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THE WEYL TRANSFORM OF A SMOOTH MEASURE ON A REAL-ANALYTIC SUBMANIFOLD

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Abstract

If μ is a smooth measure supported on a real-analytic submanifold of $\mathbb{R}^{2n}$ which is not contained in any affine hyperplane, then the Weyl transform of μ is a compact operator.

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1. Introduction

There has been long-standing interest in finding quantum or noncommutative analogues of classical results in harmonic analysis. For example, the noncommutative analogue of the Hausdorff–Young theorem (in greater generality) was proven by Kunze [6] in 1958. Subsequently, there have been more developments in this direction (see [17]). The Weyl transform is a quantum analogue of the Fourier transform. In an earlier work [7], we proved an analogue of a result of Stein about the decay of the Fourier transform of measures supported on a hypersurface of positive Gaussian curvature. Here, we prove a weaker version of that result which applies to submanifolds of arbitrary codimension.

Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted by $\hat{f}$, is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n.$$

The Riemann–Lebesgue lemma [11, Theorem 1.2] states that if $f \in L^1(\mathbb{R}^n)$, then

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

More generally, the Fourier transform of a finite Borel measure λ on $\mathbb{R}^n$ is given by

$$\hat{\lambda}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\, d\lambda(x), \quad \xi \in \mathbb{R}^n.$$

If λ is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}^n$, that is, $\lambda = fm$ for some $f \in L^1(\mathbb{R}^n)$, then the formula above reduces to the usual definition of the Fourier transform of f and

$$\lim_{|\xi|\to\infty} \hat{\lambda}(\xi) = 0.$$

In general, the Fourier transform of a measure need not vanish at infinity. For example, the Fourier transform of δ_0, the Dirac measure, is identically equal to 1. However, the decay of the Fourier transform of a measure can be deduced from certain curvature properties of the support of the measure.

Suppose M is a smooth submanifold of $\mathbb{R}^n$. By a *smooth measure on* M, we mean a measure of the form $\mu = \psi\sigma$, where σ is the measure on M induced by the Lebesgue measure on $\mathbb{R}^n$ and ψ is a smooth function on $\mathbb{R}^n$ whose support intersects M in a compact set.

It is well known that if μ is a smooth measure on a hypersurface in $\mathbb{R}^n$, $n \geq 2$, whose Gaussian curvature is nonzero everywhere, then

$$|\hat{\mu}(\xi)| \leq A|\xi|^{(1-n)/2},$$

where A is a constant independent of ξ (see, for example, [10, Theorem 1, page 348] and [5, Theorem 7.7.14]).

In [7], an analogue of this result for the Weyl transform was proved. Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and $\mathcal{B}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$. If $f \in L^1(\mathbb{R}^{2n})$, the *Weyl transform* of f is the operator $W(f) \in \mathcal{B}(\mathcal{H})$ defined by

$$(W(f)\varphi)(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x,y)e^{\pi i(x\cdot y+2y\cdot t)}\varphi(t+x)\,dx\,dy.$$

More generally, if λ is a finite Borel measure on $\mathbb{R}^{2n}$, the *Weyl transform* of λ is the operator $W(\lambda) \in \mathcal{B}(\mathcal{H})$ defined by

$$(W(\lambda)\varphi)(t) = \int_{\mathbb{R}^{2n}} e^{\pi i(x\cdot y+2y\cdot t)}\varphi(t+x)\,d\lambda(x,y).$$

The Weyl transform can be expressed in terms of the canonical representation of the Heisenberg group. Recall that the reduced Heisenberg group G is the set of triples

$$\{(x,y,z) \mid x,y \in \mathbb{R}^n, z \in \mathbb{C}, |z| = 1\}$$

with multiplication defined by

$$(x,y,z)(x',y',z') = (x+x', y+y', zz'e^{\pi i(x\cdot y'-y\cdot x')}).$$

According to the Stone–von Neumann theorem [3, Theorem 1.50], there is a unique irreducible unitary representation ρ of G such that

$$\rho(0,0,z) = zI.$$

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The standard realisation of this representation is on the Hilbert space $\mathcal{H}$ by the action

$$(\rho(x,y,z)\varphi)(t) = ze^{\pi i(x\cdot y+2y\cdot t)}\varphi(t+x). \tag{1.1}$$

Thus, the Weyl transform of λ may be expressed as

$$W(\lambda) = \int_{\mathbb{R}^{2n}} \rho(x,y,1)\, d\lambda(x,y), \tag{1.2}$$

where the integral is the weak integral as defined in [9, Definition 3.26].

