}$.

We will usually refer to an $I$ as in Definition 4.4 simply as a finite slope family. Let $\mathbb{I}^\circ = \mathcal{O}(I)$. Then $\mathbb{I}^\circ$ is a finite $\Lambda_{g,h}$-algebra that is also profinite and local. The component $I$ is determined by the surjective morphism $\theta\colon \mathbb{T}_{g,h} \twoheadrightarrow \mathbb{I}^\circ$. We sometimes refer to $\theta$ as a finite slope family. The family $I$ is equipped with maps $w_\theta\colon I \to B_{g,h}$ and $\psi_\theta\colon \mathcal{H}^N_g \to \mathbb{I}^\circ$ induced by $w_{g,h}$ and $\psi_{g,h}$, respectively. Here $^\circ$ denotes the fact that we are working with integral objects. When introducing relative Sen theory in § 7, we will need to invert $p$ and we will drop the $^\circ$ from all rings.

Proposition 2.2 implies that every family $I$ contains at least a classical point. By the accumulation property of classical point and the irreducibility of $I$, the classical points are a Zariski-dense subset of $I$. Hence, the set of classical points of $\operatorname{Spec}\mathbb{I}^\circ$ is Zariski-dense in $\operatorname{Spec}\mathbb{I}^\circ$. Every classical point of $\operatorname{Spec}\mathbb{I}^\circ$ lies over an arithmetic prime of $\operatorname{Spec}\Lambda_{g,h}$. For a family $\theta\colon \mathbb{T}_{g,h} \to \mathbb{I}^\circ$, we give the following.

Definition 4.5. We call an arithmetic prime $P_{\underline{k}} \subset \Lambda_{g,h}$ *non-critical for* $\mathbb{I}^\circ$ if:

(i) every point of $\operatorname{Spec}\mathbb{I}^\circ$ lying over $P_{\underline{k}}$ is classical;

(ii) the map $w^*_{g,B_{g,h}}\colon \Lambda_{g,h} \to \mathbb{I}^\circ$ is étale at every point of $\operatorname{Spec}\mathbb{I}^\circ$ lying over $P_{\underline{k}}$.

We call $P_{\underline{k}}$ *critical* for $\mathbb{I}^\circ$ if it is not non-critical. We also say that a classical weight $\underline{k}$ is critical or non-critical for $\mathbb{I}^\circ$ if the arithmetic prime $P_{\underline{k}}$ has that property.

*Remark* 4.6. (i) By Proposition 2.2, if $\underline{k}$ is a classical weight belonging to $B_{g,h}$ and $h < k_g - g(g+1)/2$ then $\underline{k}$ satisfies condition (i) of Definition 4.5. We do not know of a simple assumption on the weight that guarantees that the second condition is also satisfied.

(ii) The set of non-critical arithmetic primes is Zariski-dense in $\Lambda_{g,h}$. This follows from part (i) of the remark and the fact that the locus of étaleness of the morphism $\Lambda_{g,h} \to \mathbb{I}^\circ$ is Zariski-open in $\mathbb{I}^\circ$.

## 4.2 The Galois representation associated with a finite slope family

Let $\kappa$ be a point of $\mathcal{W}^\circ_g$ with $\mathbb{Z}_p$-coordinates. Let $\mathbb{T}_{2,h}$ be the genus 2, $h$-adapted Hecke algebra of centre $\kappa$. As before, we will leave the dependence on $\kappa$ implicit. For simplicity, let $\Lambda_h = \Lambda_{2,h}$, $\mathbb{T}_h = \mathbb{T}_{2,h}$. We implicitly replace $\mathbb{T}_h$ by one of its local components. Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be



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a finite slope family of $\mathrm{GSp}_4$-eigenforms. Let $\mathbb{F}_{\mathbb{T}_h}$ be the residue field of $\mathbb{T}_h$. The pseudocharacter $T_2\colon G_\mathbb{Q} \to \mathcal{O}(\mathcal{D}_2^N)^\circ$ induces pseudocharacters $T_{\mathbb{T}_h}\colon G_\mathbb{Q} \to \mathbb{T}_h$ and $\overline{T}_{\mathbb{T}_h}\colon G_\mathbb{Q} \to \mathbb{F}_{\mathbb{T}_h}$. By [Rou96, Corollary 5.2] the pseudocharacter $\overline{T}_{\mathbb{T}_h}$ is associated with a representation $\overline{\rho}_{\mathbb{T}_h}\colon G_\mathbb{Q} \to \mathrm{GL}_4(\overline{\mathbb{F}}_p)$, unique up to isomorphism. We call $\overline{\rho}_{\mathbb{T}_h}$ the residual Galois representation associated with $\mathbb{T}_h$. We assume from now on that

the representation $\overline{\rho}_{\mathbb{T}_h}$ is absolutely irreducible.

By the compactness of $G_\mathbb{Q}$ there exists a finite extension $\mathbb{F}'$ of $\mathbb{F}_{\mathbb{T}_h}$ such that $\overline{\rho}_{\mathbb{T}_h}$ is defined on $\mathbb{F}'$. Let $W(\mathbb{F}_{\mathbb{T}_h})$ and $W(\mathbb{F}')$ be the rings of Witt vectors of $\mathbb{F}_{\mathbb{T}_h}$ and $\mathbb{F}'$, respectively. Let $\mathbb{T}_h' = \mathbb{T}_h \otimes_{W(\mathbb{F}_{\mathbb{T}_h})} W(\mathbb{F}')$. We consider $T_{\mathbb{T}_h}$ as a pseudocharacter $G_\mathbb{Q} \to \mathbb{T}_h'$ via the natural inclusion $\mathbb{T}_h \hookrightarrow \mathbb{T}_h'$. Then $T_{\mathbb{T}_h}$ satisfies the hypotheses of [Rou96, Corollary 5.2], so there exists a representation $\rho_{\mathbb{T}_h'}\colon G_\mathbb{Q} \to \mathrm{GL}_4(\mathbb{T}_h')$ such that $\mathrm{Tr}\rho_{T_h'} = \mathbb{T}_{\mathbb{T}_h}$. By Proposition 2.5, for every prime $\ell$ not dividing $Np$ we have

$$\mathrm{Tr}(T_{\mathbb{T}_h})(\mathrm{Frob}_\ell) = \psi_2(T_{\ell,2}^{(2)})|_{\mathcal{D}_{2,B_h}^{M,h}}. \tag{6}$$

In particular, $\mathrm{Tr}(T_{\mathbb{T}_h})(\mathrm{Frob}_\ell)$ is an element of $\mathbb{T}_h$. As $\mathbb{T}_h$ is complete, Chebotarev's theorem implies that $T_{\mathbb{T}_h}(g)$ is an element of $\mathbb{T}_h$ for every $g \in G_\mathbb{Q}$. By a theorem of Carayol [Car94, Théorème 1] there exists a representation $\rho_{\mathbb{T}_h}\colon G_\mathbb{Q} \to \mathrm{GL}_4(\mathbb{T}_h)$ that is isomorphic to $\rho_{\mathbb{T}_h}$ over $\mathbb{T}_h'$.

The morphism $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ induces a morphism $\mathrm{GL}_4(\mathbb{T}_h) \to \mathrm{GL}_4(\mathbb{I}^\circ)$ that we still denote by $\theta$. Let $\rho_{\mathbb{I}^\circ}\colon G_\mathbb{Q} \to \mathrm{GL}_4(\mathbb{I}^\circ)$ be the representation defined by $\rho_{\mathbb{I}^\circ} = \theta \circ \rho_{\mathbb{T}_h}$. Recall that we set $\psi_\theta = \theta(\psi_2|_{\mathcal{D}_{2,B_h}^{M,h}})\colon \mathcal{H}_2^M \to \mathbb{I}^\circ$. Let

$$\mathbb{I}_{\mathrm{Tr}}^\circ = \Lambda_h[\{\mathrm{Tr}(\rho_\theta(g))\}_{g \in G_\mathbb{Q}}].$$

As $\Lambda_h \subset \mathbb{I}_{\mathrm{Tr}}^\circ \subset \mathbb{I}^\circ$, the ring $\mathbb{I}_{\mathrm{Tr}}^\circ$ is a finite $\Lambda_h$-algebra. In particular, $\mathbb{I}_{\mathrm{Tr}}^\circ$ is complete. We keep our usual notation for the reduction modulo an ideal $\mathfrak{P}$ of $\mathbb{I}_{\mathrm{Tr}}^\circ$. We say that a point $\mathfrak{P}$ of $\mathrm{Spec}\,\mathbb{I}_{\mathrm{Tr}}^\circ$ is classical if it lies under a classical point of $\mathrm{Spec}\,\mathbb{I}^\circ$.

By Proposition 2.5 we have $P_{\mathrm{char}}(\mathrm{Tr}(\rho_{\mathbb{I}^\circ}))(\mathrm{Frob}_\ell) = \psi_\theta(P_{\min}(t_{\ell,2}^{(2)};X))$, so we deduce that $\mathbb{I}_{\mathrm{Tr}}^\circ = \Lambda_h[\{\mathrm{Tr}(\rho_\theta(g))\}_{g \in G_\mathbb{Q}}]$. As the traces of $\rho_{\mathbb{I}^\circ}$ belong to $\mathbb{I}_{\mathrm{Tr}}^\circ$, another application of Carayol's theorem [Car94, Théorème 1] provides us with a representation

$$\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GL}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$$

that is isomorphic to $\rho_{\mathbb{I}^\circ}$ over $\mathbb{I}^\circ$. Thanks to the following lemma we can attach to $\theta$ a *symplectic* representation.

Lemma 4.7. *There exists a non-degenerate symplectic bilinear form on $(\mathbb{I}_{\mathrm{Tr}}^\circ)^4$ that is preserved up to a scalar by the image of $\rho_\theta$.*

*Proof.* The argument of the proof is similar to that in [GT05, Lemma 4.3.3] and [Pil12, Proposition 6.4]. We show that $\rho_\theta$ is essentially self-dual by interpolating the characters that appear in the essential self-duality conditions at the classical specializations. We deduce that $\mathrm{Im}\,\rho_\theta$ preserves a bilinear form on $(\mathbb{I}_{\mathrm{Tr}}^\circ)^4$ up to a scalar. Such a form is non-degenerate by the irreducibility of $\rho_\theta$ and it is symplectic because its specialization at a classical point is symplectic. □

Thanks to the lemma, up to replacing $\rho_\theta$ by a conjugate representation, we can suppose that it takes values in $\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$. We call $\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ the Galois representation associated with the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ_{\mathrm{Tr}}$. In the following, we work mainly with this representation, so we denote it simply by $\rho$. We write $\mathbb{F}$ for the residue field of $\mathbb{I}^\circ_{\mathrm{Tr}}$ and $\overline{\rho}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{F})$ for the residual representation associated with $\rho$.

*Remark* 4.8. Let $\varepsilon$ be the Nebentypus of the family $\theta$. By interpolating the determinants of the classical specializations of $\rho$, we obtain

$$\det \rho(g) = \varepsilon(g)(u^{-3}(1+T_1)(1+T_2))^{2\log(\chi(g))/\log(u)} \in \Lambda_{2,h}$$

for every $g \in G_\mathbb{Q}$.

## 5. Conjugate self-twists of Galois representations attached to finite slope families

Given a ring $R$, we denote by $Q(R)$ its total ring of fractions and by $R^{\mathrm{norm}}$ its normalization. Now let $R$ be an integral domain. For every homomorphism $\sigma\colon R \to R$ and every $\gamma \in \mathrm{GSp}_4(R)$, we define $\gamma^\sigma \in \mathrm{GSp}_4(R)$ by applying $\sigma$ to each coefficient of the matrix $\gamma$. This way $\sigma$ induces a group automorphism $[\cdot]^\sigma\colon G(R) \to G(R)$ for every algebraic subgroup $G \subset \mathrm{GSp}_4$ defined over $R$. For such a $G$ and any representation $\rho\colon G_\mathbb{Q} \to G(R)$, we define a representation $\rho^\sigma\colon G_\mathbb{Q} \to G(R)$ by setting $\rho^\sigma(g) = (\rho(g))^\sigma$ for every $g \in G_\mathbb{Q}$.

Let $S$ be a subring of $R$. We say that a homomorphism $\sigma\colon R \to R$ is a homomorphism of $R$ over $S$ if the restriction of $\sigma$ to $S$ is the identity. The following definition is inspired by [Rib85, § 3] and [Lan16, Definition 2.1].

Definition 5.1. Let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(R)$ be a representation. We call *conjugate self-twist for $\rho$ over $S$* an automorphism $\sigma$ of $R$ over $S$ such that there is a finite-order character $\eta_\sigma\colon G_\mathbb{Q} \to R^\times$ and an isomorphism of representations over $R$:

$$\rho^\sigma \cong \eta_\sigma \otimes \rho. \tag{7}$$

We list some basic facts about conjugate self-twists. The proofs are straightforward.

Proposition 5.2. *Let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(R)$ be a representation.*

- (i) *The conjugate self-twists for $\rho$ over $S$ form a group.*
- (ii) *Suppose that the identity of $R$ is not a conjugate self-twist for $\rho$ over $S$. Then for every conjugate self-twist $\sigma$ the character $\eta_\sigma$ satisfying the equivalence (7) is uniquely determined.*
- (iii) *Under the same hypotheses as part (ii), the association $\sigma \mapsto \eta_\sigma$ defines a cocycle on the group of conjugate self-twists with values in $R^\times$.*
- (iv) *Let $S[\mathrm{Tr}\,\mathrm{Ad}\,\rho]$ denote the ring generated over $S$ by the set $\{\mathrm{Tr}(\mathrm{Ad}(\rho)(g))\}_{g\in G_\mathbb{Q}}$. Then every element of $S[\mathrm{Tr}\,\mathrm{Ad}\,\rho]$ is fixed by all conjugate self-twists for $\rho$ over $S$.*

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family of $\mathrm{GSp}_4$-eigenforms as defined in § 4. Let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the Galois representation associated with $\theta$. Recall that $\mathbb{I}^\circ_{\mathrm{Tr}}$ is generated over $\Lambda_h$ by the traces of $\rho$. We always assume that $\overline{\rho}$ is absolutely irreducible. Let $\Gamma$ be the group of conjugate self-twists for $\rho$ over $\Lambda_h$. We omit the reference to $\Lambda_h$ from now on and we just speak of the conjugate self-twists for $\rho$.

*Remark* 5.3. An argument completely analogous to that of [Lan16, Proposition 7.1] proves that the only possible prime factors of the order of $\Gamma$ are 2 and 3.

We denote by $\mathbb{I}^\circ_0$the subring $(\mathbb{I}^\circ_{\mathrm{Tr}})^\Gamma$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$ consisting of the elements fixed by every $\sigma \in \Gamma$.

### 5.1 Lifting conjugate self-twists from classical points to families

Keep the notation as above. Let $P_{\underline{k}} \subset \Lambda_h$ be any non-critical arithmetic prime, as in Definition 4.5. The representation $\rho$ can be reduced modulo $P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}}$ to a representation $\rho_{P_{\underline{k}}}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}})$. Let $\widetilde{\sigma} \in \Gamma$ and let $\widetilde{\eta}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}})^\times$ be the character associated with $\widetilde{\sigma}$. The automorphism $\widetilde{\sigma}$ induces a ring automorphism $\widetilde{\sigma}_{P_{\underline{k}}}$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}}$. The character $\widetilde{\eta}\colon G_\mathbb{Q} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ induces a character $\widetilde{\eta}_{P_{\underline{k}}}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}})^\times$, satisfying

$$\rho_{P_{\underline{k}}}^{\widetilde{\sigma}_{P_{\underline{k}}}} \cong \widetilde{\eta}_{P_{\underline{k}}} \otimes \rho_{P_{\underline{k}}}. \tag{8}$$

As $P_{\underline{k}}$ is non-critical, $\mathbb{I}^\circ$ is étale over $\Lambda_h$ at $P_{\underline{k}}$, so $\mathbb{I}^\circ_{\mathrm{Tr}}$ is also étale over $\Lambda_h$ at $P_{\underline{k}}$. In particular, $P_{\underline{k}}$ is a product of distinct primes in $\mathbb{I}^\circ_{\mathrm{Tr}}$; denote them by $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_d$. As $\widetilde{\sigma}_{P_{\underline{k}}}$ is an automorphism of $\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}} \cong \prod_{i=1}^d \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i$, there is a permutation $s$ of the set $\{1, 2, \ldots, d\}$ and isomorphisms $\widetilde{\sigma}_{\mathfrak{P}_i}\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_{s(i)}$ for $i = 1, 2, \ldots, d$ such that $\widetilde{\sigma}|_{\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i}$ factors through $\widetilde{\sigma}_{\mathfrak{P}_i}$. The character $\widetilde{\eta}_{\sigma_{P_{\underline{k}}}}$ can be written as a product $\prod_{i=1}^d \widetilde{\eta}_{\mathfrak{P}_i}$ for some characters $\widetilde{\eta}_{\mathfrak{P}_i}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i)^\times$. From the equivalence (8), we deduce that

$$\rho_{\mathfrak{P}_i}^{\widetilde{\sigma}_{\mathfrak{P}_i}} \cong \widetilde{\eta}_{\mathfrak{P}_{s(i)}} \otimes \rho_{\mathfrak{P}_{s(i)}}.$$

The goal of this subsection is to prove that if we are given, for a single value of $i$, data $s(i)$, $\widetilde{\sigma}_{\mathfrak{P}_i}$ and $\widetilde{\eta}_{\mathfrak{P}_i}$ satisfying the isomorphism above, then there exists an element of $\Gamma$ giving rise to $\widetilde{\sigma}_{\mathfrak{P}_i}$ and $\widetilde{\eta}_{\mathfrak{P}_{s_i}}$ via reduction modulo $P_{\underline{k}}$. This result is an analogue of [Lan16, Theorem 3.1]. We state it precisely in the following proposition.

Proposition 5.4. *Let $i, j \in \{1, 2, \ldots, d\}$. Let $\sigma\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ be a ring isomorphism and let $\eta_\sigma\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)^\times$ be a character satisfying*

$$\rho_{\mathfrak{P}_i}^\sigma \cong \eta_\sigma \otimes \rho_{\mathfrak{P}_j}. \tag{9}$$

*Then there exists $\widetilde{\sigma} \in \Gamma$ with associated character $\widetilde{\eta}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}})^\times$ such that, via the construction of the previous paragraph, $s(i) = j$, $\widetilde{\sigma}_{\mathfrak{P}_i} = \sigma$ and $\widetilde{\eta}_{\mathfrak{P}_j} = \eta_\sigma$.*

To prove the proposition, we first lift $\sigma$ to an automorphism $\Sigma$ of a deformation ring for $\overline{\rho}$ and then we show that $\Sigma$ descends to a conjugate self-twist for $\rho$. This strategy is the same as that of the proof of [Lan16, Theorem 3.1], but there are various complications that have to be taken care of. We refer to [Con16, § 4.4] for the technical lemmas that generalize Lang's result, and we report here only the core elements of the proof.

Before proving Proposition 5.4 we give a corollary. Let $\mathfrak{P} \in \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_d\}$. Let $\rho_\mathfrak{P}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})$ be the reduction of $\rho$ modulo $\mathfrak{P}$ and let $\Gamma_\mathfrak{P}$ be the group of conjugate self-twists for $\rho_\mathfrak{P}$ over $\mathbb{Z}_p$. Let $\Gamma(\mathfrak{P}) = \{\sigma \in \Gamma \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}$. The reduction of the elements of $\Gamma(\mathfrak{P})$ modulo $\mathfrak{P}$ defines a morphism of groups $\Gamma(\mathfrak{P}) \to \Gamma_\mathfrak{P}$. Choosing $\mathfrak{P}_i = \mathfrak{P}_j = \mathfrak{P}$ in Proposition 5.4 gives the following result.

Corollary 5.5. *The morphism $\Gamma(\mathfrak{P}) \to \Gamma_\mathfrak{P}$ is surjective.*

5.1.1 *Lifting conjugate self-twists to the deformation ring.* We keep the notation of the beginning of this section. We write $\eta = \eta_\sigma$ for simplicity. Let $\mathbb{Q}^{Np}$ denote the maximal extension of $\mathbb{Q}$ unramified outside $Np$ and set $G_\mathbb{Q}^{Np} = \mathrm{Gal}(\mathbb{Q}^{Np}/\mathbb{Q})$. Then we can and do consider $\rho$ and $\eta$ as representations of $G_\mathbb{Q}^{Np}$ by Proposition 2.5 and (9).

Recall that we denote by $\mathfrak{m}_{\mathbb{I}^\circ_{\mathrm{Tr}}}$ the maximal ideal of $\mathbb{I}^\circ_{\mathrm{Tr}}$ and by $\mathbb{F}$ the residue field $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{m}_{\mathrm{Tr}^\circ}$. Let $W$ be the ring of Witt vectors of $\mathbb{F}$. The residual representation $\overline{\rho}\colon G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_4(\mathbb{F})$ is absolutely irreducible by assumption. By the results of [Maz89], the problem of deforming $\rho$ to a representation with coefficients in a Noetherian $W$-algebra is represented by a universal couple $(R_{\overline{\rho}}, \rho^{\mathrm{univ}})$ consisting of a Noetherian $W$-algebra $R_{\overline{\rho}}$ and a representation $\rho^{\mathrm{univ}}\colon G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_4(R_{\overline{\rho}})$.

By the universal property of $R_{\overline{\rho}}$, there exists a unique morphism of $W$-algebras $\alpha_I\colon R_{\overline{\rho}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ satisfying $\rho \cong \alpha_I \circ \overline{\rho}^{\mathrm{univ}}$. Let $\mathrm{ev}_i\colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i$ and $\mathrm{ev}_j\colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ be the two projections. The following proposition follows from arguments completely analogous to those of [Lan16, § 3.1]. The details of the proof can be found in [Con16, § 4.4.1].

Proposition 5.6.

(i) *The automorphism $\overline{\sigma}$ of $\mathbb{F}$ is trivial.*

(ii) *There is an isomorphism $\overline{\rho} \cong \overline{\eta} \otimes \overline{\rho}$.*

(iii) *There exists an automorphism $\Sigma$ of $R_{\overline{\rho}}$ such that:*

  (a) *$\Sigma$ is a lift of $\sigma$ in the sense that $\sigma \circ \mathrm{ev}_i \circ \alpha_I = \mathrm{ev}_j \circ \alpha_I \circ \Sigma$;*

  (b) $\Sigma \circ \rho^{\mathrm{univ}} = \eta \circ \rho^{\mathrm{univ}}$.

We set $\rho^\Sigma = \alpha_I \circ \Sigma \circ \rho^{\mathrm{univ}}$. Recall that $\rho$ is the Galois representation associated with the finite slope family $\theta$. Our next step consists of showing that $\rho^\Sigma$ is associated with a family of $\mathrm{GSp}_4$-eigenforms of a suitable tame level and slope bounded by $h$. Note that equality (b) in Proposition 5.6(iii) implies $\rho^\Sigma \cong \eta \otimes \rho$, so it is sufficient to show that the representation $\eta \otimes \rho$ is associated with a family with the prescribed properties.

5.1.2 *Twisting families by finite-order characters.* We show that the twist of the Galois representation associated with a family of $\mathrm{GSp}_4$-eigenforms is again associated with such a family. We deduce this by the analogous result for a single classical Siegel eigenform, proved in [Con16, § 4.5].

Let $f$ be a cuspidal $\mathrm{GSp}_4$-eigenform of weight $(k_1, k_2)$ and level $\Gamma_1(M)$ and let $\rho_{f,p}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ be the $p$-adic Galois representation attached to $f$. Let $\eta\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times$ be a character of finite order $m_0$ prime to $p$. We see $\eta$ as a Dirichlet character when convenient.

Proposition 5.7 [Con16, Corollary 4.5.5]. *Let $M' = \mathrm{lcm}(m_0, M)^2$. Let $x$ be a classical $p$-old point of $\mathcal{D}_2^M$ having weight $(k_1, k_2)$, slope $h$ and associated Galois representation $\rho_x$. Then there exists a classical $p$-old point $x_\eta$ of $\mathcal{D}_2^{M'}$ having weight $(k_1, k_2)$, slope $h$ and associated Galois representation $\rho_{x_\eta} = \eta \otimes \rho_x$.*

We remark that the proof relies on the calculations made by Andrianov in [And09, § 1]. He only considers the case $k_1 = k_2$, but his work is easily adapted to vector-valued forms.

Now consider the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ fixed in the beginning of the section. For every $p$-old classical point $x$ of $\theta$, let $x_\eta$ be the point of the eigenvariety $\mathcal{D}_2^{M'}$ provided by Proposition 5.7. Let $r'_h$ be a radius adapted to $h$ for the eigenvariety $\mathcal{D}_2^{M'}$. Let $\Lambda'_h$ be the genus 2, $h$-adapted Iwasawa algebra for $\mathcal{D}_2^{M'}$ and let $\mathbb{T}'_h$ be the genus 2, $h$-adapted Hecke algebra of level $M'$. Note that $r'_h \leqslant r_h$, so there is a natural map $\iota_h\colon \Lambda_h \to \Lambda'_h$. All tensor products with $\Lambda'_h$ over $\Lambda_h$ will be taken with respect to $\iota_h$. We can see $\rho_\theta$ as an $\mathbb{I}^\circ_{\mathrm{Tr}} \widehat{\otimes}_{\Lambda_h} \Lambda'_h$-valued representation via $\mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}} \widehat{\otimes}_{\Lambda_h} \Lambda'_h$, $a \mapsto a \otimes 1$. We do this implicitly in the following.

Lemma 5.8. *There exists a finite $\Lambda_h'$-algebra $\mathbb{J}^\circ$, a family $\theta'\colon \mathbb{T}_h' \to \mathbb{J}^\circ$ and an isomorphism $\alpha\colon \mathbb{I}^\circ_{\mathrm{Tr}} \widehat{\otimes}_{\Lambda_h} \Lambda_h' \to \mathbb{J}^\circ_{\mathrm{Tr}}$ such that the representation $\rho_{\theta'}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{J}^\circ_{\mathrm{Tr}})$ associated with $\theta'$ satisfies $\rho_{\theta'} \cong \eta \otimes (\alpha \circ \rho_\theta)$.*

*Remark* 5.9. With the notation of the proof of Lemma 5.8, all points of the set $S_\eta'$ belong to the family $\theta'$, because of the unicity of a point of $\mathcal{D}_2^{M'}$ given its associated Galois representation and slope.

By combining Lemma 5.8 and Proposition 5.6(iii) we obtain the following.

Corollary 5.10. *There exists a finite $\Lambda_h'$-algebra $\mathbb{J}^\circ$, a family $\theta'\colon \mathbb{T}_h' \to \mathbb{J}^\circ$ and an isomorphism $\alpha\colon \mathbb{I}^\circ_{\mathrm{Tr}} \widehat{\otimes}_{\Lambda_h} \Lambda_h' \to \mathbb{J}^\circ_{\mathrm{Tr}}$ such that the representation $\rho_{\theta'}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{J}^\circ_{\mathrm{Tr}})$ associated with $\theta'$ satisfies $\rho_{\theta'} \cong \alpha \circ \Sigma \circ \rho^{\mathrm{univ}}$.*

5.1.3 *Descending to a conjugate self-twist of the family.* We show that the automorphism $\Sigma$ of $R_{\overline{\rho}}$ defined in the previous subsection induces a conjugate self-twist for $\rho$. This will prove Proposition 5.4. Our argument is an analogue for $\mathrm{GSp}_4$ of that in the end of the proof of [Lan16, Theorem 3.1]; it also appears in similar forms in [Fis02, Proposition 3.12] and [DG12, Proposition A.3]. Here the non-criticality of the prime $P_{\underline{k}}$ plays an important role.

*Proof of Proposition 5.4.* Let $\overline{\rho}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{F})$ be the residual representation associated with $\rho$. Let $R_{\overline{\rho}}$ be the universal deformation ring associated with $\overline{\rho}$ and let $\overline{\rho}^{\mathrm{univ}}$ be the corresponding universal deformation. As before let $\alpha_I\colon R_{\overline{\rho}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ be the unique morphism of $W$-algebras $\alpha_I\colon R_{\overline{\rho}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ satisfying $\rho \cong \alpha_I \circ \overline{\rho}^{\mathrm{univ}}$.

Consider the morphism of $W$-algebras $\alpha_I^\Sigma = \alpha_I \circ \Sigma\colon R_{\overline{\rho}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$. We show that there exists an automorphism $\widetilde{\sigma}\colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ fitting in the following commutative diagram.

$$\begin{tikzcd} R_{\overline{\rho}} \arrow[r, "\alpha_I"] \arrow[d, "\Sigma"] & \mathbb{I}^\circ_{\mathrm{Tr}} \arrow[d, "\widetilde{\sigma}"] \\ R_{\overline{\rho}} \arrow[r, "\alpha_I^\Sigma"] & \mathbb{I}^\circ_{\mathrm{Tr}} \end{tikzcd} \tag{10}$$

We use the notation of the discussion preceding Lemma 5.8. Consider the morphism $\theta \otimes 1\colon \mathbb{T}_h \widehat{\otimes}_{\Lambda_h} \Lambda_h' \to \mathbb{I}^\circ \widehat{\otimes}_{\Lambda_h} \Lambda_h'$. For every $\Lambda_h$-algebra $A$ we denote again by $\iota_h$ the map $A \to A \widehat{\otimes}_{\Lambda_h} \Lambda_h', a \mapsto a \otimes 1$. The natural inclusion $\mathcal{D}_2^M \hookrightarrow \mathcal{D}_2^{M'}$ induces a surjection $s_h\colon \mathbb{T}_h' \to \mathbb{T}_h \widehat{\otimes}_{\Lambda_h} \Lambda_h'$. We define a family of tame level $\Gamma_1(M')$ and slope bounded by $h$ by

$$\theta^{M'} = (\theta \otimes 1) \circ s_h\colon \mathbb{T}_h' \to \mathbb{I}^\circ \widehat{\otimes}_{\Lambda_h} \Lambda_h'.$$

The Galois representation associated with $\theta^{M'}$ is $\rho_{\theta^{M'}} = \iota_h \circ \rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}} \widehat{\otimes}_{\Lambda_h} \Lambda_h')$. Let $\theta'\colon \mathbb{T}_h' \to \mathbb{J}^\circ$ be the family given by Corollary 5.10. We identify $\mathbb{I}^\circ_{\mathrm{Tr}} \widehat{\otimes}_{\Lambda_h} \Lambda_h'$ with $\mathbb{J}^\circ_{\mathrm{Tr}}$ via the isomorphism $\alpha$ given by the same corollary; in particular, the Galois representation associated with $\theta'$ is $\rho_{\theta'} = \rho^\Sigma\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}} \widehat{\otimes}_{\Lambda_h} \Lambda_h')$.

Recall that we are working under the assumptions of Proposition 5.4. In particular, we are given two primes $\mathfrak{P}_i$ and $\mathfrak{P}_j$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$, an isomorphism $\sigma\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ and a character $\eta_\sigma\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)^\times$ such that $\rho^\sigma_{\mathfrak{P}_i} \cong \eta_\sigma \otimes \rho_{\mathfrak{P}_j}$. Let $\mathfrak{P}_i'$ be the image of $\mathfrak{P}_i$ via the map $\iota_h\colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}} \widehat{\otimes}_{\Lambda_h} \Lambda_h'$. The specialization of $\rho_{\theta^{M'}}$ at $\mathfrak{P}_i'$ is $\rho_{\mathfrak{P}_i}$. Let $f'$ be an eigenform corresponding to $\mathfrak{P}_i'$. By Remark 5.9, there is a point of the family $\theta'$ corresponding to the twist of $f$ by $\eta$; let $\mathfrak{P}_{i,\eta}'$

be the prime of $\mathbb{I}^{\circ}_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h$ defining this point. The specialization of $\rho_{\theta'}$ at $\mathfrak{P}'_{i,\eta}$ is $\eta\otimes\rho_{\mathfrak{P}_i}$, which is isomorphic to $\rho^{\sigma}_{\mathfrak{P}_i}$ by assumption. Let $f'_\eta$ be an eigenform corresponding to the prime $\mathfrak{P}'_{i,\eta}$. The forms $f'$ and $f'_\eta$ have the same slope by Proposition 5.7 and their associated representations are obtained from one another via Galois conjugation (given by the isomorphism $\sigma$). Hence, $f'$ and $f'_\eta$ define the same point of the eigenvariety $\mathcal{D}_2^{M'}$. Such a point belongs to both the families $\theta^{M'}$ and $\theta'$. As $\mathfrak{P}_{\underline{k}}$ is non-critical, $\mathbb{T}'_h$ is étale at every point lying over $P_{\underline{k}}$, so the families $\theta^{M'}$ and $\theta'$ must coincide. This means that there is an isomorphism

$$\widetilde{\sigma}'\colon \mathbb{I}^{\circ}_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h \xrightarrow{\sim} \mathbb{I}^{\circ}_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h$$

such that $\rho_{\theta'}=\widetilde{\sigma}'\circ\rho^{M'}$. The isomorphism $\widetilde{\sigma}'$ induces by restriction an isomorphism $\Lambda'_h[\mathrm{Tr}(\rho^{M'})]\to\Lambda'_h[\mathrm{Tr}(\rho_{\theta'})]$. Note that $\Lambda'_h[\mathrm{Tr}(\rho^{M'})]=\iota_h(\mathbb{I}^{\circ}_{\mathrm{Tr}})$ and

$$\begin{aligned}\Lambda'_h[\mathrm{Tr}(\rho_{\theta'})]&=\Lambda'_h[\mathrm{Tr}(\widetilde{\sigma}'\circ\rho^{M'})]=\widetilde{\sigma}'(\Lambda'_h[\mathrm{Tr}(\rho^{M'})])\\&=\widetilde{\sigma}'(\Lambda'_h[\mathrm{Tr}(\iota_h\circ\rho)])=\widetilde{\sigma}'(\iota_h(\Lambda_h[\mathrm{Tr}\rho]))=\widetilde{\sigma}'(\iota_h(\mathbb{I}^{\circ}_{\mathrm{Tr}})).\end{aligned}$$

In particular, $\widetilde{\sigma}'$ induces by restriction an isomorphism $\iota_h(\mathbb{I}^{\circ}_{\mathrm{Tr}})\cong\iota_h(\mathbb{I}^{\circ}_{\mathrm{Tr}})$. As $\iota_h$ is injective, we can identify $\widetilde{\sigma}'$ with an isomorphism $\widetilde{\sigma}\colon\mathbb{I}^{\circ}_{\mathrm{Tr}}\to\mathbb{I}^{\circ}_{\mathrm{Tr}}$. By construction, $\widetilde{\sigma}$ fits in diagram (10). □

### 5.2 Rings of conjugate self-twists for representations attached to classical eigenforms

Let $f$ be a classical $\mathrm{GSp}_4$-eigenform and $\rho_{f,p}\colon G_{\mathbb{Q}}\to\mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ the $p$-adic Galois representation associated with $f$. Up to replacing $\rho_{f,p}$ with a conjugate we can suppose that it has coefficients in the ring of integers $\mathcal{O}_K$ of a $p$-adic field $K$. Suppose that $f$ satisfies the hypotheses of Theorem 3.12, i.e. that $\overline{\rho}_{f,p}$ is of $\mathrm{Sym}^3$ type, but $f$ is not the symmetric cube lift of a $\mathrm{GL}_2$-eigenform. Let $\Gamma_f$ be the group of conjugate self-twists for $\rho$ over $\mathbb{Z}_p$ and let $\mathcal{O}_K^{\Gamma_f}$ be the subring of elements of $\mathcal{O}_K$ fixed by $\Gamma_f$. As in § 3.3, we define another subring of $\mathcal{O}_K$ by $\mathcal{O}_E=\mathbb{Z}_p[\mathrm{Tr}(\mathrm{Ad}\,\rho)]$. We prove the following.

PROPOSITION 5.11. *There is an equality* $\mathcal{O}_K^{\Gamma_f}=\mathcal{O}_E$.

Before proving Proposition 5.11, we recall a result of O'Meara about isomorphisms of congruence subgroups. We denote by $\mathrm{PSp}_{2g}$ and $\mathrm{PGSp}_{2g}$ the projective symplectic groups of genus $g$. The following is a rewriting of [OMe78, Theorem 5.6.4] for the situation we are interested in.

THEOREM 5.12. *Let $F$ and $F_1$ be two fields. Let $\Delta$ and $\Delta_1$ be subgroups of* $\mathrm{PGSp}_{2g}(F)$ *and* $\mathrm{PGSp}_{2g}(F_1)$, *respectively, satisfying* $\Gamma_{\mathrm{PSp}_{2g}(F)}(\mathfrak{a})\subset\Delta$ *and* $\Gamma_{\mathrm{PSp}_{2g}(F_1)}(\mathfrak{a})\subset\Delta_1$. *Let $\Theta\colon\Delta\to\Delta_1$ be an isomorphism of groups. Then there exists an automorphism $\sigma$ of $F$ and an element $\gamma\in\mathrm{PGSp}_{2g}(F_1)$ satisfying*

$$\Theta x=\gamma x^{\sigma}\gamma^{-1}$$

*for every $x\in\Delta$.*

We fix some notation. Let $\mathrm{End}(\mathfrak{sp}_4(K))$ be the $K$-vector space of $K$-linear maps $\mathfrak{sp}_4(K)\to\mathfrak{sp}_4(K)$ and let $\mathrm{GL}(\mathfrak{sp}_4(K))$ be the subgroup consisting of the bijective ones. Let $\mathrm{Aut}(\mathfrak{gsp}_4(K))$ be the subgroup of $\mathrm{GL}(\mathfrak{sp}_4(K))$ consisting of the Lie algebra automorphisms of $\mathfrak{sp}_4(K)$. Let $\pi_{\mathrm{Ad}}$ be the natural projection $\mathrm{GSp}_4(\mathcal{O}_K)\to\mathrm{PGSp}_4(\mathcal{O}_K)$ and let $\mathrm{Ad}\colon\mathrm{PGSp}_4(K)\hookrightarrow\mathrm{GL}(\mathfrak{sp}_4(K))$ be the injective group morphism defined by the adjoint representation. As $\mathfrak{sp}_4$ admits no outer automorphisms, Ad induces an isomorphism $\mathrm{PGSp}_4(K)\cong\mathrm{Aut}(\mathfrak{sp}_4(K))$. For simplicity, we write $\rho=\rho_{f,p}$ in the following proof (but recall that in the other sections $\rho$ is the Galois representation attached to a family).

*Proof of Proposition 5.11.* The inclusion $\mathcal{O}_E \subset \mathcal{O}_K^{\Gamma_f}$ follows from Proposition 5.2(5). We prove that $\mathcal{O}_K^{\Gamma_f} \subset \mathcal{O}_E$. As $\mathcal{O}_K^{\Gamma_f}$ and $\mathcal{O}_E$ are normal, it is sufficient to show that an automorphism of $\mathcal{O}_K$ over $\mathcal{O}_E$ leaves $\mathcal{O}_K^{\Gamma_f}$ fixed. Consider such an automorphism $\sigma$. As $\mathcal{O}_E$ is fixed by $\sigma$, we have $(\mathrm{Tr}(\mathrm{Ad}\,\rho)(g))^\sigma = \mathrm{Tr}(\mathrm{Ad}\,\rho(g))$ for every $g \in G_\mathbb{Q}$, hence $\mathrm{Tr}(\mathrm{Ad}\,\rho^\sigma(g)) = \mathrm{Tr}(\mathrm{Ad}\,\rho(g))$. The equality of traces induces an isomorphism $\mathrm{Ad}\,\rho^\sigma \cong \mathrm{Ad}\,\rho$ of representations of $G_\mathbb{Q}$ with values in $\mathrm{GL}(\mathfrak{sp}_4)$. This means that there exists $\phi \in \mathrm{GL}(\mathfrak{sp}_4(K))$ satisfying

$$\mathrm{Ad}\,\rho^\sigma = \phi \circ \mathrm{Ad}\,\rho \circ \phi^{-1}. \tag{11}$$

We show that $\phi$ is actually an inner automorphism of $\mathfrak{sp}_4(K)$.

Clearly Ad induces an isomorphism $\pi_{\mathrm{Ad}}(\mathrm{Im}\,\rho) \cong \mathrm{Im}\,\mathrm{Ad}\,\rho$. For every $x \in \mathrm{GL}(\mathfrak{sp}_4(K))$, we denote by $\Theta_x$ the automorphism of $\mathrm{GL}(\mathfrak{sp}_4(K))$ given by conjugation by $x$. In particular, we write (11) as $\mathrm{Ad}\,\rho^\sigma = \Theta_\phi(\mathrm{Ad}\,\rho)$. By combining Theorems 3.12 and 5.12, we show that we can replace $\phi$ by an element $\phi' \in \mathrm{Aut}(\mathfrak{sp}_4(K))$ still satisfying $\mathrm{Ad}\,\rho^\sigma = \Theta_{\phi'}(\mathrm{Ad}\,\rho(\phi'))$.

We identify $\mathrm{PGSp}_4(\mathcal{O}_E)$ with a subgroup of $\mathrm{PGSp}_4(\mathcal{O}_K^{\Gamma_f})$ via the inclusion $\mathcal{O}_E \subset \mathcal{O}_K^{\Gamma_f}$ given in the beginning of the proof. Consider the group $\Delta = (\pi_{\mathrm{Ad}}\,\mathrm{Im}\,\rho) \cap \mathrm{PGSp}_4(\mathcal{O}_E) \subset \mathrm{PGSp}_4(\mathcal{O}_K)$ and its isomorphic image $\mathrm{Ad}(\Delta) \subset \mathrm{GL}(\mathfrak{sp}_4)$. As $f$ satisfies the hypotheses of Theorem 3.12, $\mathrm{Im}\,\rho$ contains a congruence subgroup $\Gamma_{\mathcal{O}_E}(\mathfrak{a})$ of $\mathrm{GSp}_4(\mathcal{O}_E)$ of some level $\mathfrak{a} \subset \mathcal{O}_E$. It follows that $\pi_{\mathrm{Ad}}\,\mathrm{Im}\,\rho$ contains the projective congruence subgroup $\mathrm{P}\Gamma_{\mathcal{O}_E}(\mathfrak{a})$ of $\mathrm{PGSp}_4(\mathcal{O}_E)$, so $\Delta$ also contains $\mathrm{P}\Gamma_{\mathcal{O}_E}(\mathfrak{a})$. In particular, $\Delta$ satisfies the hypotheses of Theorem 5.12. As $\mathrm{Ad}\,\rho^\sigma = \Theta_\phi(\mathrm{Ad}\,\rho)$, we have an equality $(\mathrm{Ad}(\Delta))^\sigma = \Theta_\phi(\mathrm{Ad}(\Delta))$, where we identify both sides with subgroups of $\mathrm{PGSp}_4(\mathcal{O}_E)$. Now $\sigma$ acts as the identity on $\mathrm{PGSp}_4(\mathcal{O}_E)$, so the previous equality reduces to $\mathrm{Ad}(\Delta) = \Theta_\phi(\mathrm{Ad}(\Delta))$. Let $\Theta = \mathrm{Ad}^{-1} \circ \Theta_\phi \circ \mathrm{Ad}\colon \Delta \to \Delta$. As Ad is an isomorphism, the composition $\Theta$ is an automorphism. Moreover, it satisfies

$$\Theta_\phi(\mathrm{Ad}(\delta)) = \mathrm{Ad}(\Theta(\delta)) \tag{12}$$

for every $\delta \in \Delta$. By Theorem 5.12 applied to $F = F_1 = K$, $\Delta_1 = \Delta$ and $\Theta\colon \Delta \to \Delta$, there exists an automorphism $\tau$ of $K$ and an element $\gamma \in \mathrm{GSp}_4(K)$ such that $\Theta(\delta) = \gamma\delta^\tau\gamma^{-1}$ for every $\delta \in \Delta$. We see from (12) that $\tau$ is trivial. It follows that $\Theta_\phi(y) = \mathrm{Ad}(\gamma) \circ y \circ \mathrm{Ad}(\gamma)^{-1}$ for all $y \in \mathrm{Ad}(\Delta)$. By $K$-linearity, we can extend $\Theta_\phi$ and $\Theta_{\mathrm{Ad}(\gamma)}$ to identical automorphisms of the $K$-span of $\mathrm{Ad}(\Delta)$ in $\mathrm{End}(\mathfrak{sp}_4(K))$. As $\Delta$ contains the projective congruence subgroup $\mathrm{P}\Gamma_{\mathcal{O}_E}(\mathfrak{a})$, its $K$-span contains $\mathrm{Ad}(\mathrm{GSp}_4(K))$; in particular, it contains the image of $\mathrm{Ad}\,\rho$. Hence, $\Theta_\phi$ and $\Theta_{\mathrm{Ad}(\gamma)}$ agree on $\mathrm{Ad}\,\rho$, which means that (11) implies $\mathrm{Ad}\,\rho^\sigma = \Theta_{\mathrm{Ad}(\gamma)}(\mathrm{Ad}\,\rho)$. Then, by the definition of $\Theta_{\mathrm{Ad}(\gamma)}$, we have $\mathrm{Ad}\,\rho^\sigma = \mathrm{Ad}(\gamma) \circ \mathrm{Ad}\,\rho \circ (\mathrm{Ad}(\gamma))^{-1} = \mathrm{Ad}(\gamma\rho\gamma^{-1})$. We deduce that there exists a character $\eta_\sigma\colon G_\mathbb{Q} \to \mathcal{O}_K^\times$ satisfying $\rho^\sigma(g) = \eta_\sigma(g)\gamma\rho(g)\gamma^{-1}$ for every $g \in G_\mathbb{Q}$, hence that $\rho^\sigma \cong \eta_\sigma \otimes \rho$. We conclude that $\sigma$ is a conjugate self-twist for $\rho$. In particular, $\sigma$ acts as the identity on $\mathcal{O}_K^{\Gamma_f}$, as desired. □

*Remark* 5.13. Let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the big Galois representation associated with a family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. We can define a ring $\Lambda_h[\mathrm{Tr}(\mathrm{Ad}\,\rho)]$ analogous to the ring $\mathcal{O}_E$ defined above. We have an inclusion $\Lambda_h[\mathrm{Tr}(\mathrm{Ad}\,\rho)] \subset \mathbb{I}^\circ_0$ given by Proposition 5.2(5). However, the proof of the inclusion $\mathcal{O}_K^{\Gamma_f} \subset \mathcal{O}_E$ in Proposition 5.11 relied on the fact that $\mathrm{Im}\,\rho_{f,p}$ contains a congruence subgroup of $\mathrm{GSp}_4(\mathcal{O}_E)$. As we do not know if an analogue for $\rho$ is true, we do not know whether an equality between the normalizations of $\Lambda_h[\mathrm{Tr}(\mathrm{Ad}\,\rho)]$ and $\mathbb{I}^\circ_0$ holds.

Suppose that the $\mathrm{GSp}_4$-eigenform $f$ appears in a finite slope family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. Let $\mathfrak{P}$ be the prime of $\mathbb{I}^\circ_{\mathrm{Tr}}$ associated with $f$ and suppose that $\mathfrak{P} \cap \Lambda_h$ is a non-critical arithmetic prime $P_{\underline{k}}$. Let $\mathfrak{P}_0 = \mathfrak{P} \cap \mathbb{I}^\circ_0$. The following is an analogue of [Lan16, Proposition 6.2], that results from a straightforward application of Corollary 5.5.

PROPOSITION 5.14. *There is an inclusion* $\mathbb{I}_0^\circ/\mathfrak{P}_0 \subset \mathcal{O}_K^{\Gamma_f}$.

The results of this section admit the following corollary.

COROLLARY 5.15. *Let* $\rho \cong G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ *be the representation associated with the family* $\theta$. *Let* $\mathfrak{P}$ *be a prime of* $\mathbb{I}_{\mathrm{Tr}}^\circ$ *corresponding to a classical eigenform* $f$ *that is not a symmetric cube lift of a* $\mathrm{GL}_2$*-eigenform. Let* $\mathfrak{P}_0 = \mathfrak{P} \cap \mathbb{I}_0^\circ$. *Then the image of* $\rho_{\mathfrak{P}}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P})$ *contains a non-trivial congruence subgroup of* $\mathrm{GSp}_4(\mathbb{I}_0^\circ/\mathfrak{P}_0)$.

*Proof.* As before, let $\mathcal{O}_E = \mathbb{Z}_p[\mathrm{Tr}\,\mathrm{Ad}\,\rho_{\mathfrak{P}}]$. By Theorem 3.12, the image of $\rho_{\mathfrak{P}}$ contains a congruence subgroup of $\mathrm{GSp}_4(\mathcal{O}_E)$. By combining Propositions 5.11 and 5.14, we obtain $\mathbb{I}_0^\circ/\mathfrak{P}_0 \subset \mathcal{O}_E$, hence the corollary. □

## 6. Constructing bases of lattices in unipotent subgroups

In this section, we show that the image of the Galois representation associated with a family of $\mathrm{GSp}_4$-eigenforms contains a 'sufficiently large' set of unipotent elements.

### 6.1 An approximation argument

We recall a simple generalization of the approximation argument presented in the proof of [HT15, Lemma 4.5]. We refer to [Con16, § 4.7] for the details of the proof, because there is an imprecision in the argument of [HT15]. In particular, [HT15, Lemma 4.6] does not give the inclusion (4.3) in [Con16, § 4.7]; it needs to be replaced by [Con16, Lemma 4.7.2]. Let $\mathbf{G}$ be a reductive group defined over $\mathbb{Z}$. Let $T$ and $B$ be a torus and a Borel subgroup of $\mathbf{G}$, respectively. Let $\Delta$ be the set of roots associated with $(\mathbf{G}, T)$.

PROPOSITION 6.1 [Con16, Proposition 4.7.1]. *Let* $A$ *be a profinite local ring of residual characteristic* $p$ *endowed with its profinite topology. Let* $G$ *be a compact subgroup of the level* $p$ *principal congruence subgroup* $\Gamma_{\mathbf{G}(A)}(p)$ *of* $\mathbf{G}(A)$. *Suppose that:*

(i) *the ring* $A$ *is complete with respect to the* $p$*-adic topology;*

(ii) *the group* $G$ *is normalized by a diagonal* $\mathbb{Z}_p$*-regular element of* $\mathbf{G}(A)$.

*Let* $\alpha$ *be a root of* $\mathbf{G}$. *For every ideal* $Q$ *of* $A$, *let* $\pi_Q\colon \mathbf{G}(A) \to \mathbf{G}(A/Q)$ *be the natural projection, inducing a map* $\pi_{Q,\alpha}\colon U^\alpha(A) \to U^\alpha(A/Q)$. *Then* $\pi_Q(G) \cap U^\alpha(A/Q) = \pi_Q(G \cap U^\alpha(A))$.

We give a simple corollary.

COROLLARY 6.2. *Let* $\rho\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ *be the Galois representation associated with a finite slope family* $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. *For every root* $\alpha$ *of* $\mathrm{GSp}_4$, *the group* $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}_{\mathrm{Tr}}^\circ)$ *is non-trivial.*

*Proof.* Let $\mathfrak{P}$ be a prime of $\mathbb{I}^\circ$ corresponding to a classical eigenform $f$ that is not the symmetric cube lift of a $\mathrm{GL}_2$-eigenform. Let $\mathcal{O}_E = \mathbb{Z}_p[\mathrm{Tr}(\mathrm{Ad}\,\rho_{\mathfrak{P}})]$. By Theorem 3.12 $\mathrm{Im}\,\rho_{\mathfrak{P}}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathcal{O}_E)$. In particular, $\mathrm{Im}\,\rho_{\mathfrak{P}} \cap U^\alpha(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P})$ is non-trivial for every root $\alpha$. Now we apply Proposition 6.1 to $\mathbf{G} = \mathrm{GSp}_4$, $T = T_2$, $B = B_2$ $A = \mathbb{I}_{\mathrm{Tr}}^\circ$, $G = \mathrm{Im}\,\rho$ and $Q = \mathfrak{P}$. We obtain that the projection $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}_{\mathrm{Tr}}^\circ) \to \mathrm{Im}\,\rho_{\mathfrak{P}} \cap U^\alpha(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P})$ is surjective for every $\alpha$. In particular, $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}_{\mathrm{Tr}}^\circ)$ must be non-trivial for every $\alpha$. □

### 6.2 A representation with image fixed by the conjugate self-twists

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a finite slope family with associated representation $\rho\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$. As before, we assume $\rho$ is residually irreducible and $\mathbb{Z}_p$-regular. Let $\Gamma$ be the group of conjugate self-twist of $\rho$ and let $\mathbb{I}_0^\circ$ be the subring of $\mathbb{I}_{\mathrm{Tr}}^\circ$ consisting of the elements fixed by $\Gamma$. By restricting the domain of $\rho$ and replacing it with a suitable conjugate representation, we obtain a $\mathbb{Z}_p$-regular representation with coefficients in $\mathbb{I}_0^\circ$. This is the content of the next proposition.