The analogue of the Riemann–Lebesgue lemma for the Weyl transform is the fact that $W(f)$ is a compact operator if $f \in L^1(\mathbb{R}^{2n})$ (see, for example, [3, Theorem 1.30] and [14, Theorem 1.3.3]). If λ is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}^{2n}$, that is, $\lambda = fm$ for some $f \in L^1(\mathbb{R}^{2n})$, then $W(\lambda)$ reduces to the usual definition of the Weyl transform of f and hence $W(\lambda)$ is a compact operator.

The main result of [7] is the following theorem.

THEOREM 1.1. *Suppose S is a compact connected smooth hypersurface in $\mathbb{R}^{2n}$, $n \geq 2$, whose Gaussian curvature is positive everywhere. Let μ be a smooth measure on S. Then, $W(\mu)$ is a compact operator. Moreover, when $n \geq 6$ and $p > n$,*

$$W(\mu) \in S^p(\mathcal{H}),$$

where $S^p(\mathcal{H})$ denotes the p-th Schatten class of $\mathcal{H}$.

Naturally, the question of what happens if we consider a submanifold of arbitrary codimension arises.

Let M be a smooth m-dimensional submanifold of $\mathbb{R}^n$, $1 \leq m \leq n-1$, and let μ be a smooth measure on M. It is well known that if M is of finite type, that is, at each point, M has at most a finite order of contact with any affine hyperplane, then

$$|\hat{\mu}(\xi)| \leq A|\xi|^{-1/k},$$

where k is the type of M inside the support of ψ (see [10, Theorem 2, page 351]). In particular,

$$\lim_{|\xi|\to\infty} \hat{\mu}(\xi) = 0. \tag{1.3}$$

If we consider M to be a real-analytic submanifold of $\mathbb{R}^n$, then the condition of being of finite type is equivalent to M not lying in any affine hyperplane.

The main result of this paper is the following theorem, which is an analogue of (1.3) for the Weyl transform of a smooth measure supported on a real-analytic submanifold of finite type.

THEOREM 1.2. *Suppose M is a connected real-analytic submanifold of $\mathbb{R}^{2n}$ which is not contained in an affine hyperplane. Let μ be a smooth measure on M. Then, $W(\mu)$ is a compact operator.*

In Section 2, we define and study the twisted convolution of finite Borel measures, which is an essential tool required to prove the main result which we prove in Section 3.

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In Section 4, we prove that if $n = 1$, then the conclusion of Theorem 1.2 holds for a submanifold of finite type without the additional assumption of real analyticity. This also proves Theorem 1.1 partially for $n = 1$.

In [2], Edgar and Rosenblatt proved that the translates of a nonzero function in $L^p(\mathbb{R}^n)$, $n \geq 2$, are linearly independent if and only if $p < 2n/(n-1)$. In [13, 17], the quantum translation of an operator was defined and it was shown that if X is a nonzero Hilbert–Schmidt operator, then the quantum translates of X are linearly independent. The motivation for quantum translation may be found in [13] (see also [15, 16]).

In [7], as an application of Theorem 1.1, it was shown that for $n \geq 6$, there exists a nonzero compact operator on $L^2(\mathbb{R}^n)$ whose quantum translates are linearly dependent. By using an argument similar to that in [7], Theorem 1.2 implies the following result.

THEOREM 1.3. *There exists a nonzero compact operator T on $L^2(\mathbb{R}^n)$ and distinct elements $(x_1, y_1), \ldots, (x_{4n+1}, y_{4n+1}) \in \mathbb{R}^{2n}$ such that $\{(x_1, y_1) \cdot T, \ldots, (x_{4n+1}, y_{4n+1}) \cdot T\}$ is a linearly dependent set, where $(x_i, y_i) \cdot T = \rho(x_i, y_i, 1)T\rho(x_i, y_i, 1)^{-1}$ is the quantum translation of T by (x_i, y_i), $i = 1, \ldots, 4n+1$.*

2. Twisted convolution

Recall that if $f, g \in L^1(\mathbb{R}^{2n})$, the twisted convolution of f and g, denoted by $f \natural g$, is

$$f \natural g(x, y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - x', y - y')g(x', y')e^{\pi i(x \cdot y' - y \cdot x')}\, dx'\, dy', \quad (x, y) \in \mathbb{R}^{2n}.$$

Twisted convolution turns $L^1(\mathbb{R}^{2n})$ into a noncommutative Banach algebra. It is well known that the Weyl transform is an algebra homomorphism from $L^1(\mathbb{R}^{2n})$ to $\mathcal{B}(\mathcal{H})$, that is, $W(f \natural g) = W(f)W(g)$ (see, for example, [3, page 26] and [14, page 16]).