We write $\eta_\sigma$ for the finite-order Galois character associated with $\sigma \in \Gamma$. Let $H_0 = \bigcap_{\sigma\in\Gamma} \ker \eta_\sigma$. As $\Gamma$ is finite $H_0$ is open and normal in $G_\mathbb{Q}$. Note that $\mathrm{Tr}(\rho(H_0)) \subset \mathrm{GSp}_4(\mathbb{I}_0^\circ)$. If $\overline{\rho}|_{H_0}$ is irreducible, then by Carayol's theorem [Car94, Théorème 1] there exists $g \in \mathrm{GL}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ such that the representation $\rho^g = g\rho g^{-1}$ satisfies $\mathrm{Im}\,\rho^g|_{H_0} \subset \mathrm{GL}_4(\mathbb{I}_0^\circ)$. The hypothesis of irreducibility of $\overline{\rho}|_{H_0}$ can probably be checked in the residually full or symmetric cube cases, but it would be too restrictive if we wanted to generalize our work to other interesting cases (for instance, to lifts from $\mathrm{GL}_{2/F}$ with $F/\mathbb{Q}$ real quadratic or from $\mathrm{GL}_{1/F}$ with $F/\mathbb{Q}$ CM of degree 4). For this reason we do not make the above assumption and we follow instead the approach of [CIT16, Proposition 4.14], that comes in part from the proof of [Lan16, Theorem 7.5].

Proposition 6.3. *There exists an element $g \in \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ such that:*

(i) *$g\rho g^{-1}(H_0) \subset \mathrm{GSp}_4(\mathbb{I}_0)$;*
(ii) *$g\rho g^{-1}(H_0)$ contains a diagonal $\mathbb{Z}_p$-regular element.*

*Proof.* Let $V$ be a free, rank-four $\mathbb{I}_{\mathrm{Tr}}^\circ$-module. The choice of a basis of $V$ determines an isomorphism $\mathrm{GL}_4(\mathbb{I}_{\mathrm{Tr}}^\circ) \cong \mathrm{Aut}(V)$, hence an action of $\rho$ on $V$. Let $d$ be a $\mathbb{Z}_p$-regular element contained in $\mathrm{Im}\,\rho$. We denote by $\{e_i\}_{i=1,\dots,4}$ a symplectic basis of $V$ such that $d$ is diagonal. Until further notice, we work in this basis.

By definition of conjugate self-twist, for each $\sigma \in \Gamma$ there is an equivalence $\rho^\sigma \cong \eta_\sigma \otimes \rho$. This means that there exists a matrix $C_\sigma \in \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ such that

$$\rho^\sigma(g) = \eta_\sigma C_\sigma \rho(g) C_\sigma^{-1}. \tag{13}$$

Recall that we write $\mathfrak{m}_{\mathbb{I}_{\mathrm{Tr}}^\circ}$ for the maximal ideal of $\mathbb{I}_{\mathrm{Tr}}^\circ$ and $\mathbb{F}$ for the residue field of $\mathbb{I}_{\mathrm{Tr}}^\circ$. Let $\overline{C}_\sigma$ be the image of $C_\sigma$ under the natural projection $\mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ) \to \mathrm{GSp}_4(\mathbb{F})$. We prove the following lemma.

Lemma 6.4. *For every $\sigma \in \Gamma$ the matrix $C_\sigma$ is diagonal and the matrix $\overline{C_\sigma}$ is scalar.*

*Proof.* Let $\alpha$ be any root of $\mathrm{GSp}_4$ and $u^\alpha$ be a non-trivial element of $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}_{\mathrm{Tr}}^\circ)$. Such a $u^\alpha$ exists thanks to Corollary 6.2. Let $g^\alpha$ be an element of $G_\mathbb{Q}$ such that $\rho(g^\alpha) = u^\alpha$. By evaluating (13) at $g^\alpha$ we obtain $C_\sigma u^\alpha C_\sigma^{-1} = (u^\alpha)^\sigma$, which is again an element of $U^\alpha(\mathbb{I}_{\mathrm{Tr}}^\circ)$. We deduce that $C_\sigma$ normalizes $U^\alpha(Q(\mathbb{I}_{\mathrm{Tr}}^\circ))$. This holds for every root $\alpha$, so $C_\sigma$ normalizes the Borel subgroups of upper and lower triangular matrices in $\mathrm{GSp}_4(Q(\mathbb{I}_{\mathrm{Tr}}^\circ))$. As a Borel subgroup is its own normalizer, we conclude that $C_\sigma$ is diagonal.

By Proposition 5.6(i) the action of $\Gamma$ on $\mathbb{I}_{\mathrm{Tr}}^\circ$ induces the trivial action of $\Gamma$ on $\mathbb{F}$. By evaluating (13) at $g^\alpha$ and modulo $\mathfrak{m}_{\mathbb{I}_{\mathrm{Tr}}^\circ}$, we obtain, with the obvious notation, $\overline{C}_\sigma \overline{u}^\alpha (\overline{C}_\sigma)^{-1} = (\overline{u}^\alpha)^\sigma = \overline{u}^\alpha$. As $C_\sigma$ is diagonal and $\overline{u}^\alpha \in U^\alpha(\mathbb{F})$, the left-hand side is $\alpha(\overline{C}_\sigma)\overline{u}^\alpha$. We deduce that $\alpha(\overline{C}_\sigma) = 1$. As this holds for every root $\alpha$, we conclude that $\overline{C}_\sigma$ is scalar. □

We write $C$ for the map $\Gamma \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ defined by $C(\sigma) = C_\sigma$. We show that $C$ can be modified into a 1-cocycle that still satisfies (13). For every $\sigma \in \Gamma$ and every $i$, $1 \leqslant i \leqslant 4$, let $(C_\sigma)_i$ denote the scalar matrix whose entries coincide with $i$th diagonal entry of $C_\sigma$. Define a map $C_i'\colon \Gamma \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ by $C_i'(\sigma) = (C_\sigma)_i^{-1} C_\sigma$. A simple check using (13) and the cocycle identity for the elements $\eta_\sigma$ (Proposition 5.2(iii)) shows that $C_i'$ is a 1-cocycle.



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Set $C'_\sigma = C'_i(\sigma)$. We have

$$\rho^\sigma(g) = \eta_\sigma C_\sigma \rho(g) C_\sigma^{-1} = \eta_\sigma C'_\sigma \rho(g) (C'_\sigma)^{-1}. \tag{14}$$

By Lemma 6.4 $\overline{C}_\sigma$ is scalar, so we obtain $\overline{C'_\sigma} = (\overline{C}_\sigma)_i^{-1}\overline{C}_\sigma = \mathbb{1}_4$ with the obvious notation.

Recall that $\{e_i\}_{i=1,\ldots,4}$ is our chosen basis of the free $\mathbb{I}^\circ_{\mathrm{Tr}}$-module $V$, on which $G_\mathbb{Q}$ acts via $\rho$. For every $v \in V$, we write as $v = \sum_{i=1}^4 \lambda_i(v) e_i$ its unique decomposition in the basis $(e_i)_{i=1,\ldots,4}$, with $\lambda_i(v) \in \mathbb{I}^\circ_{\mathrm{Tr}}$ for $1 \leqslant i \leqslant 4$. For every $v \in V$ and every $\sigma \in \Gamma$, we set $v^{[\sigma]} = (C'_\sigma)^{-1} \sum_{i=1}^4 \lambda_i(v)^\sigma e_i$. This defines an action of $\Gamma$ on $V$ because $C'_\sigma$ is a 1-cocycle. Let $V^{[\Gamma]}$ denote the set of elements of $V$ fixed by $\Gamma$. The action of $\Gamma$ is clearly $\mathbb{I}^\circ_0$-linear, so $V^{[\Gamma]}$ has a structure of $\mathbb{I}^\circ_0$-submodule of $V$.

Let $v \in V^{[\Gamma]}$ and $h \in H_0$. Then $\rho(h)v$ is also in $V^{[\Gamma]}$, as we see by a direct calculation using (14). We deduce that the action of $G_\mathbb{Q}$ on $V$ via $\rho$ induces an action of $H_0$ on $V^{[\Gamma]}$. We will conclude the proof of the proposition after having studied the structure of $V^{[\Gamma]}$.

Lemma 6.5. *The $\mathbb{I}^\circ_0$-submodule $V^{[\Gamma]}$ of $V$ is free of rank four and its $\mathbb{I}^\circ_{\mathrm{Tr}}$-span is $V$.*

*Proof.* Choose $i \in \{1, \ldots, 4\}$. We construct a non-zero, $\Gamma$-invariant element $w_i \in \mathbb{I}^\circ_{\mathrm{Tr}} e_i$. The submodule $\mathbb{I}^\circ_{\mathrm{Tr}} e_i$ is stable under $\Gamma$ because $C'_\sigma$ is diagonal. The action of $\Gamma$ on $\mathbb{I}^\circ_{\mathrm{Tr}} e_i$ induces an action of $\Gamma$ on the one-dimensional $\mathbb{F}$-vector space $\mathbb{I}^\circ_{\mathrm{Tr}} e_i \otimes_{\mathbb{I}^\circ_{\mathrm{Tr}}} \mathbb{F}$. Recall that the conjugate self-twists induce the identity on $\mathbb{F}$ by Proposition 5.6(i) and that the matrix $\overline{C'_\sigma}$ is trivial for every $\sigma \in \Gamma$, so $\Gamma$ acts trivially on $\mathbb{I}^\circ_{\mathrm{Tr}} e_i \otimes_{\mathbb{I}^\circ_{\mathrm{Tr}}} \mathbb{F}$.

For $x \in \mathbb{I}^\circ_{\mathrm{Tr}} e_i$, we let $\overline{x}$ be the image of $x$ via the natural projection $\mathbb{I}^\circ_{\mathrm{Tr}} e_i \to \mathbb{I}^\circ_{\mathrm{Tr}} e_i \otimes_{\mathbb{I}^\circ_{\mathrm{Tr}}} \mathbb{F}$. Choose any $v_i \in \mathbb{I}^\circ_{\mathrm{Tr}} e_i$ such that $\overline{v}_i \neq 0$. Let $w_i = \sum_{\sigma\in\Gamma} v_i^{[\sigma]}$. Clearly $w_i$ is invariant under the action of $\Gamma$. We show that $w_i \neq 0$. Then $\overline{w}_i = \sum_{\sigma\in\Gamma} \overline{v}_i^{[\sigma]} = \sum_{\sigma\in\Gamma} \overline{v}_i = \mathrm{card}(\Gamma)\cdot \overline{v}_i$ because $\Gamma$ acts trivially on $\mathbb{I}^\circ_{\mathrm{Tr}} e_i \otimes_{\mathbb{I}^\circ_{\mathrm{Tr}}} \mathbb{F}$. By Remark 5.3, the only possible prime factors of $\mathrm{card}(\Gamma)$ are 2 and 3. As we supposed that $p \geqslant 5$, we have $\mathrm{card}(\Gamma) \neq 0$ in $\mathbb{F}$. We deduce that $\overline{w}_i = \mathrm{card}(\Gamma)\overline{v}_i \neq 0$ in $\mathbb{F}$, so $w_i \neq 0$.

Note that $\{w_i\}_{i=1,\ldots,4}$ is an $\mathbb{I}^\circ_{\mathrm{Tr}}$-basis of $V$ because $\overline{w}_i \neq 0$ for every $i$. In particular, the $\mathbb{I}^\circ_0$-span of the set $\{w_i\}_{i=1,\ldots,4}$ is a free, rank-four $\mathbb{I}^\circ_0$-submodule of $V$. As $V^{[\Gamma]}$ has a structure of $\mathbb{I}^\circ_0$-module and $w_i \in V^{[\Gamma]}$ for every $i$, there is an inclusion $\sum_{i=1}^4 \mathbb{I}^\circ_0 w_i \subset V^{[\Gamma]}$. We show that this is an equality. Let $v \in V^{[\Gamma]}$. Write $v = \sum_{i=1}^4 \lambda_i w_i$ for some $\lambda_i \in \mathbb{I}^\circ_{\mathrm{Tr}}$. Then, for every $\sigma \in \Gamma$, we have $v = v^{[\sigma]} = \sum_{i=1}^4 \lambda_i^\sigma w_i^{[\sigma]} = \sum_{i=1}^4 \lambda_i^\sigma w_i$. As $\{w_i\}_{i=1,\ldots,4}$ is an $\mathbb{I}^\circ_{\mathrm{Tr}}$-basis of $V$, we must have $\lambda_i = \lambda_i^\sigma$ for every $i$. This holds for every $\sigma$, so we obtain $\lambda_i \in \mathbb{I}^\circ_0$ for every $i$. Hence, $v = \sum_{i=1}^4 \lambda_i w_i \in \sum_{i=1}^4 \mathbb{I}^\circ_0 w_i$.

The second assertion of the lemma follows immediately from the fact that the set $\{w_i\}_{i=1,\ldots,4}$ is contained in $V^{[\Gamma]}$ and is an $\mathbb{I}^\circ_{\mathrm{Tr}}$-basis of $V$. □

Now let $h \in H_0$. Let $\{w_i\}_{i=1,\ldots,4}$ be an $\mathbb{I}^\circ_0$-basis of $V^{[\Gamma]}$ satisfying $w_i \in \mathbb{I}^\circ_{\mathrm{Tr}} e_i$, such as that provided by the lemma. As $\mathbb{I}^\circ_{\mathrm{Tr}} \cdot V^{[\Gamma]} = V$, $\{w_i\}_{i=1,\ldots,4}$ is also an $\mathbb{I}^\circ_{\mathrm{Tr}}$-basis of $V$. Moreover, $\{w_i\}_{i=1,\ldots,4}$ is a symplectic basis of $V$, because $w_i \in \mathbb{I}^\circ_{\mathrm{Tr}} e_i$ for every $i$ and $\{e_i\}$ is a symplectic basis. By construction, the basis $\{w_i\}_{i=1,\ldots,4}$ has the two properties we desire. □

From now on we always work with a $\mathbb{Z}_p$-regular conjugate of $\rho$ satisfying $\rho(H_0) \subset \mathrm{GSp}_4(\mathbb{I}^\circ_0)$.

### 6.3 Lifting unipotent elements

We give a definition and a lemma that will be important in the following. Let $B \hookrightarrow A$ be an integral extension of Noetherian integral domains. We call an $A$-*lattice in* $B$ an $A$-submodule of $B$ generated by the elements of a basis of $Q(B)$ over $Q(A)$.

Lemma 6.6 [Lan16, Lemma 4.10]. *Every $A$-lattice in $B$ contains a non-zero ideal of $B$. Conversely, every non-zero ideal of $B$ contains an $A$-lattice in $B$.*

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a finite slope family of $\mathrm{GSp}_4$-eigenforms and let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the representation associated with $\theta$. For every root $\alpha$, we identify the unipotent group $U^\alpha(\mathbb{I}_0^\circ)$ with $\mathbb{I}_0^\circ$ and $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}_0^\circ)$ with a $\mathbb{Z}_p$-submodule of $\mathbb{I}_0^\circ$. The goal of this section is to show that, for every $\alpha$, $\mathrm{Im}\,\rho \cap U^\alpha$ contains a basis of a $\Lambda_h$-lattice in $\mathbb{I}_0^\circ$. Our strategy is similar to that of [CIT16, § 4.9], which in turn is inspired by [HT15] and [Lan16]. We proceed in two main steps, by showing that:

(i) there exists a non-critical arithmetic prime $P_{\underline{k}} \subset \Lambda_h$ such that $\mathrm{Im}\,\rho_{P_{\underline{k}}\mathbb{I}_0^\circ} \cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$ contains a basis of a $\Lambda_h/P_{\underline{k}}$-lattice in $\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$;

(ii) the natural morphism $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}_0^\circ) \to \mathrm{Im}\,\rho_{P_{\underline{k}}\mathbb{I}_0^\circ} \cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$ is surjective, so we can lift a basis as in point (i) to a basis of a $\Lambda_h$-lattice in $\mathbb{I}_0^\circ$.

Part (i) is proved via Theorem 3.12 and a result about the lifting of conjugate self-twists from $\rho_{P_{\underline{k}}\mathbb{I}_0^\circ}$ to $\rho$ (Proposition 5.4). Part (ii) will result from an application of Proposition 6.1.

We start by stating that we can choose an arithmetic prime with special properties. The following lemma follows from a simple Zariski-density argument, relying on the fact that the weights of the symmetric cube lifts are contained in a two-dimensional (that is, one-parameter) subscheme of $\mathrm{Spec}\,\Lambda_h$.

Lemma 6.7. *Suppose that $\overline{\rho}$ is either full or of symmetric cube type. Then there exists an arithmetic prime $P_{\underline{k}}$ of $\Lambda_h$ such that:*

(i) *$P_{\underline{k}}$ is non-critical for $\mathbb{I}^\circ$ in the sense of Definition 4.5;*

(ii) *for every prime $\mathfrak{P} \subset \mathbb{I}^\circ$ lying above $P_{\underline{k}}$, the classical eigenform corresponding to $\mathfrak{P}$ is not the symmetric cube lift of a $\mathrm{GL}_2$-eigenform.*

Let $\mathfrak{m}_0$ denote the maximal ideal of $\mathbb{I}_0^\circ$. Let $H = \{g \in H_0 \,|\, \rho(g) \equiv 1 \pmod{\mathfrak{m}_0}\}$, that is a normal open subgroup of $H_0$. We define a representation $\rho_0\colon H \to \mathrm{Sp}_4(\mathbb{I}_0^\circ)$ by setting

$$\rho_0 = \rho|_H \otimes \det(\rho|_H)^{-1/2}.$$

Here the square root is defined via the usual power series, that converges on $\rho(H)$. Even though our results are all stated for the representation $\rho$, in an intermediate step we will need to work with $\rho_0$ and its reduction modulo a prime ideal of $\mathbb{I}_0^\circ$. We now show how it will be possible to transfer our results back to $\rho_0$.

For the rest of this section, we fix an arithmetic prime $P_{\underline{k}}$ of $\Lambda_h$ satisfying conditions (i) and (ii) in Lemma 6.7. By the étaleness condition in Definition 4.5, $P_{\underline{k}}\mathbb{I}^\circ$ is an intersection of distinct primes of $\mathbb{I}^\circ$, so $P_{\underline{k}}\mathbb{I}_0^\circ$ is an intersection of distinct primes of $\mathbb{I}_0^\circ$. Let $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_d$ be the prime divisors of $P_{\underline{k}}\mathbb{I}_0^\circ$. Let $\mathfrak{I}$ be either $P_{\underline{k}}\mathbb{I}_0^\circ$ or $\mathfrak{Q}_i$ for some $i \in \{1, 2, \ldots, d\}$.

Lemma 6.8. *The group $\rho(H)$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ if and only if the group $\rho_0(H)$ does.*

*Proof.* The lemma is proved in the same way as [CIT16, Proposition 4.22], by replacing Tazhetdinov's result on subnormal subgroups of symplectic groups of genus 1 with his result in genus 2 [Taz85, Theorem]. □



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The following is a consequence of Proposition 5.15 and Lemma 6.8, together with our choice of $P_{\underline{k}}$.

Lemma 6.9. *Let $\mathfrak{Q}$ be a prime of $\mathbb{I}_0^\circ$ lying over $P_{\underline{k}}$. Then the image of $\rho_{0,\mathfrak{Q}}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q})$.*

*Proof.* By Proposition 5.15, the image of $\rho_{\mathfrak{Q}}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q})$. As $H$ is a finite index subgroup of $G_{\mathbb{Q}}$, the same is true if we replace $\rho_{\mathfrak{Q}}$ by $\rho_{\mathfrak{Q}}|_H$. Now the conclusion follows from Lemma 6.8 applied to $\mathfrak{I}=\mathfrak{Q}$. □

### 6.4 Big image in a product

Lifting the congruence subgroup of Lemma 6.9 to $\mathbb{I}^\circ$ does not provide the information we need on the image of $\rho_0$. We need the following fullness result for $\rho_{P_{\underline{k}}}$.

Proposition 6.10. *The image of the representation $\rho_{P_{\underline{k}}}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$.*

### 6.5 Unipotent subgroups and fullness

Recall that for a root $\alpha$ of $\mathrm{GSp}_4$, we denote by $U^\alpha$ the corresponding one-parameter unipotent subgroup of $\mathrm{GSp}_4$ and by $\mathfrak{u}^\alpha$ the corresponding nilpotent subalgebra of $\mathfrak{gsp}_4(R)$. For an ideal $\mathfrak{a}$ of $R$, we call 'congruence subalgebra of level $\mathfrak{a}$' of $\mathfrak{sp}_4(R)$ the Lie algebra $\mathfrak{a}\cdot\mathfrak{sp}_4(R)$. The following lemma admits two versions, for Lie algebras and for groups, that can be proved via a computation with Lie brackets and commutators, respectively.

Lemma 6.11. *Let $R$ be an integral domain. Let $\mathfrak{G}$ be a Lie subalgebra of $\mathfrak{sp}_4(R)$ and let $G$ be a subgroup of $\mathrm{Sp}_4(R)$. The following are equivalent:*

(1) *the Lie algebra $\mathfrak{G}$ contains a congruence Lie subalgebra (the group $G$ contains a congruence subgroup, respectively) of level a non-zero ideal $\mathfrak{a}$ of $R$;*
(2) *for every root $\alpha$ of $\mathrm{Sp}_4$, the nilpotent Lie algebra $\mathfrak{G}\cap\mathfrak{u}^\alpha(R)$ (the unipotent subgroup $G\cap U^\alpha(R)$, respectively) contains a non-zero ideal $\mathfrak{a}_\alpha$ of $R$ via the identification $\mathfrak{u}^\alpha(R)\cong R$ ($G\cap U^\alpha(R)\cong R$, respectively).*

*Moreover:*

(i) *if condition (1) is satisfied for an ideal $\mathfrak{a}$, then condition (2) is satisfied if we choose $\mathfrak{a}_\alpha=\mathfrak{a}$ for every $\alpha$;*
(ii) *if condition (2) is satisfied for a set of ideals $\{\mathfrak{a}_\alpha\}_\alpha$, then condition (1) is satisfied for the ideal $\mathfrak{a}=\prod_\alpha\mathfrak{a}^\alpha$, where the product is over all roots $\alpha$ of $\mathrm{Sp}_4$.*

*Remark* 6.12. In both versions of Lemma 6.11, if there is an ideal $\mathfrak{a}'$ of $R$ such that the choice $\mathfrak{a}_\alpha=\mathfrak{a}'$ for every $\alpha$ satisfies condition (2), then the choice $\mathfrak{a}=(\mathfrak{a}')^2$ satisfies condition (1).

By applying Proposition 6.10 and Lemma 6.11 to $R=\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$ and $G=\operatorname{Im}\rho_{0,P_{\underline{k}}}$ we obtain the following corollary.

Corollary 6.13. *For every root $\alpha$ of $\mathrm{GSp}_4$ the group $\operatorname{Im}\rho_{P_{\underline{k}}}\cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$ contains the image of an ideal of $\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$.*

### 6.6 Lifting the congruence subgroup

If $\alpha$ is a root of $\mathrm{GSp}_4$, $G$ is a group, $R$ is a ring and $\tau\colon G\to\mathrm{GSp}_4(R)$ is a representation, let $U^\alpha(\tau)=\tau(G)\cap U^\alpha(R)$. We always identify $U^\alpha(R)$ with $R$, hence $U^\alpha(\tau)$ with an additive subgroup of $R$.

Recall that $\rho\colon H_0\to\mathrm{GSp}_4(\mathbb{I}_0^\circ)$ is the representation associated with a finite slope family $\theta\colon\mathbb{T}_h\to\mathbb{I}^\circ$ and that $\rho_{P_{\underline{k}}}$ is the reduction of $\rho$ modulo $P_{\underline{k}}\mathbb{I}_0^\circ$. We use Corollary 6.13 together with Proposition 6.1 to obtain a result on the unipotent subgroups of the image of $\rho$.

Proposition 6.14. *For every root $\alpha$ of $\mathrm{GSp}_4$, the group $U^\alpha(\rho)$ contains a basis of a $\Lambda_h$-lattice in $\mathbb{I}_0^\circ$.*

*Proof.* Let $\pi_{\underline{k}}\colon\mathbb{I}_0^\circ\to\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$ be the natural projection. We denote also by $\pi_{\underline{k}}$ the induced map $\mathrm{GSp}_4(\mathbb{I}_0^\circ)\to\mathrm{GSp}_4(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$. For a root $\alpha$ of $\mathrm{GSp}_4$, let $\pi_{\underline{k}}^\alpha\colon U^\alpha(\mathbb{I}_0^\circ)\to U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$ be the projection induced by $\pi_{\underline{k}}$. Let $G=\operatorname{Im}\rho\cap\Gamma_{\mathrm{GSp}_4(\mathbb{I}_0^\circ)}(p)$ and $G_{P_{\underline{k}}}=\pi_{\underline{k}}(G)$. The choices $A=\mathbb{I}_0^\circ$, $\mathbf{G}=\mathrm{GSp}_4$, $T=T_2$, $B=B_2$, $G=\operatorname{Im}\rho\cap\Gamma_{\mathrm{GSp}_4(\mathbb{I}_0^\circ)}(p)$ and $Q=P_{\underline{k}}$ satisfy the hypotheses of Proposition 6.1, hence $\pi_{\underline{k}}^\alpha$ induces a surjection $G\cap U^\alpha(\mathbb{I}_0^\circ)\to G_{\underline{k}}\cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$. Let $G^\alpha=G\cap U^\alpha(\mathbb{I}_0^\circ)$ and $G_{\underline{k}}^\alpha=G_{\underline{k}}\cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$. As usual, we identify them with $\mathbb{Z}_p$-submodules of $\mathbb{I}_0^\circ$ and $\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$, respectively.

By Corollary 6.13 there exists a non-zero ideal $\mathfrak{a}_{\underline{k}}$ of $\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$ such that $\mathfrak{a}_{\underline{k}}\subset\operatorname{Im}\rho_{P_{\underline{k}}}\cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$. Set $\mathfrak{b}_{\underline{k}}=p\mathfrak{a}_{\underline{k}}$. Then $\mathfrak{b}_{\underline{k}}\subset G_{\underline{k}}^\alpha$. By the result of the previous paragraph, the map $G^\alpha\to G_{\underline{k}}^\alpha$ induced by $\pi_{\underline{k}}^\alpha$ is surjective, so we can choose a subset $A$ of $G^\alpha$ that surjects onto $\mathfrak{b}_{\underline{k}}$. Let $M$ be the $\Lambda_h$-span of $A$ in $\mathbb{I}_0^\circ$. Let $\mathfrak{b}$ be the pre-image of $\mathfrak{b}_{\underline{k}}$ via $\pi_{\underline{k}}^\alpha\colon\mathbb{I}_0^\circ\to\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$. Clearly $A\subset\mathfrak{b}$, so $M$ is a $\Lambda_h$-submodule of $\mathfrak{b}$. Moreover, $M/P_{\underline{k}}M=\mathfrak{b}_{\underline{k}}$ by the definition of $A$. As $\Lambda$ is local, Nakayama's lemma implies that the inclusion $M\hookrightarrow\mathfrak{b}$ is an equality. In particular, the $\Lambda_h$-span of $G^\alpha$ contains an ideal of $\mathbb{I}_0^\circ$. By Lemma 6.6, this implies that $G^\alpha$ contains a basis of a $\Lambda_h$-lattice in $\mathbb{I}_0^\circ$. □

## 7. Relative Sen theory

Let $\theta\colon\mathbb{T}_h\to\mathbb{I}^\circ$ be a finite slope family. We keep the notation of the previous sections. Recall that the image of the family in the connected component of unity of the weight space is a disc $B_2(\kappa,r_{h,\kappa})$ adapted to the slope $h$. To guarantee the convergence of a certain exponential series (see § 7.4), from now on we make the following assumption:

$$B_2(\kappa,r_{h,\kappa})\subset B_2(0,p^{-1/p-1}).\tag{exp}$$

In § 4, we defined a family of radii $\{r_i\}_{i\geqslant 1}$ and we let $A_{r_i}$ be the ring of rigid analytic functions bounded by 1 on $B_2(0,r_i)$. For every $i\geqslant 1$, there is a natural injection $\iota_{r_i}\colon\Lambda_h\to A_{r_i}$. Set $\mathbb{I}_{r_i,0}^\circ=\mathbb{I}_0^\circ\widehat{\otimes}_{\Lambda_h}A_{r_i}^\circ$. We endow $\mathbb{I}_{r_i,0}^\circ$ with its $p$-adic topology.

*Remark* 7.1. (i) The ring $\mathbb{I}_0^\circ$ admits two inequivalent topologies: the profinite topology and the $p$-adic topology. The representation $\rho$ is continuous with respect to the profinite topology on $\mathbb{I}_0^\circ$, but it is not necessarily continuous with respect to the $p$-adic topology.

(ii) As $\mathbb{I}_0^\circ$ is a finite $\Lambda_h$-algebra,$\mathbb{I}_{r_i,0}^\circ$ is a finite $A_{r_i}^\circ$-algebra. There is an injective ring morphism $\iota_{r_i}'\colon\mathbb{I}_0^\circ\hookrightarrow\mathbb{I}_{r_i,0}^\circ$ sending $f$ to $f\otimes 1$. This map is continuous with respect to the profinite topology on $\mathbb{I}_0^\circ$ and the $p$-adic topology on $\mathbb{I}_{r_i,0}^\circ$: this can be seen by looking at the definition of $A_{r_i}^\circ$ in § 4.1 (for $g=2$) and remarking that the $p$-adic valuation of the variables $t_1,t_2$ is positive over any disc of radius strictly smaller than 1.



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We associated with $\theta$ a representation $\rho|_{H_0}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}_0^\circ)$ that is continuous with respect to the profinite topologies on both its domain and target. By Remark 7.1(i), $\rho|_{H_0}$ needs not be continuous with respect to the $p$-adic topology on $\mathrm{GSp}_4(\mathbb{I}_0^\circ)$. This poses a problem when trying to apply Sen theory. For this reason, we introduce for every $i$ the representation $\rho_{r_i}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}_{r_i,0}^\circ)$ defined by $\rho_{r_i} = \iota'_{r_i} \circ \rho|_{H_0}$. We deduce from the continuity of $\iota'_{r_i}$ that $\rho_{r_i}$ is continuous with respect to the profinite topology on $H_0$ and the $p$-adic topology on $\mathbb{I}_{r_i,0}^\circ$. It is clear from the definition that the image of $\rho_{r_i}$ is independent of $i$ as a topological group.

There is a good notion of Lie algebra for a pro-$p$ group that is topologically of finite type. For this reason, we further restrict $H_0$ so that the image of $\rho_{r_i}$ is a pro-$p$ group. Let $H_{r_1} = \{g \in H_0 \mid \rho_{r_1}(g) \cong \mathbb{1}_4 \pmod p\}$ and set $H_{r_i} = H_{r_1}$for every $i \geqslant 1$. The subgroup $\{M \in \mathrm{GSp}_4(\mathbb{I}_{r_1,0}^\circ) \mid M \cong \mathbb{1}_4 \pmod p\}$ is of finite index in $\mathrm{GSp}_4(\mathbb{I}_{r_1,0}^\circ)$. Note that this depends on the fact that we extended the coefficients to $\mathbb{I}_{r_1,0}$, because $\{M \in \mathrm{GSp}_4(\mathbb{I}_0^\circ) \mid M \cong \mathbb{1}_4 \pmod p\}$ is *not* of finite index in $\mathrm{GSp}_4(\mathbb{I}_0)$. We deduce that $H_{r_1}$ is a normal open subgroup of $G_\mathbb{Q}$. Let $K_{H_{r_i}}$ be the subfield of $\overline{\mathbb{Q}}$ fixed by $H_{r_i}$. It is a finite Galois extension of $\mathbb{Q}$.

Recall that we fixed an embedding $G_{\mathbb{Q}_p} \hookrightarrow G_\mathbb{Q}$, identifying $G_{\mathbb{Q}_p}$ with a decomposition subgroup of $G_\mathbb{Q}$ at $p$. Let $H_{r_i,p} = H_{r_i} \cap G_{\mathbb{Q}_p}$. Let $K_{H_{r_i},p}$ be the subfield of $\overline{\mathbb{Q}}_p$ fixed by $H_{r_i,p}$. The field $K_{H_{r_i},p}$ will play a role when we apply Sen theory. For every $i$, let $G_{r_i} = \rho_{r_i}(H_{r_i})$ and $G_{r_i}^{\mathrm{loc}} = \rho_{r_i}(H_{r_i,p})$.

*Remark* 7.2. The topological Lie groups $G_r$ and $G_r^{\mathrm{loc}}$ are independent of $r$, in the following sense. For positive integers $i,j$ with $i \leqslant j$, let $\iota_{r_j}^{r_i}\colon \mathbb{I}_{r_j,0} \to \mathbb{I}_{r_i,0}$ be the natural morphism induced by the restriction of analytic functions $A_{r_j} \to A_{r_i}$. As $H_{r_i} = H_{r_j} = H_{r_1}$ by definition, $\iota_{r_j}^{r_i}$ induces isomorphisms $\iota_{r_j}^{r_i}\colon G_{r_j} \xrightarrow{\sim} G_{r_i}$ and $\iota_{r_j}^{r_i}\colon G_{r_j}^{\mathrm{loc}} \xrightarrow{\sim} G_{r_i}^{\mathrm{loc}}$.

### 7.1 Big Lie algebras

As before, let $r$ be a radius among the $r_i$, $i \in \mathbb{N}^{>0}$. We will associate with $\rho_r(H_r)$ a Lie algebra that will give the context in which to apply Sen's results. Our methods require that we work with $\mathbb{Q}_p$-Lie algebras, so we define the rings $A_r = A_r^\circ[p^{-1}]$ and $\mathbb{I}_{r,0} = \mathbb{I}_{r,0}^\circ[p^{-1}]$.

Let $\mathfrak{a}$ be a height-two ideal of $\mathbb{I}_{r,0}$. The quotient $\mathbb{I}_{r,0}/\mathfrak{a}$ is a finite-dimensional $\mathbb{Q}_p$-algebra. Let $\pi_\mathfrak{a}\colon \mathbb{I}_{r,0} \to \mathbb{I}_{r,0}/\mathfrak{a}$ be the natural projection. We still denote by $\pi_\mathfrak{a}$ the induced map $\mathrm{GSp}_4(\mathbb{I}_{r,0}) \to \mathrm{GSp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$. Consider the subgroups $G_{r,\mathfrak{a}} = \pi_\mathfrak{a}(G_r)$ and $G_{r,\mathfrak{a}}^{\mathrm{loc}} = \pi_\mathfrak{a}(G_r^{\mathrm{loc}})$ of $\mathrm{GSp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$. They are both pro-$p$ groups and they are topologically of finite type because $\mathrm{GSp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$ is. It makes sense to consider the logarithm of an element of $G_{r,\mathfrak{a}}$ because this group is contained in $\{M \in \mathrm{GSp}_4(\mathbb{I}_{r,0}/\mathfrak{a}) \mid M \cong \mathbb{1}_4 \pmod p\}$.

We attach to $G_{r,\mathfrak{a}}$ and $G_{r,\mathfrak{a}}^{\mathrm{loc}}$ the $\mathbb{Q}_p$-vector subspaces $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}}$of $\mathfrak{gsp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$ defined by

$$\mathfrak{G}_{r,\mathfrak{a}} = \mathbb{Q}_p \cdot \log G_{r,\mathfrak{a}} \quad \text{and} \quad \mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}} = \mathbb{Q}_p \cdot \log G_{r,\mathfrak{a}}^{\mathrm{loc}}.$$

The $\mathbb{Q}_p$-Lie algebra structure of $\mathfrak{gsp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$ restricts to a $\mathbb{Q}_p$-Lie algebra structure on $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}}$. These two Lie algebras are finite-dimensional over $\mathbb{Q}_p$ because $\mathfrak{gsp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$ is.

*Remark* 7.3. The Lie algebras $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}}$ are independent of $r$, in a sense analogous to that of Remark 7.2.

Recall that there is a natural injection $\Lambda_2 \hookrightarrow \Lambda_h$, hence an injection $\Lambda_2[p^{-1}] \hookrightarrow \Lambda_h[p^{-1}]$. For every $\underline{k} = (k_1,k_2)$, the ideal $P_{\underline{k}}\Lambda_h[p^{-1}]$ is either prime in $\Lambda_h[p^{-1}]$ or equal to $\Lambda_h[p^{-1}]$. We define the set of 'bad' ideals $S_\Lambda^{\mathrm{bad}}$ of $\Lambda_2[p^{-1}]$ as

$$S_\Lambda^{\mathrm{bad}} = \{(1+T_1-u), (1+T_2-u^2), (1+T_2-u(1+T_1)), ((1+T_1)(1+T_2)-u^3)\}.$$



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Then we define the set of bad prime ideals of $\Lambda_h[p^{-1}]$ as

$$S^{\mathrm{bad}} = \{P \text{ prime of } \Lambda_h[p^{-1}] \mid P \cap \Lambda_2[p^{-1}] \in S^{\mathrm{bad}}_{\Lambda}\}.$$

We will take care to define rings where the images of the ideals in $S^{\mathrm{bad}}$ consist of invertible elements. The reason for this will be clear in Proposition 7.13. Let $S_2$ be the set of ideals $\mathfrak{a}$ of $\mathbb{I}_{r,0}$ of height two such that $\mathfrak{a}$ is prime to $P$ for every $P \in S^{\mathrm{bad}}$. Let $S_2'$ be the subset of prime ideals in $S_2$. We define the ring

$$\mathbb{B}_r = \varprojlim_{\mathfrak{a}\in S_2} \mathbb{I}_{r,0}/\mathfrak{a},$$

where the limit of finite-dimensional $\mathbb{Q}_p$-Banach spaces is taken with respect to the natural transition maps $\mathbb{I}_{r,0}/\mathfrak{a}_1 \to \mathbb{I}_{r,0}/\mathfrak{a}_2$ defined for every inclusion of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2$. We equip $\mathbb{I}_{r,0}/\mathfrak{a}$ with the $p$-adic topology for every $\mathfrak{a}$ and $\mathbb{B}_r$ with the projective limit topology. There is a natural injection $\iota_{\mathbb{B}_r} \colon \mathbb{I}_{r,0} \hookrightarrow \mathbb{B}_r$ with dense image.

Now consider the sets

$$S^{\mathrm{bad}}_A = \{P \cap A_r \mid P \in S^{\mathrm{bad}}\}, \quad S_{2,A} = \{\mathfrak{a} \cap A_r \mid \mathfrak{a} \in S_2\}, \quad S_{2,A}' = \{\mathfrak{a} \cap A_{r,} \mid \mathfrak{a} \in S_2'\}.$$

For later use, we define a ring

$$B_r = \varprojlim_{\mathfrak{a}\in S_{2,A}} A_r/\mathfrak{a},$$

where the limit of finite-dimensional $\mathbb{Q}_p$-Banach spaces is taken with respect to the natural transition maps $A_r/\mathfrak{a}_1 \to A_r/\mathfrak{a}_2$ defined for every inclusion of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2$. We equip $A_r/\mathfrak{a}$ with the $p$-adic topology for every $\mathfrak{a}$ and $B_r$ with the projective limit topology. There is a natural injection $\iota_{B_r} \colon A_r \hookrightarrow B_r$ with dense image. The natural inclusion $B_r \hookrightarrow \mathbb{B}_r$ gives $\mathbb{B}_r$ a structure of finite $B_r$-algebra, as we can deduce from the fact that $\mathbb{I}_{r,0}$ is a finite $A_r$-algebra.

*Remark* 7.4. For every $P \in S^{\mathrm{bad}}$ we have $P \cdot \mathbb{B}_r = \mathbb{B}_r$, because the limit defining $\mathbb{B}_r$ is over ideals prime to $P$. In the same way, we have $P \cdot B_r = B_r$ for every $P \in S^{\mathrm{bad}}_A$.

We attach to the groups $G_r$ and $G_r^{\mathrm{loc}}$ two $\mathbb{Q}_p$-Lie subalgebras of $\mathfrak{gsp}_4(\mathbb{B}_r)$. Let

$$\mathfrak{G}_r = \varprojlim_{\mathfrak{a}\in S_2} \mathfrak{G}_{r,\mathfrak{a}} \quad \text{and} \quad \mathfrak{G}_r^{\mathrm{loc}} = \varprojlim_{\mathfrak{a}\in S_2} \mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}},$$

where $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}}$ are the Lie algebras we attached to $G_{r,\mathfrak{a}}$ and $G_{r,\mathfrak{a}}^{\mathrm{loc}}$. The $\mathbb{Q}_p$-Lie algebra structures on $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}}$ induce $\mathbb{Q}_p$-Lie algebra structures on $\mathfrak{G}_r$ and $\mathfrak{G}_r^{\mathrm{loc}}$. We endow $\mathfrak{G}_r$ and $\mathfrak{G}_r^{\mathrm{loc}}$ with the $p$-adic topology induced by that on $\mathfrak{gsp}_4(\mathbb{B}_r)$.

When we introduce the Sen operators, we will have to extend the scalars of the various rings and Lie algebras to $\mathbb{C}_p$. We denote this operation by adding a lower index $\mathbb{C}_p$ to the objects previously defined. We still endow all the rings with their $p$-adic topology. Clearly $\mathbb{I}_{r,0,\mathbb{C}_p}$ has a structure of finite $A_{r,\mathbb{C}_p}$-algebra and $\mathbb{B}_{r,\mathbb{C}_p}$ has a structure of finite $B_{r,\mathbb{C}_p}$-algebra.

*Remark* 7.5. The $\mathbb{Q}_p$-Lie algebras $\mathfrak{G}_r$ and $\mathfrak{G}_r^{\mathrm{loc}}$ do not have *a priori* any $\mathbb{B}_r$ or $B_r$-module structure. As a crucial step in our arguments we will use Sen theory to induce a $B_{r,\mathbb{C}_p}$-vector space (hence a $B_{r,\mathbb{C}_p}$-Lie algebra) structure on suitable subalgebras of $\mathfrak{G}_{r,\mathbb{C}_p}$.



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### 7.2 The Sen operator associated with a $p$-adic Galois representation

Let $L$ be a $p$-adic field and let $\mathcal{R}$ be a Banach $L$-algebra. Let $K$ be another $p$-adic field, $m$ be a positive integer and $\tau\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathcal{R})$ be a continuous representation. We recall the construction of the Sen operator associated with $\tau$, following [Sen93].

We fix embeddings of $K$ and $L$ in $\overline{\mathbb{Q}}_p$. The constructions that follow will depend on these choices. We suppose that the Galois closure $L^{\mathrm{Gal}}$ of $L$ over $\mathbb{Q}_p$ is contained in $K$. If this is not the case, we restrict $\tau$ to the open subgroup $\mathrm{Gal}(\overline{K}/KL^{\mathrm{Gal}}) \subset \mathrm{Gal}(\overline{K}/K)$. We denote by $\chi\colon \mathrm{Gal}(\overline{L}/L) \to \mathbb{Z}_p^\times$ the $p$-adic cyclotomic character. Let $L_\infty$ be a totally ramified $\mathbb{Z}_p$-extension of $L$. Let $\gamma$ be a topological generator of $\Gamma = \mathrm{Gal}(L_\infty/L)$. For a positive integer $n$, let $\Gamma_n \subset \Gamma$ be the subgroup generated by $\gamma^{p^n}$ and $L_n = L_\infty^{\langle\gamma^{p^n}\rangle}$ be the subfield of $L_\infty$ fixed by $\Gamma_n$. We have $L_\infty = \bigcup_n L_n$. Let $L'_n = L_nK$ and $G'_n = \mathrm{Gal}(\overline{L}/L'_n)$.

Write $\mathcal{R}^m$ for the $\mathcal{R}$-module over which $\mathrm{Gal}(\overline{K}/K)$ acts via $\tau$. We define an action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathcal{R}^m \widehat{\otimes}_L \mathbb{C}_p$ by letting $\sigma \in \mathrm{Gal}(\overline{K}/K)$ send $x \otimes y$ to $\tau(\sigma)(x) \otimes \sigma(y)$. Then by [Sen93], there exists a matrix $M \in \mathrm{GL}_m\big(\mathcal{R} \widehat{\otimes}_L \mathbb{C}_p\big)$, an integer $n \geqslant 0$ and a representation $\delta\colon \Gamma_n \to \mathrm{GL}_m(\mathcal{R} \otimes_L L'_n)$ such that for all $\sigma \in G'_n$ we have

$$M^{-1}\tau(\sigma)\sigma(M) = \delta(\sigma). \tag{15}$$

Definition 7.6. The Sen operator associated with $\tau$ is the element

$$\phi = \lim_{\sigma\to 1} \frac{\log(\delta(\sigma))}{\log(\chi(\sigma))}$$

of $\mathrm{M}_m(\mathcal{R} \widehat{\otimes}_L \mathbb{C}_p)$.

The limit in the definitions always exists and is independent of the choice of $\delta$ and $M$.

Now suppose that $\mathcal{R} = L$ and that $\tau$ is a Hodge–Tate representation with Hodge–Tate weights $h_1, h_2, \dots, h_m$. Let $\phi$ be the Sen operator associated with $\tau$; it is an element of $\mathrm{M}_m(\mathbb{C}_p)$. The following theorem is a consequence of the results of [Sen80].

Theorem 7.7. *The characteristic polynomial of $\phi$ is $\prod_{i=1}^m (X - h_i)$.*

We restrict now to the case $L = \mathcal{R} = \mathbb{Q}_p$, so that $\tau$ is a continuous representation $\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathbb{Q}_p)$. Define a $\mathbb{Q}_p$-Lie algebra $\mathfrak{g} \subset \mathrm{M}_m(\mathbb{Q}_p)$ by $\mathfrak{g} = \mathbb{Q}_p \cdot \log(\tau(\mathrm{Gal}(\overline{K}/K)))$. We say that $\mathfrak{g}$ is the Lie algebra of $\tau(\mathrm{Gal}(\overline{K}/K))$. Let $\phi$ be the Sen operator associated with $\tau$.

Theorem 7.8 [Sen73, Theorem 1]. *The Sen operator $\phi$ is an element of $\mathfrak{g} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$.*

*Remark* 7.9. The proof of Theorem 7.8 relies on the fact that $\tau(\mathrm{Gal}(\overline{K}/K))$ is a finite-dimensional Lie group. It is doubtful that this proof can be generalized to the relative case.

### 7.3 The relative Sen operator associated with $\rho_r$

Fix a radius $r$ in the set $\{r_i\}_{i\in\mathbb{N}^{>0}}$. Consider as usual the representation $\rho_r\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}_{r,0})$. We defined earlier a $p$-adic field $K_{H_r,p}$. Write $G_{K_{H_r,p}}$ for its absolute Galois group. We look at the restriction $\rho_r|_{G_{K_{H_r,p}}}\colon G_{K_{H_r,p}} \to \mathrm{GSp}_4(\mathbb{I}_{r,0})$ as a representation with values in $\mathrm{GL}_4(\mathbb{I}_{r,0})$. Recall that $\mathfrak{G}_r^{\mathrm{loc}}$ is the Lie algebra associated with the image of $\rho_r|_{G_{K_{H_r,p}}}$. The goal of this section is to prove an analogue of Theorem 7.8 for this representation, i.e. to attach to $\rho_r|_{G_{K_{H_r,p}}}$ a '$\mathbb{B}_r$-Sen operator' belonging to $\mathfrak{G}^{\mathrm{loc}}_{r,\mathbb{C}_p}$. We start by constructing various Sen operators via Definition 7.6.

(i) The $\mathbb{Q}_p$-algebra $\mathbb{I}_{r,0}$ is complete for the $p$-adic topology. We associate with $\rho_r|_{G_{K_{H_r,p}}}$ a Sen operator $\phi_r \in \mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$.

(ii) Let $\mathfrak{a} \in S_2$. Then $\mathbb{I}_{r,0}/\mathfrak{a}$ is a finite-dimensional $\mathbb{Q}_p$-algebra. As usual write $\pi_\mathfrak{a}\colon \mathbb{I}_{r,0} \to \mathbb{I}_{r,0}/\mathfrak{a}$ for the natural projection. Denote by $\rho_{r,\mathfrak{a}}$ the representation $\pi_\mathfrak{a} \circ \rho_r|_{G_{K_{H_r,p}}} \colon G_{K_{H_r,p}} \to \mathrm{GL}_4(\mathbb{I}_{r,0}/\mathfrak{a})$. We associate with $\rho_{r,\mathfrak{a}}$ a Sen operator $\phi_{r,\mathfrak{a}} \in \mathrm{M}_4((\mathbb{I}_{r,0}/\mathfrak{a}) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)$.

(iii) Let $\mathfrak{a} \in S_2$. Let $d$ be the $\mathbb{Q}_p$-dimension of $\mathbb{I}_{r,0}/\mathfrak{a}$. Let $k$ be a positive integer. An $\mathbb{I}_{r,0}/\mathfrak{a}$-linear endomorphism of $(\mathbb{I}_{r,0}/\mathfrak{a})^k$ defines a $\mathbb{Q}_p$-linear endomorphism of the underlying $\mathbb{Q}_p$-vector space $\mathbb{Q}_p^{kd}$. This gives natural maps $\alpha_{\mathbb{Q}_p}\colon \mathrm{M}_k(\mathbb{I}_{r,0}/\mathfrak{a}) \to \mathrm{M}_{kd}(\mathbb{Q}_p)$ and $\alpha_{\mathbb{Q}_p}^\times\colon \mathrm{GL}_k(\mathbb{I}_{r,0}/\mathfrak{a}) \to \mathrm{GL}_{kd}(\mathbb{Q}_p)$ (we leave the dependence of these morphisms on $k$ implicit). Choose $k = 4$ and consider the representation $\rho_{r,\mathfrak{a}}^{\mathbb{Q}_p} = \alpha_{\mathbb{Q}_p}^\times \circ \rho_{r,\mathfrak{a}}\colon G_\mathbb{Q} \to \mathrm{GL}_{4d}(\mathbb{Q}_p)$. We associate with $\rho_{r,\mathfrak{a}}^{\mathbb{Q}_p}$ a Sen operator $\phi_{r,\mathfrak{a}}^{\mathbb{Q}_p} \in \mathrm{M}_{4d}(\mathbb{C}_p)$.

The operators constructed in constructions (i), (ii) and (iii) are related by the following lemma. We write $\pi_{\mathfrak{a},\mathbb{C}_p} = \pi_\mathfrak{a} \otimes 1\colon \mathbb{I}_{r,0,\mathbb{C}_p} \to \mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p}$. As above, we let $d$ be the $\mathbb{Q}_p$-dimension of $\mathbb{I}_{r,0}/\mathfrak{a}$. For every positive integer $k$, we set $\alpha_{\mathbb{C}_p} = \alpha_{\mathbb{Q}_p} \otimes 1\colon \mathrm{M}_k(\mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p}) \to \mathrm{M}_{kd}(\mathbb{C}_p)$ and $\alpha_{\mathbb{C}_p}^\times = \alpha_{\mathbb{Q}_p}^\times \otimes 1\colon \mathrm{GL}_k(\mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p}) \to \mathrm{GL}_{kd}(\mathbb{C}_p)$.

Lemma 7.10. *For every $\mathfrak{a} \in S_2$, the following relations hold:*

(i) $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a},\mathbb{C}_p}(\phi_r)$;

(ii) $\phi_{r,\mathfrak{a}}^{\mathbb{Q}_p} = \alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$.

*Proof.* This can be checked directly from the construction of the Sen operator presented in § 7.2. □

Recall that there is a natural inclusion $\iota'_{\mathbb{B}_r,\mathbb{C}_p}\colon \mathbb{I}_{r,0,\mathbb{C}_p} \hookrightarrow \mathbb{B}_{r,\mathbb{C}_p}$. We define the $\mathbb{B}_r$-Sen operator attached to $\rho_r|_{G_{K_{H_r,p}}}$ as

$$\phi_{\mathbb{B}_r} = \iota'_{\mathbb{B}_r,\mathbb{C}_p}(\phi_r).$$

By definition, $\phi_{\mathbb{B}_r}$ is an element of $\mathrm{M}_4(\mathbb{B}_{r,\mathbb{C}_p})$. Since $\mathbb{B}_{r,\mathbb{C}_p} = \varprojlim_{\mathfrak{a}\in S_2} \mathbb{I}_{r,0}/\mathfrak{a}$, it is clear that $\phi_{\mathbb{B}_r} = \varprojlim_{\mathfrak{a}\in S_2} \pi_{\mathfrak{a},\mathbb{C}_p}(\phi_r)$. Then Lemma 7.10(i) implies that

$$\phi_{\mathbb{B}_r} = \varprojlim_{\mathfrak{a}\in S_2} \phi_{r,\mathfrak{a}}. \tag{16}$$

Note that Theorem 7.8 can be applied only to representations with coefficients in $\mathbb{Q}_p$, hence to construction (iii) above. However, we can use Lemma 7.10(ii) to show the following.