DEFINITION 2.1. Let μ and ν be finite Borel measures on $\mathbb{R}^{2n}$. The twisted convolution of μ and ν is the measure on $\mathbb{R}^{2n}$, denoted by $\mu \natural \nu$, given by

$$\mu \natural \nu(E) = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_E(x + x', y + y')e^{\pi i(x \cdot y' - y \cdot x')}\, d\mu(x, y)\, d\nu(x', y'),$$

where E is a Borel subset of $\mathbb{R}^{2n}$.

It follows from Definition 2.1 that if $f \in C_0(\mathbb{R}^{2n})$, then

$$\int_{\mathbb{R}^{2n}} f(x, y)\, d(\mu \natural \nu)(x, y) = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} f(x + x', y + y')e^{\pi i(x \cdot y' - y \cdot x')}\, d\mu(x, y)\, d\nu(x', y'). \tag{2.1}$$

Let $f, g \in L^1(\mathbb{R}^{2n})$. Let m denote the Lebesgue measure on $\mathbb{R}^{2n}$. Let $\mu_f = fm$ and $\mu_g = gm$. If E is a Borel subset of $\mathbb{R}^{2n}$, then

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$$\begin{aligned}\mu_f \natural \mu_g(E) &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_E(x+x', y+y') e^{\pi i(x\cdot y' - y\cdot x')}\, d\mu(x,y)\, d\nu(x',y') \\ &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_E(x+x', y+y') e^{\pi i(x\cdot y' - y\cdot x')} f(x,y) g(x',y')\, dm(x,y)\, dm(x',y') \\ &= \int_{\mathbb{R}^{2n}} \chi_E(x,y) \int_{\mathbb{R}^{2n}} e^{\pi i(x\cdot y' - y\cdot x')} f(x-x', y-y') g(x',y')\, dm(x',y')\, dm(x,y) \\ &= \int_{\mathbb{R}^{2n}} \chi_E(x,y)(f \natural g)(x,y)\, dm(x,y) = \mu_{f\natural g}(E).\end{aligned}$$

Therefore, the two definitions of twisted convolution coincide for L^1 functions.

Twisted convolution turns $M(\mathbb{R}^{2n})$, the set of finite Borel measures on $\mathbb{R}^{2n}$, into a noncommutative Banach algebra. The following theorem shows that the Weyl transform is an algebra homomorphism from $M(\mathbb{R}^{2n})$ to $\mathcal{B}(\mathcal{H})$.

THEOREM 2.2. *Let μ and ν be finite Borel measures on $\mathbb{R}^{2n}$. Then,*

$$W(\mu \natural \nu) = W(\mu)W(\nu).$$

PROOF. By (1.2) and (2.1),

$$\begin{aligned}W(\mu \natural \nu) &= \int_{\mathbb{R}^{2n}} \rho(x,y,1)\, d(\mu \natural \nu)(x,y) \\ &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \rho(x+x', y+y', 1) e^{\pi i(x\cdot y' - y\cdot x')}\, d\mu(x,y)\, d\nu(x',y') \\ &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \rho(x+x', y+y', e^{\pi i(x\cdot y' - y\cdot x')})\, d\mu(x,y)\, d\nu(x',y') \quad \text{(by (1.1))} \\ &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \rho((x,y,1)(x',y',1))\, d\mu(x,y)\, d\nu(x',y') \\ &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \rho(x,y,1)\rho(x',y',1)\, d\nu(x',y')\, d\mu(x,y) \\ &= \int_{\mathbb{R}^{2n}} \rho(x,y,1) W(\nu)\, d\mu(x,y) = W(\mu)W(\nu).\end{aligned}$$

□

3. The proof

To prove Theorem 1.2, we need a result analogous to a result of Ragozin about the absolute continuity of the convolution of measures supported on analytic submanifolds of $\mathbb{R}^n$ (see [8, Theorem 5.1]). In particular, Ragozin proved the absolute continuity of the convolution square of the surface measure on a compact analytic hypersurface of $\mathbb{R}^n$. Later, Thangavelu proved that the twisted convolution of the surface measure on a unit sphere in $\mathbb{R}^{2n}$ with itself is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2n}$ (see [12, Proposition 4.3]).