Proposition 7.11. *The operator $\phi_{\mathbb{B}_r}$ belongs to the Lie algebra $\mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}}$. In particular, it belongs to $\mathfrak{G}_{r,\mathbb{C}_p}$.*

*Proof.* For every $\mathfrak{a} \in S_2$, let $d_\mathfrak{a}$ be the degree of the extension $\mathbb{I}_{r,0}/\mathfrak{a}$ over $\mathbb{Q}_p$. Let $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc},\mathbb{Q}_p}$ be the $\mathbb{Q}_p$-Lie subalgebra of $\mathrm{M}_{4d_\mathfrak{a}}(\mathbb{Q}_p)$ associated with the image of $\rho_{r,\mathfrak{a}}^{\mathbb{Q}_p}$, defined by $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc},\mathbb{Q}_p} = \mathbb{Q}_p \cdot \log(\mathrm{Im}\, \rho_{r,\mathfrak{a}}^{\mathbb{Q}_p})$. Let $\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc},\mathbb{Q}_p} = \mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc},\mathbb{Q}_p} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$. As $\mathrm{Im}\, \rho_{r,\mathfrak{a}}^{\mathbb{Q}_p} = \alpha_{\mathbb{Q}_p}^\times(\mathrm{Im}\, \rho_{r,\mathfrak{a}})$, we can write

$$\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc},\mathbb{Q}_p} = \alpha_{\mathbb{C}_p}(\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc}}). \tag{17}$$



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The representation $\rho_{r,\mathfrak{a}}^{\mathbb{Q}_p}$ satisfies the assumptions of Theorem 7.8, so the Sen operator $\phi_{r,0,\mathfrak{a}}^{\mathbb{Q}_p}$ belongs to $\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc},\mathbb{Q}_p}$. By Lemma 7.10(ii) $\phi_{r,\mathfrak{a}}^{\mathbb{Q}_p} = \alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$. Then (17) and the injectivity of $\alpha_{\mathbb{C}_p}$ give

$$\phi_{r,\mathfrak{a}} \in \mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc}}. \tag{18}$$

As $\mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}} = \varprojlim_{\mathfrak{a}\in S_2} \mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc}}$, (16) and (18) imply that $\phi_{\mathbb{B}_r} \in \mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}}$. □

### 7.4 The exponential of the Sen operator

We use the work of the previous section to construct an element of $\mathrm{GL}_4(\mathbb{B}_r)$ that has some specific eigenvalues and normalizes the Lie algebra $\mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}}$. Such an element will be used in § 8 to induce a $B_{r,\mathbb{C}_p}$-module structure on some subalgebra of $\mathfrak{G}_{r,\mathbb{C}_p}$, thus replacing the matrix '$\rho(\sigma)$' of [HT15] that is not available in the non-ordinary setting.

Let $\phi_r \in \mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$be the Sen operator defined in the previous section. We rescale it to define an element $\phi_r' = \log(u)\phi_r$, where $u = 1+p$. Let $(T_1, T_2)$ be the images in $A_r$ of the coordinate functions on the weight space. The logarithms and the exponentials in the following proposition are defined via the usual power series, that converge because of the assumption (exp) we made in the beginning of § 7.

Lemma 7.12. *The eigenvalues of $\phi_r'$ are $0$, $\log(u^{-2}(1+T_2))$, $\log(u^{-1}(1+T_1))$ and $\log(u^{-3}(1+T_1)(1+T_2))$. In particular, the exponential series defines an element $\exp(\phi_r') \in \mathrm{GL}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$, of eigenvalues $1$, $u^{-2}(1+T_2)$, $u^{-1}(1+T_1)$ and $u^{-3}(1+T_1)(1+T_2)$.*

*Proof.* The $p$-adic Galois representation $\rho_f$ associated with a classical eigenform $f$ of weight $(k_1, k_2)$ is Hodge–Tate with Hodge–Tate weights $(0, k_2-2, k_1-1, k_1+k_2-3)$. By Theorem 7.7 these weights are the eigenvalues of the Sen operator $\phi_f$ associated with $\rho_f$. By Lemma 7.10(i), the eigenvalues of $\phi_r$ interpolate those of the operators $\phi_f$ when $f$ varies in the set of classical points of $A_r$. As such points form a Zariski-dense subset of $\operatorname{Spec} A_r$, the interpolation is unique and can be easily computed. The lemma follows from this. □

Let $\Phi_{\mathbb{B}_r} = \iota_{\mathbb{B}_{r,\mathbb{C}_p}}(\exp(\phi_{r,0}'))$. By definition, $\Phi_{\mathbb{B}_r}$ is an element of $\mathrm{GL}_4(\mathbb{B}_{r,\mathbb{C}_p})$. We show that it has the two properties we need. We define a matrix $C_{T_1,T_2} \in \mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ by

$$C_{T_1,T_2} = \mathrm{diag}\big(u^{-3}(1+T_1)(1+T_2), u^{-1}(1+T_1), u^{-2}(1+T_2), 1\big).$$

Proposition 7.13.

(i) *There exists $\gamma \in \mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ satisfying $\Phi_{\mathbb{B}_r} = \gamma C_{T_1,T_2}\gamma^{-1}$.*

(ii) *The element $\Phi_{\mathbb{B}_r}$ normalizes the Lie algebra $\mathfrak{G}_{r,\mathbb{C}_p}$.*

*Proof.* The matrices $\Phi_{\mathbb{B}_r}$ and $C_{T_1,T_2}$ have the same eigenvalues by Lemma 7.12. Hence, there exists $\gamma \in \mathrm{GL}_4(\mathbb{B}_{r,\mathbb{C}_p})$ satisfying the equality of part (i) if and only if the difference between any two of the eigenvalues of $\Phi_{\mathbb{B}_r}$ is invertible in $\mathbb{B}_r$. We check by a direct calculation that each of these differences belongs to an ideal of the form $P\cdot\mathbb{B}_r$ with $P \in S^{\mathrm{bad}}$, hence it is invertible in $\mathbb{B}_r$ by Remark 7.4. As both $\Phi_{\mathbb{B}_r}$ and $C_{T_1,T_2}$ are elements of $\mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$, we can take $\gamma \in \mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$.

Part (ii) follows from Proposition 7.11. □



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## 8. Existence of a Galois level in the residual symmetric cube and full cases

We have all the ingredients we need to state and prove our first main theorem. Let $h \in \mathbb{Q}^{+,\times}$. Let $\mathbb{T}_h$ be a local component of the $h$-adapted Hecke algebra of genus 2 and level $\Gamma_1(M) \cap \Gamma_0(p)$. Suppose that condition (exp) of § 7 is satisfied and that the residual Galois representation $\overline{\rho}_{\mathbb{T}_h}$ associated with $\mathbb{T}_h$ is either full or of symmetric cube type in the sense of Definition 3.11. Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family, i.e. the morphism of finite $\Lambda_h$-algebras describing an irreducible component of $\mathbb{T}_h$. Let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the Galois representation associated with $\theta$. Suppose that $\rho$ is $\mathbb{Z}_p$-regular in the sense of Definition 3.10. For every radius $r$ in the set $\{r_i\}_{i\in\mathbb{N}^{>0}}$ defined in § 4, let $\mathfrak{G}_r$ be the Lie algebra that we attached to $\mathrm{Im}\,\rho$ in § 7.1.

Theorem 8.1. *There exists a non-zero ideal $\mathfrak{l}$ of $\mathbb{I}_0$ such that*

$$\mathfrak{l} \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r \tag{19}$$

*for every $r \in \{r_i\}_{i\in\mathbb{N}^{>0}}$.*

Let $\Delta$ be the set of roots of $\mathrm{GSp}_4$ with respect to our choice of maximal torus. Recall that for $\alpha \in \Delta$ we denote by $\mathfrak{u}^\alpha$ the nilpotent subalgebra of $\mathfrak{gsp}_4$ corresponding to $\alpha$. Let $r$ be a radius in the set $\{r_i\}_{i\geqslant 1}$. We set $\mathfrak{U}^\alpha_r = \mathfrak{G}_r \cap \mathfrak{u}^\alpha(\mathbb{B}_r)$ and $\mathfrak{U}^\alpha_{r,\mathbb{C}_p} = \mathfrak{G}_{r,\mathbb{C}_p} \cap \mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})$, which coincides with $\mathfrak{U}^\alpha_r \,\widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$. Via the isomorphisms $\mathfrak{u}^\alpha(\mathbb{B}_r) \cong \mathbb{B}_r$ and $\mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p}) \cong \mathbb{B}_{r,\mathbb{C}_p}$, we see $\mathfrak{U}^\alpha_r$ as a $\mathbb{Q}_p$-vector subspace of $\mathbb{B}_r$ and $\mathfrak{U}^\alpha_{r,\mathbb{C}_p}$ as a $\mathbb{C}_p$-vector subspace of $\mathbb{B}_{r,\mathbb{C}_p}$.

Recall that $U^\alpha$ denotes the one-parameter unipotent subgroup of $\mathrm{GSp}_4$ associated with the root $\alpha$. Let $H_r$ be the normal open subgroup of $G_\mathbb{Q}$ defined in the beginning of § 7. Note that Proposition 6.14 holds with $\rho|_{H_0}$ replaced by $\rho|_{H_r}$ because $H_r$ is open in $G_\mathbb{Q}$. Let $U^\alpha(\rho|_{H_r}) = U^\alpha(\mathbb{I}^\circ_0) \cap \rho(H_r)$ and $U^\alpha(\rho_r) = U^\alpha \cap \rho_r(H_r)$. Via the isomorphisms $U^\alpha(\mathbb{I}_0) \cong \mathbb{I}_0$ and $U^\alpha(\mathbb{I}_{r,0}) \cong \mathbb{I}_{r,0}$ we identify $U^\alpha(\rho|_{H_0})$ and $U^\alpha(\rho_r)$ with $\mathbb{Z}_p$-submodules of $\mathbb{I}_0$ and $\mathbb{I}_{r,0}$, respectively. Note that the injection $\mathbb{I}^\circ_0 \hookrightarrow \mathbb{I}^\circ_{r,0}$ induces an isomorphism of $\mathbb{Z}_p$-modules $U^\alpha(\rho|_{H_0}) \cong U^\alpha(\rho_r)$.

We define a nilpotent subalgebra of $\mathfrak{gsp}_4(\mathbb{I}_{r,0})$ by $\mathfrak{U}^\alpha_{\mathbb{I}_{r,0}} = \mathbb{Q}_p \cdot \log(U^\alpha(\rho_r))$. We identify $\mathfrak{U}^\alpha_{\mathbb{I}_{r,0}}$ with a $\mathbb{Q}_p$-vector subspace of $\mathbb{I}_{r,0}$. Note that the natural injection $\iota_{\mathbb{B}_r}\colon \mathbb{I}_{r,0} \hookrightarrow \mathbb{B}_r$ induces an injection $\mathfrak{U}^\alpha_{\mathbb{I}_{r,0}} \hookrightarrow \mathfrak{U}^\alpha_r$ for every $\alpha$.

Lemma 8.2. *For every $\alpha \in \Delta$ and every $r$, there exists a non-zero ideal $\mathfrak{l}^\alpha$ of $\mathbb{I}_0$, independent of $r$, such that the $B_r$-span of $\mathfrak{U}^\alpha_r$ contains $\mathfrak{l}^\alpha\mathbb{B}_r$.*

*Proof.* Let $d$ be the dimension of $Q(\mathbb{I}^\circ_0)$ over $Q(\Lambda_h)$. Let $\alpha \in \Delta$. By Proposition 6.14 the unipotent subgroup $U^\alpha(\rho|_{H_r})$ contains a basis $E = \{e_i\}_{i=1,\dots,d}$ of a $\Lambda_h$-lattice in $\mathbb{I}^\circ_0$. Lemma 6.6 implies that the $\Lambda_h[p^{-1}]$-span of $E$ contains a non-zero ideal $\mathfrak{l}^\alpha$ of $\mathbb{I}_0$. Consider the map $\iota^\alpha\colon U^\alpha(\mathbb{I}_0) \to \mathfrak{u}^\alpha(\mathbb{B}_r)$ given by the composition

$$U^\alpha(\mathbb{I}_0) \hookrightarrow U^\alpha(\mathbb{I}_{r,0}) \xrightarrow{\log} \mathfrak{u}^\alpha(\mathbb{I}_{r,0}) \hookrightarrow \mathfrak{u}^\alpha(\mathbb{B}_r),$$

where all the maps have been introduced above. Note that $\iota^\alpha(U^\alpha(\rho|_{H_0})) \subset \mathfrak{U}^\alpha_r$. Let $E_{\mathbb{B}_r} = \iota^\alpha(E)$. As $\iota^\alpha$ is a morphism of $\mathbb{I}_0$-modules we have

$$B_r \cdot \mathfrak{U}^\alpha_r \supset B_r \cdot E_{\mathbb{B}_r} = B_r \cdot (\Lambda_h[p^{-1}] \cdot E_{\mathbb{B}_r}) = B_r \cdot \iota^\alpha(\Lambda_h[p^{-1}] \cdot E) \supset \mathbb{B}_r \cdot \iota^\alpha(\mathfrak{l}^\alpha) = \mathfrak{l}^\alpha\mathbb{B}_r.$$

By construction and by Remark 7.2, the ideal $\mathfrak{l}^\alpha$ can be chosen independently of $r$. □



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Let $\gamma$ be an element of $\mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ such that $\Phi_{\mathbb{B}_r} = \gamma C_{T_1,T_2}\gamma^{-1}$; it exists by Proposition 7.13(i). Let $\mathfrak{G}^{\gamma}_{r,\mathbb{C}_p} = \gamma^{-1}\mathfrak{G}_{r,\mathbb{C}_p}\gamma$. For each $\alpha \in \Delta$, let $\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p} = \mathfrak{u}^{\alpha}(\mathbb{B}_{r,\mathbb{C}_p}) \cap \mathfrak{G}^{\gamma}_{r,\mathbb{C}_p}$. We prove the following lemma by an argument similar to that of [HT15, Theorem 4.8].

LEMMA 8.3. *For every $\alpha \in \Delta$, the Lie algebra $\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}$ is a $B_{r,\mathbb{C}_p}$-submodule of $\mathbb{B}_{r,\mathbb{C}_p}$.*

*Proof.* By Proposition 7.13(ii), the operator $\Phi_{\mathbb{B}_r}$ normalizes $\mathfrak{G}_{r,\mathbb{C}_p}$, hence $C_{T_1,T_2}$ normalizes $\mathfrak{G}^{\gamma}_{r,\mathbb{C}_p}$. Let $\alpha_1$ and $\alpha_2$ be the roots sending $\mathrm{diag}(t_1, t_2, \nu t_2^{-1}, \nu t_1^{-1}) \in T_2$ to $t_1/t_2$ and $\nu^{-1}t_2^2$, respectively. With respect to our choice of Borel subgroup, the set of positive roots of $\mathrm{GSp}_4$ is $\{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$. Conjugation by $C_{T_1,T_2}$ on the $\mathbb{C}_p$-vector space $\mathfrak{u}^{\alpha_1}(\mathbb{B}_{r,\mathbb{C}_p})$ induces multiplication by $\alpha_1(C_{T_1,T_2}) = u^{-2}(1+T_2)$. As $u^{-2} \in \mathbb{Z}_p^{\times}$ and $\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}$ is stable under $\mathrm{Ad}(C_{T_1,T_2})$, multiplication by $1+T_2$ on $\mathfrak{u}^{\alpha_1}(\mathbb{B}_{r,\mathbb{C}_p})$ leaves $\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}$ stable. Now we compute

$$(1+T_2)\cdot\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p} = (1+T_2)\cdot[\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}] = [(1+T_2)\cdot\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}] \subset [\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}] = \mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p},$$

where the inclusion $(1+T_2)\cdot\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p} \subset \mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}$ is the result of the previous sentence. Similarly, conjugation by $C_{T_1,T_2}$ on the $\mathbb{C}_p$-vector space $\mathfrak{u}^{\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ induces multiplication by $\alpha_2(C_{T_1,T_2}) = u\cdot(1+T_1)/(1+T_2)$. As $u \in \mathbb{Z}_p^{\times}$ and $\mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}$ is stable under $\mathrm{Ad}(C_{T_1,T_2})$, multiplication by $1+T_2$ on $\mathfrak{u}^{\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ leaves $\mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}$ stable. The same calculation as above shows that multiplication by $(1+T_1)/(1+T_2)$ on $\mathfrak{u}^{\alpha_1+\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ leaves $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ stable. We deduce that multiplication by $1+T_1$ also leaves $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ stable. As $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ is a $\mathbb{C}_p$-vector space, the $\mathbb{C}_p[T_1,T_2]$-module structure on $\mathfrak{u}^{\alpha_1+\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ induces a $\mathbb{C}_p[T_1,T_2]$-module structure on $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$. With respect to the $p$-adic topology $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ is complete and $\mathbb{C}_p[T_1,T_2]$ is dense in $B_{r,\mathbb{C}_p}$, so the $B_{r,\mathbb{C}_p}$-module structure on $\mathfrak{u}^{\alpha_1+\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ induces a $B_{r,\mathbb{C}_p}$-module structure on $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$.

If $\beta$ is any root, we can write

$$\begin{aligned} B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\beta}_{r,\mathbb{C}_p} &= B_{r,\mathbb{C}_p}\cdot[\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\beta-\alpha_1-\alpha_2}_{r,\mathbb{C}_p}] \\ &\subset [B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\beta-\alpha_1-\alpha_2}_{r,\mathbb{C}_p}] \subset [\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\beta-\alpha_1-\alpha_2}_{r,\mathbb{C}_p}] = \mathfrak{U}^{\gamma,\beta}_{r,\mathbb{C}_p}, \end{aligned}$$

where the inclusion $B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p} \subset \mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ is the result of the previous paragraph. □

*Proof of Theorem 8.1.* Let $E_{\mathbb{B}_r} \subset \mathfrak{U}^{\alpha}_r$ be the set defined in the proof of Lemma 8.2. Let $E_{\mathbb{B}_r,\mathbb{C}_p} = \{e\otimes 1 \mid e \in E_{\mathbb{B}_r}\} \subset \mathfrak{U}^{\alpha}_{r,\mathbb{C}_p}$. Consider the Lie subalgebra $B_{r,\mathbb{C}_p}\cdot\mathfrak{G}_{r,\mathbb{C}_p}$ of $\mathfrak{gsp}_4(\mathbb{B}_{r,\mathbb{C}_p})$. For every $\alpha \in \Delta$ we have $B_{r,\mathbb{C}_p}\cdot\mathfrak{G}_{r,\mathbb{C}_p} \cap \mathfrak{u}^{\alpha}(B_{r,\mathbb{C}_p}) = B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\alpha}_r$. By Lemma 8.2 there exists an ideal $\mathfrak{l}^{\alpha}$ of $\mathbb{I}_0$, independent of $r$, such that $\mathfrak{l}^{\alpha}\cdot\mathbb{B}_{r,\mathbb{C}_p} \subset B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\alpha}_r$. Let $\mathfrak{l}_0 = \prod_{\alpha\in\Delta}\mathfrak{l}^{\alpha}$. Then Lemma 6.11 gives an inclusion

$$\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p}) \subset B_{r,\mathbb{C}_p}\cdot\mathfrak{G}_{r,\mathbb{C}_p}. \tag{20}$$

As before, let $\gamma$ be an element of $\mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ satisfying $\Phi_{\mathbb{B}_r} = \gamma C_{T_1,T_2}\gamma^{-1}$. The Lie algebra $\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ is stable under $\mathrm{Ad}(\gamma^{-1})$, so (20) implies that $\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p}) = \gamma^{-1}(\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p}))\gamma \subset \gamma^{-1}(B_{r,\mathbb{C}_p}\cdot\mathfrak{G}_r)\gamma = B_{r,\mathbb{C}_p}\cdot\gamma^{-1}\mathfrak{G}_r\gamma = B_{r,\mathbb{C}_p}\cdot\mathfrak{G}^{\gamma}_r$. We deduce that, for every $\alpha \in \Delta$,

$$\begin{aligned} \mathfrak{l}_0\cdot\mathfrak{u}^{\alpha}(\mathbb{B}_{r,\mathbb{C}_p}) &= \mathfrak{u}^{\alpha}(\mathbb{B}_{r,\mathbb{C}_p}) \cap \mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p}) \subset \mathfrak{u}^{\alpha}(\mathbb{B}_{r,\mathbb{C}_p}) \cap B_{r,\mathbb{C}_p}\cdot\mathfrak{G}^{\gamma}_{r,\mathbb{C}_p} \\ &= B_{r,\mathbb{C}_p}\cdot(\mathfrak{u}^{\alpha}(\mathbb{B}_{r,\mathbb{C}_p}) \cap \mathfrak{G}^{\gamma}_{r,\mathbb{C}_p}) = B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}. \end{aligned} \tag{21}$$



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By Lemma 8.3 $\mathfrak{U}^{\alpha,\gamma}_{r,\mathbb{C}_p}$ is a $B_{r,\mathbb{C}_p}$-submodule of $\mathfrak{u}_r(\mathbb{B}_{r,\mathbb{C}_p})$, so $B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}=\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}$. Hence, (21) gives

$$\mathfrak{l}_0\cdot\mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})\subset\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p} \tag{22}$$

for every $\alpha$. Set $\mathfrak{l}_1=\mathfrak{l}_0^2$. By Lemma 6.11 and Remark 6.12, applied to the Lie algebra $\mathfrak{G}_{r,\mathbb{C}_p}$ and the set of ideals $\{\mathfrak{l}_1\mathbb{B}_r\}_{\alpha\in\Delta}$, (22) implies that $\mathfrak{l}_1\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})\subset\mathfrak{G}^\gamma_{r,\mathbb{C}_p}$. Observe that the left-hand side of the last equation is stable under $\mathrm{Ad}(\gamma)$, so we can write

$$\mathfrak{l}_1\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})=\gamma(\mathfrak{l}_1\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p}))\gamma^{-1}\subset\gamma\mathfrak{G}^\gamma_{r,\mathbb{C}_p}\gamma^{-1}=\mathfrak{G}_{r,\mathbb{C}_p}. \tag{23}$$

The same argument as in the end of the proof of [CIT16, Theorem 6.2] shows that the extension of scalars to $\mathbb{C}_p$ in (23) is unnecessary, up to restricting the ideal $\mathfrak{l}_1$. This amounts essentially to consider the inclusion $\mathfrak{l}_1\cdot\mathbb{B}_{r,\mathbb{C}_p}\subset\mathfrak{U}^\alpha_{r,\mathbb{C}_p}$ modulo an ideal $\mathfrak{a}$ of $\mathbb{I}_{r,0}$, for every root $\alpha$, and rewrite it in terms of well-chosen $\mathbb{Q}_p$-bases for the $\mathbb{Q}_p$-structures of the two sides. The projective limit over $\mathfrak{a}$ then gives the inclusion $\mathfrak{l}_1\cdot\mathbb{B}_r\subset\mathfrak{U}^\alpha_r$. For $\mathfrak{l}=\mathfrak{l}_1^2$, Lemma 6.11 and Corollary 6.12 give

$$\mathfrak{l}\cdot\mathfrak{sp}_4(\mathbb{B}_r)\subset\mathfrak{G}_r.$$

By definition, we have $\mathfrak{l}=\mathfrak{l}_1^2=\mathfrak{l}_0^4=\left(\prod_{\alpha\in\Delta}\mathfrak{l}^\alpha\right)^4$. For every $\alpha$ the ideal $\mathfrak{l}^\alpha$ provided by Lemma 8.2 is independent of $r$, so $\mathfrak{l}$ is also independent of $r$. This concludes the proof of Theorem 8.1. □

Definition 8.4. We call *Galois level* of $\theta$ and denote by $\mathfrak{l}_\theta$the largest ideal of $\mathbb{I}_0$ satisfying the inclusion (19).

### 8.1 The Galois level of ordinary families

We explain how, for an ordinary family of $\mathrm{GSp}_4$-eigenforms, we can use our arguments to prove a stronger result than Theorem 8.1. Let $M$ be a positive integer. Let $\mathbb{T}^{\mathrm{ord}}$ be a local component of the big ordinary cuspidal Hecke algebra of level $\Gamma_1(M)\cap\Gamma_0(p)$ for $\mathrm{GSp}_4$; it is a finite and flat $\Lambda_2$-algebra. With the terminology of § 4 we consider $\mathbb{T}^{\mathrm{ord}}$ as the genus 2, 0-adapted Hecke algebra of the given level. Suppose that the residual representation $\overline{\rho}_{\mathbb{T}^{\mathrm{ord}}}$ associated with $\mathbb{T}^{\mathrm{ord}}$ is absolutely irreducible and of $\mathrm{Sym}^3$ type in the sense of Definition 3.11. Let $\theta\colon\mathbb{T}^{\mathrm{ord}}\to\mathbb{I}^\circ$ be a family, i.e. the morphism of finite $\Lambda_2$ algebras describing an irreducible component of $\mathbb{T}^{\mathrm{ord}}$. By taking $h=0$ and $r_h=1$ in the construction in § 4 we obtain $\mathbb{T}_h=\mathbb{T}^{\mathrm{ord}}$. Note that all of our arguments and constructions are valid for this algebra; none of them relied on the fact that the slope of the family was positive.

We keep all the notation we introduced for the family $\theta$. Let $\rho\colon G_\mathbb{Q}\to\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the Galois representation associated with $\theta$. Suppose that $\rho$ is $\mathbb{Z}_p$-regular in the sense of Definition 3.10. Then we have the following.

Theorem 8.5. *There exists a non-zero ideal $\mathfrak{l}$ of $\mathbb{I}_0^\circ$ and an element $g$ of $\mathrm{GSp}_4(\mathbb{I}_0^\circ)$ such that*

$$g\Gamma_{\mathbb{I}_0^\circ}(\mathfrak{l})g^{-1}\subset\operatorname{Im}\rho. \tag{24}$$

The main difference with respect to the proof of Theorem 8.1 is that relative Sen theory is not necessary anymore, because the exponential of the Sen operator defined in § 7.4 is replaced by an element provided by the ordinarity of $\rho$. This is the reason why we do not need the Lie-theoretic constructions and we obtain a group-theoretic result. Note that this also makes the inversion of $p$ unnecessary. Theorem 8.5 is an analogue of [Lan16, Theorem 2.4], which deals with ordinary



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families of $\mathrm{GL}_2$-eigenforms, and a generalization to the case where $\mathbb{I}^\circ \neq \Lambda_2$ of [HT15, Theorem 4.8] for $n = 2$ and families of residual symmetric cube type.

We only sketch the proof of the theorem, pointing out the differences with respect to that of Theorem 8.1.

*Proof.* Let $u = 1+p$, let $\chi$ be the $p$-adic cyclotomic character and, for $\sigma \in \mathbb{I}_0^{\circ,\times}$, let $\mathrm{ur}(\sigma)\colon G_{\mathbb{Q}_p} \to \mathbb{I}_0^{\circ,\times}$ be the unramified character sending a lift of the Frobenius automorphism to $\sigma$. By Hida theory, the ordinarity of $\theta$ implies the ordinarity of the Galois representation $\rho$, in the sense that the restriction of $\rho$ to a decomposition group at $p$ is a conjugate of an upper triangular representation with diagonal entries given by

$$\begin{aligned}(\chi^{-3} \cdot ((1+T_1)(1+T_2))^{\log(\chi)/\log(u)}\,\mathrm{ur}(\alpha), \chi^{-1} \cdot (1+T_1)^{\log(\chi)/\log(u)}\,\mathrm{ur}(\beta), \chi^{-2}\\ \cdot (1+T_2)^{\log(\chi)/\log(u)}\,\mathrm{ur}(\gamma), \mathrm{ur}(\delta))\end{aligned}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{I}_0^{\circ,\times}$. Consider a conjugate of $\rho$ that has the form displayed above. Up to conjugation by an upper triangular matrix, we can suppose that $\mathrm{Im}\,\rho$ contains a diagonal $\mathbb{Z}_p$-regular element. By Proposition 6.3, we can further replace the representation with a conjugate by a diagonal matrix such that $\rho(H_0) \subset \mathrm{GSp}_4(\mathbb{I}_0^\circ)$. This is true because the basis we start with in the proof of Proposition 6.3 is replaced by a collinear one.

We work from now on with the last one of the conjugates of the original $\rho$ mentioned in the previous paragraph; this choice gives the element $g$ appearing in Theorem 8.5. It is clear from the form of $\rho$ that there exists an element $\sigma$ in the inertia subgroup at $p$ such that $\rho(\sigma) = C_{T_1,T_2}$, where $C_{T_1,T_2}$ is the matrix defined in § 7.4. Hence, $\mathrm{Im}\,\rho$ is stable under $\mathrm{Ad}\,C_{T_1,T_2}$. The same argument as in Lemma 8.3, with the nilpotent algebra $\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}$ replaced by the unipotent subgroup $U^\alpha(\mathrm{Im}\,\rho)$ and the extension of rings $B_r \subset \mathbb{B}_r$ replaced by $\Lambda_2 \subset \mathbb{I}_0^\circ$, gives $U^\alpha(\mathrm{Im}\,\rho)$ a structure of $\Lambda_2$-module for every root $\alpha$ of $\mathrm{Sp}_4$. By Proposition 6.14, $U^\alpha(\mathrm{Im}\,\rho)$ contains a basis of a $\Lambda_2$-lattice in $\mathbb{I}_0^\circ$ for every $\alpha$. Hence, by Lemma 6.6, $U^\alpha(\mathrm{Im}\,\rho)$ contains a non-zero ideal of $\mathbb{I}_0^\circ$ for every $\alpha$. By Lemma 6.11 the group $\mathrm{Im}\,\rho$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ)$. □

## 9. The symmetric cube morphisms of Hecke algebras

Let $\mathrm{Sym}^3\colon \mathrm{GL}_2 \to \mathrm{GSp}_4$ be the morphism of group schemes over $\mathbb{Z}$ defined by the symmetric cube representation of $\mathrm{GL}_2$. If $R$ is a ring, we still denote by $\mathrm{Sym}^3$ the morphism $\mathrm{GL}_2(R) \to \mathrm{GSp}_4(R)$ induced by the morphism of group schemes. For every representation $\rho$ of a group with values in $\mathrm{GL}_2(R)$, we set $\mathrm{Sym}^3\rho = \mathrm{Sym}^3 \circ \rho$.

Kim and Shahidi [KS02, Theorem B] proved the existence of a Langlands functoriality transfer from $\mathrm{GL}_2$ to $\mathrm{GL}_4$ associated with $\mathrm{Sym}^3\colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_4(\mathbb{C})$. Thanks to an unpublished result by Jacquet, Piatetski-Shapiro and Shalika [KS02, Theorem 9.1], this transfer descends to $\mathrm{GSp}_4$. If $\pi$ is an automorphic representation of $\mathrm{GL}_2$, then the automorphic representation $\Pi = \mathrm{Sym}^3\pi$ of $\mathrm{GSp}_4$ given by the above construction is globally generic. In particular, it does not correspond to a holomorphic modular form for $\mathrm{GSp}_4$. However, Ramakrishnan and Shahidi showed that, when $\pi$ is associated with a modular form, the component at infinity $\Pi_\infty$ of $\Pi$ can be replaced by a holomorphic representation $\Pi_\infty^{\mathrm{hol}}$ such that $\Pi_f \otimes \Pi_\infty^{\mathrm{hol}}$ belongs to the $L$-packet of $\Pi$. This is the content of [RS07, Theorem A′], that we recall below. Note that in [RS07, Theorem A′], the theorem is stated only for $\pi$ associated with a form $f$ of level $\Gamma_0(N)$ and even weight $k \geqslant 2$, but Ramakrishnan pointed out that the proof also works when $f$ has level $\Gamma_1(N)$ and arbitrary weight $k \geqslant 2$.

Let $\pi$ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated with a cuspidal, non-CM eigenform $f$ of weight $k \geqslant 2$ and level $\Gamma_1(N)$ for some $N \geqslant 1$. Let $p$ be a prime not dividing $N$ and let $\rho_{f,p}$ be the $p$-adic Galois representation attached to $f$.

If $K$ is a compact open subgroup of $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$, we call the *level of* $K$ the smallest integer $M$ such that $K$ contains the principal congruence subgroup of $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$ of level $M$. Given an automorphic representation $\Pi$ of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, we call the level of $\Pi$ the smallest integer $M$ such that the finite component of $\Pi$ admits an invariant vector by a compact open subgroup of $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$ of level $M$.

Theorem 9.1 (See [RS07, Theorem A′]). *There exists a cuspidal automorphic representation* $\Pi^{\mathrm{hol}} = \bigotimes_v \Pi_v^{\mathrm{hol}}$ *of* $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$*, satisfying:*

(i) $\Pi_\infty^{\mathrm{hol}}$ *is in the holomorphic discrete series;*
(ii) $L(s, \Pi^{\mathrm{hol}}) = L(s, \pi, \mathrm{Sym}^3)$*;*
(iii) $\Pi^{\mathrm{hol}}$ *is unramified at primes not dividing* $N$*;*
(iv) $\Pi^{\mathrm{hol}}$ *admits an invariant vector by a compact open subgroup* $K$ *of* $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ *of level* $N(\mathrm{Sym}^3 \rho_{f,p})$.

A simple computation of Hodge–Tate weights gives the following corollary.

Corollary 9.2. *Let* $f$ *be a cuspidal, non-CM* $\mathrm{GL}_2$*-eigenform of weight* $k \geqslant 2$*. For every prime* $\ell$*, let* $\rho_{f,\ell}$ *be the* $\ell$*-adic Galois representation associated with* $f$*. There exists a cuspidal* $\mathrm{GSp}_4$*-eigenform* $F$ *of weight* $(2k-1, k+1)$ *with associated* $\ell$*-adic Galois representation* $\mathrm{Sym}^3 \rho_{f,\ell}$ *for every prime* $\ell$*. For every prime* $p$ *not dividing* $N$*, the level of* $F$ *is a divisor of the prime-to-*$p$ *conductor of* $\mathrm{Sym}^3 \rho_{f,p}$*.*

We denote by $\mathrm{Sym}^3 f$ the cuspidal Siegel eigenform given by the corollary. Let $N(f)$ and $N(\mathrm{Sym}^3 f)$ be the levels of $f$ and $\mathrm{Sym}^3 f$, respectively. We give an upper bound for $N(\mathrm{Sym}^3 f)$ in terms of $N(f)$. Let $N(f) = \prod_{i=1}^d \ell_i^{a_i}$ be the decomposition of $N(f)$ in powers of distinct prime factors. For $i \in \{1, 2, \ldots, d\}$ set

$$a_i' = \begin{cases} 1 & \text{if } f \text{ is Steinberg at } \ell_i \text{ and } a_i = 1, \\ 3a_i & \text{otherwise.} \end{cases}$$

We define $M(f) = \prod_{i=1}^d \ell_i^{a_i'}$.

Corollary 9.3. *We have* $N(\mathrm{Sym}^3 f) \mid M(f)$*. In particular,* $N(\mathrm{Sym}^3 f) \mid N(f)^3$*.*

*Proof.* At places where the automorphic representation $\pi$ giving rise to $f$ is Steinberg and Iwahori-spherical, we look at regular unipotent elements in the image of an inertia subgroup at $p$ to check that the symmetric cube of $\pi$ is also Iwahori-spherical. At the other places, we give a bound on the conductor of the local Galois representation via Livné's formula [Liv89, Proposition 1.1] and apply Theorem 9.1(iv). $\square$

Borrowing the terminology of [Lud14, § 4.3], we say that $\Gamma_1^{(1)}(N)$ and $\Gamma_1^{(2)}(N^3)$ are compatible levels for the symmetric cube transfer for all $N \in \mathbb{N}$.



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## 9.1 Constructing the morphisms of Hecke algebras

As usual, we fix an integer $N \geqslant 1$ and a prime $p$ not dividing $N$. We work with the abstract Hecke algebras $\mathcal{H}_1^N$, $\mathcal{H}_2^N$ spherical outside $N$ and Iwahoric dilating at $p$. Let $M = N^3$. If $f$ is a non-CM $\mathrm{GL}_2$-eigenform of level $\Gamma_1(N)$, we denote by $\mathrm{Sym}^3 f$ the classical, cuspidal $\mathrm{GSp}_4$-eigenform of level $\Gamma_1(M)$ given by Corollary 9.2. In the following, we define some morphisms of Hecke algebras that allow to recover the system of Hecke eigenvalues of the $p$-stabilizations of $\mathrm{Sym}^3 f$ in terms of that of a $p$-stabilization of $f$. If $\chi$ is a system of Hecke eigenvalues, we write $\chi_\ell$ for its local component at the prime $\ell$.

DEFINITION 9.4. For every prime $\ell \nmid Np$, let

$$\lambda_\ell \colon \mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_\ell), \mathrm{GSp}_4(\mathbb{Z}_\ell)) \to \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell), \mathrm{GL}_2(\mathbb{Z}_\ell))$$

be the morphism defined by

$$\begin{aligned}
&T_{\ell,0}^{(2)} \mapsto (T_{\ell,0}^{(1)})^3,\\
&T_{\ell,1}^{(2)} \mapsto -(T_{\ell,1}^{(1)})^6 + (4\ell-2)T_{\ell,0}^{(1)}(T_{\ell,1}^{(1)})^4 + (6\ell-4\ell^2)(T_{\ell,0}^{(1)})^2(T_{\ell,1}^{(1)})^2 - 3\ell^2(T_{\ell,0}^{(1)})^3,\\
&T_{\ell,2}^{(2)} \mapsto (T_{\ell,1}^{(1)})^3 - 2\ell T_{\ell,1}^{(1)}T_{\ell,0}^{(1)}.
\end{aligned}$$

Let $\lambda^{Np} \colon \mathcal{H}_2^{Np} \to \mathcal{H}_1^{Np}$be the morphism defined by $\lambda^{Np} = \bigotimes_{\ell \nmid Np} \lambda_\ell$.

DEFINITION 9.5. For $i \in \{1, 2, \ldots, 8\}$ we define morphisms

$$\lambda_{i,p} \colon \mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p))^- \to \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p)).$$

For $i \in \{1, 2, 3, 4\}$, the morphism $\lambda_{i,p}$ is defined on a set of generators of $\mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p))^-$ as follows:

(1) $\lambda_{1,p}$ maps $\quad t_{p,0}^{(2)} \mapsto (t_{p,0}^{(1)})^3, \quad t_{p,1}^{(2)} \mapsto t_{p,0}^{(1)}(t_{p,1}^{(1)})^4, \quad t_{p,2}^{(2)} \mapsto (t_{p,1}^{(1)})^3;$

(2) $\lambda_{2,p}$ maps $\quad t_{p,0}^{(2)} \mapsto (t_{p,0}^{(1)})^3, \quad t_{p,1}^{(2)} \mapsto (t_{p,0}^{(1)})^2(t_{p,1}^{(1)})^2, \quad t_{p,2}^{(2)} \mapsto (t_{p,1}^{(1)})^3;$

(3) $\lambda_{3,p}$ maps $\quad t_{p,0}^{(2)} \mapsto (t_{p,0}^{(1)})^3, \quad t_{p,1}^{(2)} \mapsto t_{p,0}^{(1)}(t_{p,1}^{(1)})^4, \quad t_{p,2}^{(2)} \mapsto t_{p,0}^{(1)}t_{p,1}^{(1)};$

(4) $\lambda_{4,p}$ maps $\quad t_{p,0}^{(2)} \mapsto (t_{p,0}^{(1)})^3, \quad t_{p,1}^{(2)} \mapsto (t_{p,0}^{(1)})^4(t_{p,1}^{(1)})^{-2}, \quad t_{p,2}^{(2)} \mapsto t_{p,0}^{(1)}t_{p,1}^{(1)}.$

For $i \in \{5, 6, 7, 8\}$, the morphism $\lambda_{i,p} \colon \mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p)) \to \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ is given by

$$\lambda_{i,p} = \delta \circ \lambda_{i-4,p},$$

where $\delta$ is the automorphism of $\mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ defined on a set of generators of the subalgebra $\mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))^-$ by

$$\delta(t_{p,0}^{(1)}) = t_{p,0}^{(1)}, \quad \delta(t_{p,1}^{(1)}) = t_{p,0}^{(1)}(t_{p,1}^{(1)})^{-1} \tag{25}$$

and extended in the unique way.

Let $f$ be as in the beginning of the section and let $f^{\mathrm{st}}$ be one of its $p$-stabilizations. Let $\chi_{1,p} \colon \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p)) \to \overline{\mathbb{Q}}_p$ and $\chi_{1,p}^{\mathrm{st}} \colon \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), I_{1,p})^- \to \overline{\mathbb{Q}}_p$ be the systems of Hecke eigenvalues at $p$ of $f$ and $f^{\mathrm{st}}$, respectively. Note that $\chi_{1,p}$ is the restriction of $\chi_{1,p}^{\mathrm{st}}$ to the abstract spherical Hecke algebra at $p$.

Recall from § 2.2 that for $g = 1, 2$ there is an isomorphism of $\mathbb{Q}$-algebras

$$\iota_{I_{2,p}}^{T_2} \colon \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_p), I_{g,p})^- \to \mathcal{H}(T_g(\mathbb{Q}_p), T_g(\mathbb{Z}_p))^-.$$



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Let $\iota_{T_2}^{I_{2,p}}$ be its inverse. Then $\chi_g^{\mathrm{st}}\circ\iota_{T_g}^{I_{g,p}}$ is a character $\mathcal{H}(T_g(\mathbb{Q}_p),T_g(\mathbb{Z}_p))^-\to\overline{\mathbb{Q}}_p$, that can be extended uniquely to a character $(\chi_{g,p}^{\mathrm{st}}\circ\iota_{T_g}^{I_{g,p}})^{\mathrm{ext}}\colon\mathcal{H}(T_g(\mathbb{Q}_p),T_g(\mathbb{Z}_p))\to\overline{\mathbb{Q}}_p$.

The morphism $\lambda_{1,p}$ factors as $\iota_{1,p}^-\circ\lambda_{1,p}^-$ for some morphism

$$\lambda_{1,p}^-\colon\mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_p),I_{2,p})^-\to\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p),I_{1,p})^-$$

and the natural inclusion $\iota_{g,p}^-\colon\mathcal{H}(T_g(\mathbb{Q}_p),T_g(\mathbb{Z}_p))^-\to\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_p),I_{g,p})$.

Definition 9.6. We set $\lambda_1=\lambda^{Np}\otimes\lambda_{1,p}^-$.

Let $\chi_2^{\mathrm{st},i}\colon\mathcal{H}_2^N\to\overline{\mathbb{Q}}_p$ be the character defined by:

(i) $\chi_{2,\ell}^{\mathrm{st},i}=\chi_{1,\ell}^{\mathrm{st}}\circ\lambda_i$ for every prime $\ell\nmid Np$;

(ii) $\chi_{2,p}^{\mathrm{st},i}=(\chi_{1,p}^{\mathrm{st}}\circ\iota_{I_{1,p}}^{T_1})^{\mathrm{ext}}\circ\lambda_{i,p}\circ\iota_{T_2}^{I_{2,p}}$.

Proposition 9.7. *For every $i\in\{1,2,\dots,8\}$, the form $\mathrm{Sym}^3 f$ has a $p$-stabilization $(\mathrm{Sym}^3 f)_i^{\mathrm{st}}$ with associated system of Hecke eigenvalues $\chi_2^{\mathrm{st},i}$. Conversely, if $(\mathrm{Sym}^3 f)^{\mathrm{st}}$ is a $p$-stabilization of $\mathrm{Sym}^3 f$ with associated system of Hecke eigenvalues $\chi_2^{\mathrm{st}}$, then there exists $i\in\{1,2,\dots,8\}$ such that $\chi_2^{\mathrm{st}}=\chi_2^{\mathrm{st},i}$.*

*Proof.* In this proof, we leave the composition with the isomorphism $\iota_{I_{1,p}}^{T_1}$ and $\iota_{I_{2,p}}^{T_2}$ implicit and we consider $\chi_{1,p}^{\mathrm{st}}$ and $\chi_{2,p}^{\mathrm{st}}$ as characters of $\mathcal{H}(T_1(\mathbb{Q}_p),T_1(\mathbb{Z}_p))^-$ and $\mathcal{H}(T_2(\mathbb{Q}_p),T_2(\mathbb{Z}_p))^-$, respectively, for notational ease. Let $\rho_{f,p}\colon G_{\mathbb{Q}}\to\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the $p$-adic Galois representation associated with $f$, so that the $p$-adic Galois representation associated with $\mathrm{Sym}^3 f$ is $\mathrm{Sym}^3\rho_{f,p}$. Via $p$-adic Hodge theory, we attach to $\rho_{f,p}$ a two-dimensional $\overline{\mathbb{Q}}_p$-vector space $\mathbf{D}_{\mathrm{cris}}(\rho_{f,p})$ endowed with a $\overline{\mathbb{Q}}_p$-linear Frobenius endomorphism $\varphi_{\mathrm{cris}}(\rho_{f,p})$ satisfying $\det(1-X\varphi_{\mathrm{cris}}(\rho_{f,p}))=\chi_{1,p}(P_{\min}(t_{p,2}^{(2)};X))$.

We use the notation of § 2.2.1 for the elements of the Weyl groups of $\mathrm{GL}_2$ and $\mathrm{GSp}_4$. Let $\alpha_p$ and $\beta_p$ be the two roots of $\chi_{1,p}(P_{\min}(t_{p,2}^{(2)};X))$, ordered so that $\chi_{1,p}^{\mathrm{st}}(t_{p,1}^{(1)})=\alpha_p$ and $\beta_p=\chi_{1,p}^{\mathrm{st}}((t_{p,1}^{(1)})^w)$.

Let $\mathbf{D}_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p})$ be the four-dimensional $\overline{\mathbb{Q}}_p$-vector space attached to $\rho_{\mathrm{Sym}^3 f,p}$ by $p$-adic Hodge theory. Denote by $\varphi_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p})$ the Frobenius endomorphism acting on $\mathbf{D}_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p})$. It satisfies $\det(1-X\varphi_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p}))=\chi_{2,p}(P_{\min}(t_{p,2}^{(2)};X))$ by [Urb05, Théorème 1]. The coefficients of $P_{\min}(t_{p,2}^{(2)};X)$ belong to the spherical Hecke algebra at $p$, so we have $\chi_{2,p}^{\mathrm{st}}(P_{\min}(t_{p,2}^{(2)};X))=\chi_{2,p}(P_{\min}(t_{p,2}^{(2)};X))$. From the relation $\rho_{\mathrm{Sym}^3 f,p}=\mathrm{Sym}^3\rho_{f,p}$ and (4) we deduce that

$$\begin{aligned}&(X-\chi_{2,p}^{\mathrm{st}}(t_{p,2}^{(2)}))(X-\chi_{2,p}^{\mathrm{st}}((t_{p,2}^{(2)})^{w_1}))\cdot(X-\chi_{2,p}^{\mathrm{st}}((t_{p,2}^{(2)})^{w_2}))(X-\chi_{2,p}^{\mathrm{st}}((t_{p,2}^{(2)})^{w_1w_2}))\\&\quad=(X-\alpha_p^3)(X-\alpha_p^2\beta_p)(X-\alpha_p\beta_p^2)(X-\beta_p^3).\end{aligned}$$

In particular, the sets of roots of the two sides must coincide. As $t_{\ell,2}^{(2)}(t_{\ell,2}^{(2)})^{w_1w_2}=(t_{\ell,2}^{(2)})^{w_1}(t_{\ell,2}^{(2)})^{w_2}$ we have eight possible choices. Four choices for the 4-tuple

$$(\chi_{2,p}^{\mathrm{st}}(t_{\ell,2}^{(2)}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_1}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_2}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_1w_2}))$$

are

$$(\alpha_p^3,\alpha_p^2\beta_p,\alpha_p\beta_p^2,\beta_p^3),(\alpha_p^3,\alpha_p\beta_p^2,\alpha_p^2\beta_p,\beta_p^3),(\alpha_p^2\beta_p,\alpha_p^3,\beta_p^3,\alpha_p\beta_p^2),(\alpha_p^2\beta_p,\beta_p^3,\alpha_p^3,\alpha_p\beta_p^2).$$

The other four choices are obtained by exchanging $\alpha_p$ with $\beta_p$ in the above.



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By writing $\alpha_p = \chi_{1,p}^{\mathrm{st}}(t_{p,1}^{(1)})$, $\beta_p = \chi_{1,p}^{\mathrm{st}}((t_{p,1}^{(1)})^w)$ and recalling the relations $t_{p,1}^{(2)} = t_{p,2}^{(2)}(t_{p,2}^{(2)})^{w_1}$ and $t_{p,0}^{(2)} = t_{p,2}^{(2)}(t_{p,2}^{(2)})^{w_1 w_2}$, we find that the first four choices for the triple

$$(\chi_{2,p}^{\mathrm{st}}(t_{\ell,0}^{(2)}), \chi_{2,p}^{\mathrm{st}}(t_{p,1}^{(2)}), \chi_{2,p}^{\mathrm{st}}(t_{p,2}^{(2)}))$$

are

$$\begin{gathered}(\chi_{1,p}^{\mathrm{st}}(t_{p,0}^{(1)})^3, \chi_{1,p}^{\mathrm{st}}(t_{p,0}^{(1)}(t_{p,1}^{(1)})^4), \chi_{1,p}^{\mathrm{st}}((t_{p,1}^{(1)})^3)), (\chi_{1,p}^{\mathrm{st}}(t_{p,0}^{(1)})^3, \chi_{1,p}^{\mathrm{st}}((t_{p,0}^{(1)})^2(t_{p,1}^{(1)})^2), \chi_{1,p}^{\mathrm{st}}((t_{p,1}^{(1)})^3)),\\ (\chi_{1,p}^{\mathrm{st}}(t_{p,0}^{(1)})^3, \chi_{1,p}^{\mathrm{st}}(t_{p,0}^{(1)}(t_{p,1}^{(1)})^4), \chi_{1,p}^{\mathrm{st}}(t_{p,0}^{(1)}t_{p,1}^{(1)})), (\chi_{1,p}^{\mathrm{st}}(t_{p,0}^{(1)})^3, \chi_{1,p}^{\mathrm{st}}((t_{p,0}^{(1)})^4(t_{p,1}^{(1)})^{-2}), \chi_{1,p}^{\mathrm{st}}(t_{p,0}^{(1)}t_{p,1}^{(1)})).\end{gathered} \tag{26}$$

The four other choices are obtained by replacing $t_{p,0}^{(1)}$ and $t_{p,1}^{(1)}$ in the triples above by their images via the automorphism $\delta$ of $\mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ defined by (25).

Let $\lambda_p\colon \mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p))^- \to \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ be a morphism satisfying $\chi_2^{\mathrm{st}} = (\chi_1^{\mathrm{st}})^{\mathrm{ext}} \circ \lambda_p \circ \iota_{2,p}^-$ (recall that we leave the maps $\iota_{I_{g,p}}^{T_g}$ implicit). By the arguments of the previous paragraph, this happens if and only if the triple $(\lambda_{i,p}(t_{p,0}^{(2)}), \lambda_{i,p}(t_{p,1}^{(2)}), \lambda_{i,p}(t_{p,2}^{(2)}))$ coincides with one of the four listed in (26) or the four derived from those by applying $\delta$. A simple check shows that these triples correspond to the choices $\lambda_p = \lambda_{i,p}$ for $i \in \{1, 2, \ldots, 8\}$. $\square$

Let $f_\alpha^{\mathrm{st}}$ be a $p$-stabilization of a classical, cuspidal, non-CM $\mathrm{GL}_2$-eigenform $f$. Let $h$ be the slope of $f$. For $i \in \{1, 2, 3, 4\}$, denote by $\mathrm{Sym}^3(f_\alpha^{\mathrm{st}})_i$ the $\mathrm{GSp}_4$-eigenform $(\mathrm{Sym}^3 f)_i^{\mathrm{st}}$ given by Proposition 9.7. The forms $(\mathrm{Sym}^3 f)_i^{\mathrm{st}}$ with $5 \leqslant i \leqslant 8$ coincide with $(\mathrm{Sym}^3 f_\beta^{\mathrm{st}})_i$, $1 \leqslant i \leqslant 4$, where $f_\beta^{\mathrm{st}}$ is the $p$-stabilization of $f$ different from $f_\alpha^{\mathrm{st}}$. Let $k$ and $h$ be the weight and slope, respectively, of $f_\alpha^{\mathrm{st}}$. The following corollary is derived from Proposition 9.7 via some simple calculations.

COROLLARY 9.8. *The slopes of the forms* $\mathrm{Sym}^3(f_\alpha^{\mathrm{st}})_i$, *with* $1 \leqslant i \leqslant 4$, *are*

$$\begin{aligned}&\mathrm{sl}(\mathrm{Sym}^3(f_\alpha^{\mathrm{st}})_1) = 7h, \mathrm{sl}(\mathrm{Sym}^3(f_\alpha^{\mathrm{st}})_2) = \mathrm{sl}(\mathrm{Sym}^3(f_\alpha^{\mathrm{st}})_3) = k - 1 + 5h,\\ &\mathrm{sl}(\mathrm{Sym}^3(f_\alpha^{\mathrm{st}})_4) = 4(k-1) - h.\end{aligned}$$

If $f^{\mathrm{st}}$ is a $p$-old $\mathrm{GL}_2$-eigenform of level $\Gamma_1(N) \cap \Gamma_0(p)$, we write $\chi_{2,f^{\mathrm{st}}}^i$ for the system of Hecke eigenvalues of $\mathrm{Sym}^3(f^{\mathrm{st}})_i$, $1 \leqslant i \leqslant 4$. For a $\overline{\mathbb{Q}}_p$-point $x$ of $\mathcal{D}_2^M$ let $\chi_x\colon \mathcal{H}_2^N \to \overline{\mathbb{Q}}_p$ be the system of Hecke eigenvalues associated with $x$. For $1 \leqslant i \leqslant 4$, let $S_i^{\mathrm{Sym}^3}$ be the set of $\overline{\mathbb{Q}}_p$-points $x$ of $\mathcal{D}_2^M$ defined by the condition

$$x \in S_i^{\mathrm{Sym}^3} \iff \exists \text{ a } p\text{-old } \mathrm{GL}_2\text{-eigenform } f^{\mathrm{st}} \text{ of level } \Gamma_1(N) \cap \Gamma_0(p) \text{ such that } \chi_x = \chi_{2,f^{\mathrm{st}}}^i.$$

By combining Corollary 9.8 with the fact that the slope is bounded on an affinoid domain, we obtain the following.

COROLLARY 9.9. *If* $i \neq 1$, *then the set* $S_i^{\mathrm{Sym}^3}$ *is discrete in* $\mathcal{D}_2^M$.

*Remark* 9.10. As a consequence of Corollary 9.9, the only symmetric cube lifts that we can hope to interpolate $p$-adically are those in the set $S_1^{\mathrm{Sym}^3}$.



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### 9.2 The symmetric cube morphism of eigenvarieties

We fix until the end of the paper a continuous representation $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. We let $\mathcal{D}^N_{1,\overline{\rho}}$ be the union of the connected components of $\mathcal{D}^N_1$ having $\overline{\rho}$ as associated residual representation. From now on we replace $\mathcal{D}^N_1$ implicitly with $\mathcal{D}^N_{1,\overline{\rho}}$ (that is, we write $\mathcal{D}^N_1$ for $\mathcal{D}^N_{1,\overline{\rho}}$). The only purpose of this choice is to assure that the symmetric cube morphism of eigenvarieties of Proposition 9.11 is a closed immersion. We also replace implicitly $\mathcal{D}^M_2$ with $\mathcal{D}^M_{2,\mathrm{Sym}^3\overline{\rho}}$, because the symmetric cube morphism from $\mathcal{D}^N_{1,\overline{\rho}}$ will land into this connected component of $\mathcal{D}^M_2$.

There is a map $\mathrm{Sym}^3_1$ from the set of classical, non-CM, $p$-old points of $\mathcal{D}^N_{1,\overline{\rho}}$ to the set $S_1^{\mathrm{Sym}^3}$ of Corollary 9.9; it maps a point $x$ corresponding to an eigenform $f$ to the point of $S_1^{\mathrm{Sym}^3}$ corresponding to $(\mathrm{Sym}^3 f_x)^{\mathrm{st}}_1$. As we are fixing the residual representation $\overline{\rho}$, the map $\mathrm{Sym}^3_1$ is injective. Indeed, one has $\mathrm{Sym}^3\rho \cong \mathrm{Sym}^3\rho'$ for $\rho,\rho'\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ if and only if $\rho$ is a twist of $\rho'$ with a character of order 3, but this can be checked to be incompatible with them having the same residual representation.

We remind the reader that we only work on the connected components of identity of the various weights spaces. The association $k \mapsto (2k-1, k+1)$ is interpolated by the morphism of rigid analytic spaces

$$\begin{gathered}\iota\colon \mathcal{W}^\circ_1 \hookrightarrow \mathcal{W}^\circ_2,\\ T \mapsto (u^{-1}(1+T)^2 - 1, u(1+T) - 1).\end{gathered}$$

Recall the morphism $\lambda_1\colon \mathcal{H}^N_2 \to \mathcal{H}^N_1$ of Definition 9.6. The following proposition gives a morphism of eigenvarieties that interpolates the map $\mathrm{Sym}^3_1$.

Proposition 9.11 [Con16, Propositions 3.9.5-7 and Definition 3.9.8]. *There exists a closed immersion $\xi\colon \mathcal{D}^{N,\mathcal{G}}_1 \to \mathcal{D}^M_2$ of rigid analytic spaces over $\mathbb{Q}_p$ such that the following diagrams commute.*

$$\begin{tikzcd} \mathcal{D}^{N,\mathcal{G}}_1 \arrow[r, "\xi"] \arrow[d] & \mathcal{D}^M_2 \arrow[d] \\ \mathcal{W}^\circ_1 \arrow[r, "\iota"] & \mathcal{W}^\circ_2 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}^N_2 \arrow[r, "\lambda_1"] \arrow[d] & \mathcal{H}^N_1 \arrow[d] \\ \mathcal{O}(\mathcal{D}^M_2) \arrow[r, "\xi^*"] & \mathcal{O}(\mathcal{D}^{N,\mathcal{G}}_1) \end{tikzcd} \tag{27}$$

The proof of Proposition 9.11 presented in [Con16] relies on the results of [BC09, § 7.2.3]. One can give an alternative proof using [Han17, Theorem 5.1.6] instead.

*Remark* 9.12. Let $f$ be a classical, cuspidal, CM $\mathrm{GL}_2$-eigenform of level $\Gamma_1(N)$. As $f$ is CM, the $\mathrm{GSp}_4$-eigenform $\mathrm{Sym}^3 f$ provided by Corollary 9.2 may not be cuspidal. Suppose that it is not. Let $x$ be a point of $\mathcal{D}^N_1$ corresponding to a positive slope $p$-stabilization of $f$. By [CIT16, Corollary 3.6], $x$ is a CM point of a non-CM component $I$ of $\mathcal{D}^N_1$. Then $\xi(x)$ belongs to the cuspidal eigenvariety $\mathcal{D}^M_2$, but it is not cuspidal since $\mathrm{Sym}^3 f$ is not. This means that $\xi(x)$ is a non-cuspidal specialization of a cuspidal family of $\mathrm{GSp}_4$-eigenforms. Brasca and Rosso [BR16] constructed an eigenvariety for $\mathrm{GSp}_4$ parametrizing the systems of Hecke eigenvalues associated with the non-cuspidal overconvergent eigenforms and they glued it with $\mathcal{D}^M_2$. It should be possible to show that a cuspidal and a non-cuspidal component of this glued eigenvariety cross at $\xi(x)$.

## 10. The symmetric cube locus on the $\mathrm{GSp}_4$-eigenvariety

The goal of this section is to give two definitions (a Galois-theoretic one and an automorphic one) of a symmetric cube locus on the $\mathrm{GSp}_4$-eigenvariety and to show that they coincide. This is the content of Theorem 10.1. The main ingredient of the proof is Theorem 3.8. To apply it, we assume from now on that the representation $\overline{\rho}$ fixed in the beginning of § 9.2 satisfies conditions $(*_{\overline{\rho}})$.

In the following, $p$ is a prime number, $N$ is a positive integer prime to $p$ and $M = N^3$. Let $T_1\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{D}_1^N)$ and $T_2\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{D}_2^M)$ be the pseudocharacters provided by Proposition 2.5. By an abuse of notation, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are subvarieties of $\mathcal{D}_1^N$ and $\mathcal{D}_2^M$, respectively, we still write $\psi_1\colon \mathcal{H}_1^N \to \mathcal{O}(\mathcal{V}_1)$ and $\psi_2\colon \mathcal{H}_2^N \to \mathcal{O}(\mathcal{V}_2)$ for the compositions of $\psi_1$ and $\psi_2$ with the restrictions of analytic functions to $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. We also write $T_{\mathcal{V}_1}\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V}_1)$ and $T_{\mathcal{V}_2}\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V}_2)$ for the compositions of $T_1$ and $T_2$ with the restrictions of analytic functions to $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively.

THEOREM 10.1. *Let $\mathcal{V}_2$ be a rigid analytic subvariety of $\mathcal{D}_2^M$. Consider the following four conditions.*

(1a) *There exists a morphism of rings $\psi_2^{(1)}\colon \mathcal{H}_1^{Np} \to \mathcal{O}(\mathcal{V}_2)$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{H}_2^{Np} \arrow[r, "\lambda^{Np}"] \arrow[d, "\psi_2"] & \mathcal{H}_1^{Np} \arrow[dl, "\psi_2^{(1)}"] \\
\mathcal{O}(\mathcal{V}_2) &
\end{tikzcd} \tag{28}$$

(1b) *There exists a pseudocharacter $T_{\mathcal{V}_2,1}\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V}_2)$ of dimension two such that*

$$T_{\mathcal{V}_2} = \mathrm{Sym}^3\, T_{\mathcal{V}_2,1}. \tag{29}$$

(2a) *There exists a rigid analytic subvariety $\mathcal{V}_1$ of $\mathcal{D}_1^N$ and a morphism of rings $\phi\colon \mathcal{O}(\mathcal{V}_1) \to \mathcal{O}(\mathcal{V}_2)$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{H}_2^{Np} \arrow[r, "\lambda^{Np}"] \arrow[rrr, bend left, "\psi_2"] & \mathcal{H}_1^{Np} \arrow[r, "\psi_1"] & \mathcal{O}(\mathcal{V}_1) \arrow[r, "\phi"] & \mathcal{O}(\mathcal{V}_2)
\end{tikzcd} \tag{30}$$

(2b) *There exists a rigid analytic subvariety $\mathcal{V}_1$ of $\mathcal{D}_1^N$ and a morphism of rings $\phi\colon \mathcal{O}(\mathcal{V}_1) \to \mathcal{O}(\mathcal{V}_2)$ such that*

$$T_{\mathcal{V}_2} = \mathrm{Sym}^3(\phi \circ T_{\mathcal{V}_1}). \tag{31}$$

*Then:*

(i) *conditions (1a) and (1b) are equivalent;*
(ii) *conditions (2a) and (2b) are equivalent;*
(iii) *condition (2b) implies condition (1b);*
(iv) *when $\mathcal{V}_2$ is a point, the four conditions are equivalent.*

*Proof.* We prove statements (i), (ii) and (iii) for an arbitrary rigid analytic subvariety $\mathcal{V}_2$ of $\mathcal{D}_2^M$.

(1a) $\implies$ (1b). Let $\psi_2^{(1)}\colon \mathcal{H}_1^{Np} \to \mathcal{O}(\mathcal{V}_2)$ be a morphism of rings making diagram (28) commute. By the argument in the proof of Proposition 9.7, the commutativity of diagram (28) gives an equality

$$\psi_2(P_{\min}(t_{\ell,2}^{(2)}; X)) = \mathrm{Sym}^3(\psi_2^{(1)}(P_{\min}(t_{\ell,1}^{(1)}; X))). \tag{32}$$

Choose a character $\varepsilon_1$ satisfying $\varepsilon_1^6 = \varepsilon$. For every $\ell$ not dividing $Np$, let $P_\ell$ be a polynomial in $\mathcal{H}_2^{Np}[X]^{\deg=2}$ satisfying

$$\operatorname{Sym}^3 P_\ell(X) = \psi_2(P_{\min}(t_{\ell,2}^{(2)}; X)); \tag{33}$$

$$P_\ell(0) = \varepsilon_1 \cdot (1+T)^{\log(\chi(g))/\log(u)}. \tag{34}$$

Such a polynomial exists thanks to (32) and to Remark 4.8, and it is clearly unique. The roots of $P_\ell$ differ from those of $\psi_2(P_{\min}(t_{\ell,2}^{(2)}; X))$ by a factor equal to a cubic root of 1. The map

$$\begin{gathered} P\colon \{\gamma \operatorname{Frob}_\ell \gamma^{-1}\}_{\ell \nmid Np;\, \gamma \in G_{\mathbb{Q}}} \to \mathcal{O}(\mathcal{V}_2)[X]^{\deg=2}, \\ \gamma \operatorname{Frob}_\ell \gamma^{-1} \mapsto P_\ell, \end{gathered}$$

is continuous with respect to the restriction of the profinite topology on $G_{\mathbb{Q}}$. This follows from the fact that the maps

$$\begin{gathered} \{\gamma \operatorname{Frob}_\ell \gamma^{-1}\}_{\ell \nmid Np;\, \gamma \in G_{\mathbb{Q}}} \to \mathcal{O}(\mathcal{V}_2)[X]^{\deg=4} \\ \gamma \operatorname{Frob}_\ell \gamma^{-1} \mapsto \psi_2(P_{\min}(t_{\ell,2}^{(2)}; X)) = \operatorname{Sym}^3 P(\gamma \operatorname{Frob}_\ell \gamma^{-1})(X) \end{gathered}$$

and

$$\begin{gathered} \{\gamma \operatorname{Frob}_\ell \gamma^{-1}\}_{\ell \nmid Np;\, \gamma \in G_{\mathbb{Q}}} \to \mathcal{O}(\mathcal{V}_2)^\times \\ \gamma \operatorname{Frob}_\ell \gamma^{-1} \mapsto P(\gamma \operatorname{Frob}_\ell \gamma^{-1})(0) = \varepsilon_1 \cdot (1+T)^{\log(\chi(g))/\log(u)} \end{gathered}$$

are continuous on $\{\gamma \operatorname{Frob}_\ell \gamma^{-1}\}_{\ell \nmid Np;\, \gamma \in G_{\mathbb{Q}}}$. By Chebotarev's theorem $P$ can be extended to a continuous map $P\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V}_2)[X]^{\deg=2}$. Now define a map $T_{\mathcal{V}_2,1}\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V}_2)$ by $T_{\mathcal{V}_2,1}(g) = (P(g)(-1)P(g)(1))/2$. We can check that $T_{\mathcal{V}_2,1}$ is a pseudocharacter of dimension two. Its characteristic polynomial is $P$, so the fact that $T_{\mathcal{V}_2} = \operatorname{Sym}^3 T_{\mathcal{V}_2,1}$ follows from (33).

(1b) $\implies$ (1a). Suppose that there exists a pseudocharacter $T_{\mathcal{V}_2,1}\colon G_{\mathbb{Q}} \to \mathcal{O}_{\mathcal{V}_2}$ such that $T_{\mathcal{V}_2} = \operatorname{Sym}^3 T_{\mathcal{V}_2,1}$. Then $P_{\mathrm{char}}(T_{\mathcal{V}_2}) = \operatorname{Sym}^3 P_{\mathrm{char}}(T_{\mathcal{V}_2,1})$. By evaluating the two polynomials at $\operatorname{Frob}_\ell$ we obtain

$$\begin{aligned} \psi_2(P_{\min}(t_{\ell,2}^{(2)}; X)) &= P_{\mathrm{char}}(T_{\mathcal{V}_2})(\operatorname{Frob}_\ell) = \operatorname{Sym}^3 P_{\mathrm{char}}(T_{\mathcal{V}_2,1})(\operatorname{Frob}_\ell) \\ &= \operatorname{Sym}^3 \left( X^2 - T_{\mathcal{V}_2,1}(\operatorname{Frob}_\ell)X + \frac{T_{\mathcal{V}_2,1}(\operatorname{Frob}_\ell)^2 - T_{\mathcal{V}_2,1}(\operatorname{Frob}_\ell^2)}{2} \right), \end{aligned} \tag{35}$$

where the first equality is given by Proposition 2.5 and the last one comes from a trivial calculation. Let $\psi_2^{(1)}\colon \mathcal{H}_1^{Np} \to \mathcal{O}(\mathcal{V}_2)$ be a morphism of rings satisfying

$$X^2 - T_{\mathcal{V}_2,1}(\operatorname{Frob}_\ell)X + \frac{T_{\mathcal{V}_2,1}(\operatorname{Frob}_\ell)^2 - T_{\mathcal{V}_2,1}(\operatorname{Frob}_\ell^2)}{2} = X^2 - \psi_2^{(1)}(T_{\ell,1}^{(1)})X + \ell\psi_2^{(1)}(T_{\ell,0}^{(1)}) \tag{36}$$

for every $\ell \nmid Np$. It is clear that such a morphism exists and is unique. Note that the right-hand side of (36) is $\psi_2^{(1)}(P_{\min}(t_{\ell,1}^{(1)}; X))$. Then (35) gives

$$\psi_2(P_{\min}(t_{\ell,2}^{(2)}; X)) = \operatorname{Sym}^3(\psi_2^{(1)}(P_{\min}(t_{\ell,1}^{(1)}; X))).$$

Exactly as in the proof of Proposition 9.7, by developing the two polynomials and comparing their coefficients we obtain that $\psi_2 = \psi_2^{(1)} \circ \lambda^{Np}$. Hence $\psi_2^{(1)}$ fits into diagram (28).



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(2a) $\iff$ (2b). Let $\mathcal{V}_1$ be a subvariety of $\mathcal{D}_1^N$ and let $\phi\colon \mathcal{O}(\mathcal{V}_1)\to\mathcal{O}(\mathcal{V}_2)$ be a morphism of rings. We show that the couple $(\mathcal{V}_1,\phi)$ satisfies condition (2a) if and only if it satisfies condition (2b). For $g=1,2$ and every prime $\ell\nmid Np$ Proposition 2.5 gives

$$P_{\mathrm{char}}(T_{\mathcal{V}_g})(\mathrm{Frob}_\ell)=\psi_g(P_{\min}(t_{\ell,g}^{(g)};X)). \tag{37}$$

The argument in the proof of Proposition 9.7 gives an equality

$$\lambda^{Np}(P_{\min}(t_{\ell,2}^{(2)};X))=\mathrm{Sym}^3(P_{\min}(t_{\ell,1}^{(1)};X)). \tag{38}$$

As the set $\{\gamma\,\mathrm{Frob}_\ell\,\gamma^{-1}\}_{\ell\nmid Np;\,\gamma\in G_{\mathbb{Q}}}$ is dense in $G_{\mathbb{Q}}$, the pseudocharacters $\mathrm{Sym}^3(\phi\circ T_{\mathcal{V}_1})$ and $T_{\mathcal{V}_2}$ coincide if and only if their characteristic polynomials coincide on $\mathrm{Frob}_\ell$ for every $\ell\nmid Np$. By (37), the condition above is equivalent to

$$\mathrm{Sym}^3(\phi\circ\psi_1(P_{\min}(t_{\ell,1}^{(1)};X)))=\psi_2(P_{\min}(t_{\ell,2}^{(2)};X))$$

for every $\ell\nmid Np$. Thanks to (38) the left-hand side can be rewritten as

$$\mathrm{Sym}^3(\phi\circ\psi_1(P_{\min}(t_{\ell,1}^{(1)};X)))=\phi\circ\psi_1(\mathrm{Sym}^3(P_{\min}(t_{\ell,1}^{(1)};X)))=\phi\circ\psi_1\circ\lambda^{Np}(P_{\min}(t_{\ell,2}^{(2)};X)).$$

When $\ell$ varies over the primes not dividing $Np$ the coefficients of the polynomials $P_{\min}(t_{\ell,2}^{(2)};X)$ generate the Hecke algebra $\mathcal{H}_2^{Np}$. Hence, the equality of the right-hand sides of the last two equations holds if and only if $\phi\circ\psi_1\circ\lambda^{Np}=\psi_2$.

(2b) $\implies$ (1b). Suppose that condition (2b) is satisfied by some closed subvariety $\mathcal{V}_1$ of $\mathcal{D}_1^N$ and some morphism of rings $\phi\colon\mathcal{O}(\mathcal{V}_1)\to\mathcal{O}(\mathcal{V}_2)$. Consider the pseudocharacter $T_{\mathcal{V}_2,1}=\phi\circ T_{\mathcal{V}_1}\colon G_{\mathbb{Q}}\to\mathcal{O}(\mathcal{V}_2)$. Clearly $T_{\mathcal{V}_2,1}$ satisfies condition (1b).

It remains to prove that condition (1b) $\implies$ condition (2b) when $\mathcal{V}_2$ is a $\overline{\mathbb{Q}}_p$-point of $\mathcal{D}_2^M$. Here, we need to apply Theorem 3.8. Write $x_2$ for the point $\mathcal{V}_2$; the system of eigenvalues $\psi_{x_2}$ is that of a classical $\mathrm{GSp}_4$-eigenform. By Remark 2.6(i) $T_{x_2}$ is the pseudocharacter associated with a representation $\rho_{x_2}\colon G_{\mathbb{Q}}\to\mathrm{GL}_4(\overline{\mathbb{Q}}_p)$. Let $E$ be a finite extension of $\mathbb{Q}_p$ over which $\rho_{x_2}$ is defined. Suppose that $x_2$ satisfies condition (1b). Let $T_{x_2,1}\colon G_{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$ be a pseudocharacter such that $T_{x_2}\cong\mathrm{Sym}^3T_{x_2,1}$. By Taylor's theorem in [Tay91] there exists a representation $\rho_{x_2,1}\colon G_{\mathbb{Q}}\to\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that $T_{x_2,1}=\mathrm{Tr}(\rho_{x_2,1})$. From the definition of the symmetric cube of a pseudocharacter, we deduce that $\rho_{x_2}\cong\mathrm{Sym}^3\rho_{x_2,1}$. As $\rho_{x_2}$ is attached to an overconvergent $\mathrm{GSp}_4$-eigenform, Theorem 3.8(ii) implies that $\rho_{x_2,1}$ is the $p$-adic Galois representation attached to an overconvergent $\mathrm{GL}_2$-eigenform $f$. Such a form defines a point $x_1$ of the eigencurve $\mathcal{D}_1^N$. Thus, the subvariety $\mathcal{V}_1=x_1$ satisfies condition (2b). □

In light of Theorem 10.1, we give the following definitions.

Definition 10.2.

(i) We say that a subvariety $\mathcal{V}_2$ of $\mathcal{D}_2^M$ is of $\mathrm{Sym}^3$ *type* if it satisfies the equivalent conditions (2a) and (2b) of Theorem 10.1.

(ii) The $\mathrm{Sym}^3$-*locus* of $\mathcal{D}_2^M$ is the set of points of $\mathcal{D}_2^M$ of $\mathrm{Sym}^3$ type.



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Recall the closed immersion $\iota\colon \mathcal{W}_1^\circ \to \mathcal{W}_2^\circ$ defined in § 9.2. Let $\mathcal{D}_{2,\iota}^M$ be the one-dimensional subvariety of $\mathcal{D}_2^M$ fitting in the following Cartesian diagram.

$$\begin{tikzcd} \mathcal{D}_{2,\iota}^M \arrow[r] \arrow[d] & \mathcal{D}_2^M \arrow[d, "w_2"] \\ \iota(\mathcal{W}_1^\circ) \arrow[r, "\iota"] & \mathcal{W}_2^\circ \end{tikzcd}$$

The following lemma follows from a simple computation involving the generalized Hodge–Tate weights of a point of $\mathrm{Sym}^3$ type.

Lemma 10.3. *The* $\mathrm{Sym}^3$*-locus of* $\mathcal{D}_2^M$ *is contained in the one-dimensional subvariety* $\mathcal{D}_{2,\iota}^M$.

The $\mathrm{Sym}^3$-locus of $\mathcal{D}_2^M$ admits a Hecke-theoretic definition thanks to condition (2b) of Theorem 10.1.

We define an ideal $\mathcal{I}_{\mathrm{Sym}^3}$ of $\mathcal{O}(\mathcal{D}_2^M)$ by

$$\mathcal{I}_{\mathrm{Sym}^3} = \psi_2(\ker(\psi_1 \circ \lambda^{Np})) \cdot \mathcal{O}(\mathcal{D}_2^M).$$

We denote by $\mathcal{D}_{2,\mathrm{Sym}^3}^M$ the analytic Zariski subvariety of $\mathcal{D}_2^M$ defined as the zero locus of the ideal $\mathcal{I}_{\mathrm{Sym}^3}$.

Proposition 10.4.

(i) *The* $\mathrm{Sym}^3$*-locus of* $\mathcal{D}_2^M$ *is the set of points underlying* $\mathcal{D}_{2,\mathrm{Sym}^3}^M$.

(ii) *The variety* $\mathcal{D}_{2,\mathrm{Sym}^3}^M$ *is of* $\mathrm{Sym}^3$ *type.*

(iii) *A rigid analytic subvariety* $\mathcal{V}_2$ *of* $\mathcal{D}_2^M$ *is of* $\mathrm{Sym}^3$ *type if and only if it is a subvariety of* $\mathcal{D}_{2,\mathrm{Sym}^3}^M$.

(iv) *A rigid analytic subvariety* $\mathcal{V}_2$ *of* $\mathcal{D}_2^M$ *satisfies conditions (1a) and (1b) of Theorem 10.1 if and only if it is a subvariety of* $\mathcal{D}_{2,\mathrm{Sym}^3}^M$.

*Proof.* Statements (i) and (ii), together with the 'if' implications of statements (iii) and (iv), follow immediately from the definition of $\mathcal{D}_{2,\mathrm{Sym}^3}^M$. If $\mathcal{V}_2$ satisfies condition (2a) of Theorem 10.1, then one has $\psi_2 = \phi \circ \psi_1 \circ \lambda^{Np}$ for some $\phi\colon \mathcal{O}(\mathcal{V}_1) \to \mathcal{O}(\mathcal{V}_2)$, so $\psi_2$ must vanish on $\ker(\psi_1 \circ \lambda^{Np})$, giving the 'only if' implication of statement (iii).

For the remaining direction of statement (iv), let $\mathcal{V}_2$ be a rigid analytic subvariety of $\mathcal{D}_2^M$ satisfying conditions (1a) and (1b) of Theorem 10.1. Let $x_2$ by a point of $\mathcal{V}_2$. Then $x_2$ satisfies conditions (1a) and (1b). By Theorem 10.1, $x_2$ also satisfies conditions (2a) and (2b), so it is a point of $\mathcal{D}_{2,\mathrm{Sym}^3}^M$. We conclude that $\mathcal{V}_2$ is a subvariety of $\mathcal{D}_{2,\mathrm{Sym}^3}^M$. □

*Remark* 10.5. By Proposition 10.4 the $\mathrm{Sym}^3$-locus in $\mathcal{D}_2^M$ can be given the structure of a Zariski-closed rigid analytic subspace. From now on, we will always consider the $\mathrm{Sym}^3$-locus as equipped with this structure and we will identify it with the subvariety $\mathcal{D}_{2,\mathrm{Sym}^3}^M$ of $\mathcal{D}_2^M$.

Proposition 10.4(i) and Lemma 10.3 give the following.

Corollary 10.6. *The* $\mathrm{Sym}^3$*-locus intersects each irreducible component of* $\mathcal{D}_2^M$ *in a proper analytic subvariety of dimension at most* 1.

Propositions 10.4(iii) and (iv) allow us to improve the result of Theorem 10.1.

Corollary 10.7. *For every rigid analytic subvariety $\mathcal{V}_2$ of $\mathcal{D}_2^M$, conditions (1a), (1b), (2a) and (2b) of Theorem 10.1 are equivalent.*

## 11. An automorphic description of the Galois level

### 11.1 The fortuitous $\mathrm{Sym}^3$-congruence ideal of a finite slope family

Let $\theta\colon \mathbb{T}_h \twoheadrightarrow \mathbb{I}^\circ$ be a finite slope family and let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the representation associated with $\theta$ in the previous section. Recall that $\overline{\rho}$ is absolutely irreducible by assumption. We also assume that $\rho$ is $\mathbb{Z}_p$-regular and of residual $\mathrm{Sym}^3$ type, as in Definitions 3.10 and 3.11. In this section, we define a 'fortuitous congruence ideal' for the family $\theta$. It is the ideal describing the intersection of the $\mathrm{Sym}^3$-locus of $\mathcal{D}_2^M$ with the family $\theta$. Recall that the $\mathrm{Sym}^3$-locus is the zero locus of the ideal $\mathcal{I}_{\mathrm{Sym}^3}$ of $\mathcal{O}(\mathcal{D}_2^M)^\circ$ defined in § 10 and that $r_{\mathcal{D}^{M,h}_{2,B_h}}\colon \mathcal{O}(\mathcal{D}_2^M)^\circ \to \mathbb{T}_h$ denotes the restriction of analytic functions.

Definition 11.1. The *fortuitous* $\mathrm{Sym}^3$-*congruence ideal* for the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ is the ideal of $\mathbb{I}^\circ_0$ defined by

$$\mathfrak{c}_\theta = (\theta \circ r_{\mathcal{D}^{M,h}_{2,B_h}})(\mathcal{I}_{\mathrm{Sym}^3}) \cdot \mathbb{I}^\circ \cap \mathbb{I}^\circ_0.$$

In most cases, we will simply refer to $\mathfrak{c}_\theta$ as the 'congruence ideal'. The next proposition describes its main properties. Let $\mathfrak{I}$ be an ideal of $\mathbb{I}^\circ$ and let $\mathfrak{I}_{\mathrm{Tr}} = \mathfrak{I} \cap \mathbb{I}^\circ_{\mathrm{Tr}}$. Let $\rho_\mathfrak{I}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{I}_{\mathrm{Tr}})$ be the reduction of $\rho$ modulo $\mathfrak{I}$. If $\theta_1\colon \mathbb{T}_{h,1} \to \mathbb{J}$ is a finite slope family of $\mathrm{GL}_2$-eigenforms we denote by $\rho_{\theta_1}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{J})$ the associated Galois representation. For an ideal $\mathcal{J}$ of $\mathbb{J}$ we let $\rho_{\theta_1,\mathcal{J}}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{J}/\mathcal{J})$ be the reduction of $\rho_{\theta_1}$ modulo $\mathcal{J}$.

For an ideal $\mathfrak{I}$ of $\mathbb{I}^\circ_0$ we denote by $\rho_\mathfrak{I}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}^\circ_0/\mathfrak{I})$ the reduction of $\rho|_{H_0}$ modulo $\mathfrak{I}$. We give the following characterization of $\mathfrak{c}_\theta$.

Lemma 11.2. *Let $P_0$ be a prime ideal of $\mathbb{I}^\circ_0$. The following are equivalent:*

(i) *$P_0 \supset \mathfrak{c}_\theta$;*

(ii) *there exists a finite extension $\mathbb{I}'$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/P_0\mathbb{I}^\circ_{\mathrm{Tr}}$ and a representation $\rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}},1}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}}} \cong \mathrm{Sym}^3 \rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}},1}$ over $\mathbb{I}'$;*

(iii) *for one prime $P$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$ lying above $P_0$, there exists a finite extension $\mathfrak{I}'$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/P$ and a representation $\rho_{P,1}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_P \cong \mathrm{Sym}^3 \rho_{P,1}$ over $\mathbb{I}'$;*

(iv) *there exists a representation $\rho_{P_0,1}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}^\circ_0/\mathfrak{I})$ such that $\rho_{P_0} \cong \mathrm{Sym}^3 \rho_{P_0,1}$ over $\mathbb{I}^\circ_0/\mathfrak{I}$.*

Note that we did not specify the image in the weight space of the admissible subdomain of $\mathcal{D}_1^N$ associated with the family $\theta_1$. It is the preimage in $\mathcal{W}_1^\circ$ of the disc $B_{2,h}$ via the immersion $\iota\colon \mathcal{W}_1^\circ \to \mathcal{W}_2^\circ$ defined in § 9.2.

*Proof.* As all the coefficient rings are local and all the residual representations are absolutely irreducible, we can apply the results of § 10 by replacing pseudocharacters everywhere with the associated representations, that exist by [Rou96, Corollary 5.2] and are defined over the ring of coefficients of the pseudocharacter by Carayol's theorem [Car94, Théorème 1].



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Now the equivalence (i) $\iff$ (ii) follows from Proposition 10.4(iv) applied to the rigid analytic variety $\mathcal{V}_2 = I$. The equivalence (ii) $\iff$ (iii) follows from Proposition 10.4(iii) by checking that the slopes satisfy the required inequality: this is a consequence of Corollary 9.8 and Remark 9.10.

If condition (iii) is satisfied by some $\rho_{P_0,1}$, then $\rho_{P_0,1} = \rho_{P,1}|_{H_0}$ satisfies condition (iv). If condition (iv) holds, then the image of $\rho_{P_0}$ is contained in $\mathrm{Sym}^3\,\mathrm{GL}_2(\mathbb{I}_0^\circ/\mathfrak{I})$. As $\rho_{P_0} = \rho_{P_0\mathbb{I}_{\mathrm{Tr}}^\circ}|_{H_0}$ Lemma 3.14 implies that, after extending the coefficients to a finite extension $\mathbb{I}_0'$ of $\mathbb{I}_{\mathrm{Tr}}^\circ/P_0\mathbb{I}_{\mathrm{Tr}}^\circ$ the image of $\rho_{P_0\mathbb{I}_{\mathrm{Tr}}^\circ}$ is contained in $\mathrm{Sym}^3\,\mathrm{GL}_2(\mathbb{I}_0')$. This gives condition (ii), completing the proof. □

We gather some information on the congruence ideal.

Lemma 11.3.

(i) *The ideal $\mathfrak{c}_\theta$ is of height one. In particular, it is non-zero.*

(ii) *If there is no representation $\overline{\rho}_1\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ satisfying $\overline{\rho} \cong \mathrm{Sym}^3\,\overline{\rho}_1$, then $\mathfrak{c}_\theta = \mathbb{I}_0^\circ$.*

(iii) *Suppose that there exists a non-CM classical point $x \in \mathcal{D}_1^N$ of weight $k$ such that $\mathrm{sl}(x) \leqslant h/7$ and $\iota(k) \in B_{2,h}$ and $k > h-4$. Then there exists a family $\theta$ of $\mathrm{GSp}_4$-eigenforms of slope bounded by $h$ such that $\mathfrak{c}_\theta$ has a prime divisor of height 1.*

*Proof.* Part (i) is an immediate consequence of Corollary 10.6. Part (ii) follows trivially from the definition of $\mathfrak{c}_\theta$. We prove part (iii). Let $x$ be a point satisfying the assumptions of part (iii) and let $f$ be the corresponding classical $\mathrm{GL}_2$-eigenform. Let $\mathrm{Sym}^3 x$ be the point of $\mathcal{D}_2^M$ that corresponds to the form $(\mathrm{Sym}^3 f)_1^{\mathrm{st}}$ defined in Proposition 9.7. Let $\xi\colon \mathcal{D}_1^{N,\mathcal{G}} \to \mathcal{D}_2^M$ be the map of rigid analytic spaces given by Proposition 9.11. The image of an irreducible component $J$ of $\mathcal{D}_1^{N,\mathcal{G}}$ containing $x$ is an irreducible component of $\mathcal{D}_2^M$ that contains $\xi(J)$. By Corollary 9.8, we have $\mathrm{sl}(\mathrm{Sym}^3 x) \leqslant h$. As $k+1 > h-3$, the weight map is étale at the point $\mathrm{Sym}^3 x$, so there exists only one irreducible component of $\mathcal{D}_2^M$ containing $\mathrm{Sym}^3 x$. We denote by $\theta$ the finite slope family supported in such a component and containing $\mathrm{Sym}^3 x$, and by $I$ the support of $\theta$. By our last remark, the space $\xi(J)$ intersects the admissible domain $I$ in a one-dimensional subspace. The ideal of $\mathbb{I}^\circ = \mathcal{O}(I)^\circ$ consisting of elements that vanish on $\xi(J)$ is a height-one ideal of $I$ that divides the congruence ideal $\mathfrak{c}_\theta$. In particular, $\mathfrak{c}_\theta$ admits a height-one prime divisor. □

The fortuitous $\mathrm{Sym}^3$-congruence ideal is an analogue of the congruence ideal of [CIT16, Definition 3.10]. There is an important difference between the situation studied here and in [CIT16] and those treated in [Hid15, HT15]. In [Hid15, HT15], the congruence ideal describes the locus of intersection between a fixed 'general' family (i.e. such that its specializations are not lifts of forms from a smaller group) and the 'non-general' families. Such non-general families are obtained as the $p$-adic lift of families of overconvergent eigenforms for smaller groups (e.g. $\mathrm{GL}_{1/K}$ for an imaginary quadratic field $K$ in the case of CM families of $\mathrm{GL}_2$-eigenforms, as in [Hid15], and $\mathrm{GL}_{2/F}$ for a real quadratic field $F$ in the case of 'twisted Yoshida type' families of $\mathrm{GSp}_4$-eigenforms, as in [HT15]). In our setting, there are no non-general families: the overconvergent $\mathrm{GSp}_4$-eigenforms that are lifts of overconvergent eigenforms for smaller groups must be of $\mathrm{Sym}^3$ type by Lemma 3.15 and Theorem 3.8, and we know that the $\mathrm{Sym}^3$-locus on the $\mathrm{GSp}_4$-eigenvariety does not contain any two-dimensional irreducible component by Lemma 11.3(i). Hence, the ideal $\mathfrak{c}_\theta$ measures the locus of points that are of $\mathrm{Sym}^3$ type, without belonging to a two-dimensional family of $\mathrm{Sym}^3$ type. For this reason, we call it the 'fortuitous' $\mathrm{Sym}^3$-congruence ideal. This is a higher-dimensional analogue of the situation of [CIT16], where it is shown that the positive slope CM points do not form one-dimensional families, but appear as isolated points on the irreducible components of the eigencurve (see [CIT16, Corollary 3.6]).

## 11.2 Comparison of the Galois level with the congruence ideal

In § 8, we attached a Galois level to a family of finite slope $\mathrm{GSp}_4$-eigenforms. The goal of this section is to compare this Galois theoretic objects with the congruence ideal introduced in § 11.1, that is an object defined in terms of congruences of overconvergent automorphic forms.

We work in the setting of Theorem 8.1. In particular, $h$ is a positive rational number, $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ is a family of $\mathrm{GSp}_4$-eigenforms of slope bounded by $h$ and $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ is the Galois representation associated with $\theta$. We make the same assumptions on $\theta$ and $\rho$ as in Theorem 8.1; in particular, $\rho$ is $\mathbb{Z}_p$-regular and the residual representation $\overline{\rho}$ is either full or of symmetric cube type. With the family $\theta$ we associate two ideals of $\mathbb{I}_0$:

- the ideal $\mathfrak{c}_\theta \cdot \mathbb{I}_0$, where $\mathfrak{c}_\theta$ is the fortuitous $(\mathrm{Sym}^3, \mathbb{I}_0^\circ)$-congruence ideal (see Definition 11.1);
- the Galois level $\mathfrak{l}_\theta$ (see Definition 8.4).

To simplify notation, we write $\mathfrak{c}_\theta$ for $\mathfrak{c}_\theta \cdot \mathbb{I}_0$. For every ring $R$ and every ideal $\mathfrak{I}$ of $R$, we denote by $V_R(\mathfrak{I})$ the set of primes of $R$ containing $\mathfrak{I}$. The theorem below is an analogue of [CIT16, Theorem 6.2]. The set $S^{\mathrm{bad}}$ of 'bad' primes of $\mathbb{I}_0^\circ$ was defined in § 7.1. Note that $V_{\mathbb{I}_0}(\mathfrak{l}_\theta) \cap S^{\mathrm{bad}}$ is empty because the property defining the Galois level only involves $\mathfrak{l}_\theta \cdot \mathbb{B}_r$, and the primes in $S^{\mathrm{bad}}$ are invertible in $\mathbb{B}_r$.

Theorem 11.4. *There is an equality of sets $V_{\mathbb{I}_0}(\mathfrak{c}_\theta) - S^{\mathrm{bad}} = V_{\mathbb{I}_0}(\mathfrak{l}_\theta)$.*

Recall that there is a natural inclusion $\iota_r\colon \mathbb{I}_0 \hookrightarrow \mathbb{I}_{r,0}$.

*Proof.* First we prove that $V_{\mathbb{I}_0}(\mathfrak{c}_\theta) - S^{\mathrm{bad}} \subset V_{\mathbb{I}_0}(\mathfrak{l}_\theta) - S^{\mathrm{bad}}$. Choose a radius $r$ in the set $\{r_i\}_{i\in\mathbb{N}^{>0}}$ defined in § 4. Let $P \in V_{\mathbb{I}_0}(\mathfrak{c}_\theta) - S^{\mathrm{bad}}$ and let $\rho_P$ be the reduction of $\rho|_{H_0}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}_0)$ modulo $P$. By Proposition 11.2 there exists a representation $\rho_{P,1}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_0/P)$ such that $\rho_P \cong \mathrm{Sym}^3 \rho_{P,1}$. Let $\rho_{r,P} = \iota_r \circ \rho_P$ and $\rho_{r,P,1} = \iota_r \circ \rho_{P,1}$. The isomorphism above gives $\rho_{r,P} \cong \mathrm{Sym}^3 \rho_{r,P,1}$.

Suppose by contradiction that $\mathfrak{l}_\theta \not\subset P$. By the definition of $\mathfrak{l}_\theta$, we have $\mathfrak{G}_r \supset \mathfrak{l}_\theta \cdot \mathfrak{sp}_4(\mathbb{B}_r)$. Recall that $\mathbb{B}_r/P = \mathbb{I}_{r,0}/P$ by the construction of $\mathbb{B}_r$. By looking at the previous inclusion modulo $P$, we obtain

$$\mathfrak{G}_{r,P} \supset (\mathfrak{l}_\theta/(P \cap \mathfrak{l}_\theta)) \cdot \mathfrak{sp}_4(\mathbb{I}_{r,0}/P). \tag{39}$$

As $\mathfrak{l}_\theta \not\subset P$, we have $\mathfrak{l}_\theta/(P \cap \mathfrak{l}_\theta) \neq 0$. By definition, $\mathfrak{G}_{r,P} = \mathbb{Q}_p \cdot \log \mathrm{Im}\, \rho_{r,P}$. By our previous argument $\mathrm{Im}\, \rho_{r,P} \subset \mathrm{Sym}^3 \mathrm{GL}_2(\mathbb{I}_{r,0}/P\mathbb{I}_{r,0})$, so $\log \mathrm{Im}\, \rho_{r,P}$ cannot contain a subalgebra of the form $\mathfrak{I} \cdot \mathfrak{sp}_4(\mathbb{I}_{r,0}/P\mathbb{I}_{r,0})$ for a non-zero ideal $\mathfrak{I}$ of $\mathbb{I}_{r,0}/P\mathbb{I}_{r,0}$. This contradicts (39).

We prove the inclusion $V_{\mathbb{I}_0}(\mathfrak{l}_\theta) - S^{\mathrm{bad}} \subset V_{\mathbb{I}_0}(\mathfrak{c}_\theta) - S^{\mathrm{bad}}$. Let $P$ be a prime of $\mathbb{I}_0$. We have to show that if $P \notin S^{\mathrm{bad}}$ and $\mathfrak{l}_\theta \subset P$, then $\mathfrak{c}_\theta \subset P$. Every prime of $\mathbb{I}_0$ is the intersection of the maximal ideals that contain it, so it is sufficient to show the previous implication when $P$ is a maximal ideal.

Let $P$ be a maximal ideal of $\mathbb{I}_0$ such that $P \notin S^{\mathrm{bad}}$ and $\mathfrak{l}_\theta \subset P$. Let $\kappa_P$ be the residue field $\mathbb{I}_0/P$. We define two ideals of $\mathbb{I}_{r,0}$ by $\mathfrak{l}_{\theta,r} = \iota_r(\mathfrak{l}_\theta)\mathbb{I}_{r,0}$ and $P_r = \iota_r(P)\mathbb{I}_{r,0}$. Note that $\iota_r$ induces an isomorphism $\mathbb{I}_0/P \cong \mathbb{I}_{r,0}/P_r$. In particular, $P_r$ is maximal in $\mathbb{I}_{r,0}$ and $\mathbb{I}_{r,0}/P_r \cong \kappa_P$, which is a local field.

As before, let $\rho_{r,P} = \iota_r \circ \rho_P$. The residual representation $\overline{\rho}_{r,P}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}^\circ_{r,0}/\mathfrak{m}_{\mathbb{I}^\circ_{r,0}})$ associated with $\rho_{r,P}$ coincides with $\overline{\rho}|_{H_0}$. In particular, $\rho_{r,P}$ is of residual $\mathrm{Sym}^3$ type in the sense of Definition 3.11. Let $G_{r,P} = \mathrm{Im}\, \rho_{r,P}$ and $G^\circ_{r,P}$ be the connected component of the identity in $G_{r,P}$. Let $\overline{G^\circ_{r,P}}^{\mathrm{Zar}}$ be the Zariski-closure of $G^\circ_{r,P}$ in $\mathrm{GSp}_4(\mathbb{I}_{r,0}/P_r)$. As $\rho_{r,P}$ is residually either



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full or of symmetric cube type, by the classification preceding Lemma 3.15 one of the following must hold:

(i) the algebraic group $\overline{G^{\circ}_{r,P}}^{\mathrm{Zar}}$ is isomorphic to $\mathrm{Sym}^3\,\mathrm{SL}_2$ over $\mathbb{I}_{r,0}/P_r$;

(ii) the algebraic group $\overline{G^{\circ}_{r,P}}^{\mathrm{Zar}}$ is isomorphic to $\mathrm{Sp}_4$ over $\mathbb{I}_{r,0}/P_r$.

In both cases, let $H^0$ denote the normal open subgroup of $H_0$ satisfying $\mathrm{Im}\,\rho_{r,P}|_{H^0} = G^{\circ}_{r,P}$. As $H_0$ is open and normal in $G_{\mathbb{Q}}$, $H^0$ is also open and normal in $G_{\mathbb{Q}}$. In case (i), there exists a representation $\rho^0_{r,P}\colon H^0 \to \mathrm{GL}_2(\mathbb{I}_{r,0}/P_r)$ such that $\rho_{r,P}|_{H^0} \cong \mathrm{Sym}^3\,\rho^0_{r,P}$. As the image of $\rho_{r,P}|_{H^0}$ is Zariski-dense in the copy of $\mathrm{SL}_2(\mathbb{I}_{r,0}/P_r)$ embedded via the symmetric cube map, the image of $\rho^0_{r,P}$ is Zariski-dense in $\mathrm{SL}_2(\mathbb{I}_{r,0}/P_r)$. From Lemma 3.14, we deduce that $\mathrm{Im}\,\rho^0_{r,P}$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_{r,0}/P_r)$. Now the hypotheses of Lemma 3.14 are satisfied by the representation $\rho^0_{r,P}$ and the group $H^0$, so we conclude that there exists a representation $\rho'_{H_0,r,P}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_{r,0}/P_r)$ such that $\rho_{H_0,r,P} \cong \mathrm{Sym}^3\,\rho'_{H_0,r,P}$. By Lemma 11.2, the prime $P$ must contain $\mathfrak{c}_\theta$, as desired.

We show that case (ii) never occurs. Suppose by contradiction that $\overline{G^{\circ}_{H_0,r,P}}^{\mathrm{Zar}} \cong \mathrm{Sp}_4$ over $\mathbb{I}_{r,0}/P_r$. By Propositions 5.11 and 5.14, we know that the field $\mathbb{I}_0/P$ is generated over $\mathbb{Q}_p$ by the traces of $\mathrm{Ad}(\rho_P|_{H_0})$. Hence, the field $\mathbb{I}_{r,0}/P_r$ is generated over $\mathbb{Q}_p$ by the traces of $\mathrm{Ad}\,\rho_{r,P}$. By Theorem 3.13 applied to $\mathrm{Im}\,\rho_{r,P}$, there exists a non-zero ideal $\mathfrak{l}_{r,P}$ of $\mathbb{I}_{r,0}/P_r$ such that $G_{r,P}$ contains the principal congruence subgroup $\Gamma_{\mathbb{I}_{r,0}/P_r}(\mathfrak{l}_{r,P})$ of $\mathrm{Sp}_4(\mathbb{I}_{r,0}/P_r)$. By definition $\mathfrak{G}_{r,P} = \mathbb{Q}_p \cdot \log(\mathrm{Im}\,\rho_{r,P}|_{H_r})$, where $H_r$ is open in $G_{\mathbb{Q}}$, so up to replacing $\mathfrak{l}_{r,P}$ by a smaller non-zero ideal we have

$$\mathfrak{l}_{r,P} \cdot \mathfrak{sp}_4(\mathbb{I}_{r,0}/P_r) \subset \log(\Gamma_{\mathbb{I}_{r,0}/P_r}(\mathfrak{l}_{r,P})) \subset \log(\iota_{r,0}(G_P)) \subset \mathfrak{G}_{r,P}. \tag{40}$$

The algebras $\mathfrak{G}_{r,P}$ are independent of $r$ in the sense of Remark 7.3, so there exists an ideal $\mathfrak{l}_P$ of $\mathbb{I}_0/P$ such that, for every $r$ in the set $\{r_i\}_{i\geqslant 1}$, the ideal $\mathfrak{l}_{r,P} = \iota_r(\mathfrak{l}_P)$ satisfies (40). We choose the ideals $\mathfrak{l}_{r,P}$ of this form.

As before, $\Delta$ is the set of roots of $\mathrm{GSp}_4$ with respect to the chosen maximal torus. Let $\alpha \in \Delta$. Let $\mathfrak{U}^\alpha_r$ and $\mathfrak{U}^\alpha_{r,P_r}$ be the nilpotent Lie subalgebras of $\mathfrak{G}_r$ and $\mathfrak{G}_{r,P_r}$, respectively, corresponding to $\alpha$. We denote by $\pi_{P_r}$ the projection $\mathfrak{gsp}_4(\mathbb{B}_r) \to \mathfrak{gsp}_4(\mathbb{B}_r/P_r\mathbb{B}_r)$. Clearly $\mathfrak{G}_{r,P_r} = \pi_{P_r}(\mathfrak{G}_r)$, so $\mathfrak{U}^\alpha_{r,P_r} = \pi_{P_r}(\mathfrak{U}^\alpha_r)$. Equation (40) gives $\mathfrak{l}_{r,P}\mathfrak{u}^\alpha(\mathbb{I}_{r,0}/P_r) \subset \mathfrak{U}^\alpha_{r,P_r}$. Choose a subset $A^\alpha_P$ of $\mathfrak{u}^\alpha(\mathbb{I}_0)$ such that, for every $r$, $\iota_r(A^\alpha_P) \subset \mathfrak{U}^\alpha_r$ and $\pi_{P_r}(\iota_r(A^\alpha_P)) = \mathfrak{l}_{r,P}\mathfrak{u}^\alpha(\mathbb{I}_{r,0}/P_r)$. Such a set exists because the algebras $\mathfrak{U}^\alpha_r$ are independent of $r$ by Remark 7.3 and the ideals $\mathfrak{l}_{r,P}$ have been chosen of the form $\iota_r(\mathfrak{l}_P)$. Let $\mathfrak{A}_P$ be the ideal generated by $A_P$ in $\mathfrak{u}^\alpha(\mathbb{I}_0)$ and set $\mathfrak{A}_P = \left(\prod_{\alpha\in\Delta}\mathfrak{A}^\alpha_P\right)^4$. By the same argument as in the proof of Theorem 8.1, the ideal $\mathfrak{A}_P$ satisfies

$$\iota_r(\mathfrak{A}_P) \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r.$$

As $\mathfrak{l}_\theta \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r$ for every $r$, we also have $(\mathfrak{l}_\theta + \mathfrak{A}^\alpha_P)\mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r$ for every $r$.

By assumption $\mathfrak{l}_\theta \subset P$, so $\pi_P(\mathfrak{l}_\theta) = 0$. By the definition of $\mathfrak{A}_P$, we have $\pi_P(\mathfrak{A}_P) \supset \pi_P(A_P) = \mathfrak{l}_P$, so $\pi_P(\mathfrak{l}_\theta + \mathfrak{A}_P) = \mathfrak{l}_P$. We deduce that $\mathfrak{l}_\theta + \mathfrak{A}_P$ is strictly larger than $\mathfrak{l}_\theta$. This contradicts the fact that $\mathfrak{l}_\theta$ is the largest among the ideals $\mathfrak{l}$ of $\mathbb{I}_0$ satisfying $\mathfrak{l} \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r$. □

By combining Theorem 11.4 and Lemma 11.3(ii), we obtain the following.

Corollary 11.5. *When the residual representation $\overline{\rho}$ is full, the Galois level $\mathfrak{l}_\theta$ is trivial.*



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