The following theorem is an analogue of [8, Theorem 5.1] for twisted convolutions.

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THEOREM 3.1. *Let $M_1, \ldots, M_k$ be connected real-analytic submanifolds of $\mathbb{R}^{2n}$ such that $T_{p_1}M_1 + \cdots + T_{p_k}M_k = \mathbb{R}^{2n}$ for some choice of points $p_i \in M_i$, $i = 1, \ldots, k$, where $T_{p_i}M_i$ denotes the tangent space at p_i of M_i. If μ_i is a smooth measure on M_i, $i = 1, \ldots, k$, then $\mu_1 \natural \cdots \natural \mu_k$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2n}$.*

Let $\Sigma_k : M_1 \times \cdots \times M_k \to \mathbb{R}^{2n}$ be the map given by $\Sigma_k(p_1, \ldots, p_k) = p_1 + \cdots + p_k$. We need the following lemma to prove Theorem 3.1.

LEMMA 3.2. *There exists a smooth function $\varphi_k : M_1 \times \cdots \times M_k \to \mathbb{C}$ such that, for a Borel set $E \subseteq \mathbb{R}^{2n}$,*

$$\mu_1 \natural \cdots \natural \mu_k(E) = (\varphi_k \, \mu_1 \times \cdots \times \mu_k)(\Sigma_k^{-1}(E)),$$

that is, $\mu_1 \natural \cdots \natural \mu_k$ is the push-forward of the measure $\varphi_k \, \mu_1 \times \cdots \times \mu_k$ by Σ_k.

PROOF. We will prove the result by induction on k. First, consider $k = 2$. Observe that by Definition 2.1,

$$\mu_1 \natural \mu_2(E) = (\varphi_2 \, \mu_1 \times \mu_2)(\Sigma_2^{-1}(E)), \tag{3.1}$$

where $\varphi_2 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{C}$ is given by $\varphi_2((x_1, y_1), (x_2, y_2)) = e^{\pi i (x_1 \cdot y_2 - y_1 \cdot x_2)}$.

Assume that there exists a function $\varphi_{k-1} : M_1 \times \cdots \times M_{k-1} \to \mathbb{C}$ such that

$$\mu_1 \natural \cdots \natural \mu_{k-1}(E) = (\varphi_{k-1} \, \mu_1 \times \cdots \times \mu_{k-1})(\Sigma_{k-1}^{-1}(E)).$$

Then,

$$\begin{aligned}
&((\mu_1 \natural \cdots \natural \mu_{k-1}) \natural \mu_k)(E) \\
&\quad = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_E(x + x_k, y + y_k) e^{\pi i (x \cdot y_k - y \cdot x_k)} \, d(\mu_1 \natural \cdots \natural \mu_{k-1})(x, y) \, d\mu_k(x_k, y_k) \\
&\quad = \int_{\mathbb{R}^{2n}} \cdots \int_{\mathbb{R}^{2n}} [\chi_E(x_1 + \cdots + x_k, y_1 + \cdots + y_k) e^{\pi i ((x_1 + \cdots + x_{k-1}) \cdot y_k - (y_1 + \cdots + y_{k-1}) \cdot x_k)} \\
&\qquad\qquad \cdot \varphi_{k-1}((x_1, y_1), \ldots, (x_{k-1}, y_{k-1}))] \, d\mu_1(x_1, y_1) \cdots d\mu_k(x_k, y_k) \\
&\quad = (\varphi_k \, \mu_1 \times \cdots \times \mu_k)(\Sigma_k^{-1}(E)),
\end{aligned}$$

where $\varphi_k : M_1 \times \cdots \times M_k \to \mathbb{C}$ is given by

$$\varphi_k((x_1, y_1), \ldots, (x_k, y_k)) = \varphi_{k-1}((x_1, y_1), \ldots, (x_{k-1}, y_{k-1})) e^{\pi i ((x_1 + \cdots + x_{k-1}) \cdot y_k - (y_1 + \cdots + y_{k-1}) \cdot x_k)}.$$

□

Theorem 3.1 now follows by the argument in [8, Theorem 5.1]. Here, we give a more streamlined proof using the co-area formula.

PROOF OF THEOREM 3.1. Let $M_1, \ldots, M_k$ be connected real-analytic submanifolds of $\mathbb{R}^{2n}$. Then, $M_1 \times \cdots \times M_k$ is a connected real-analytic manifold. Let $\tau_{(M_1 \times \cdots \times M_k)}$ denote the Riemannian measure on $M_1 \times \cdots \times M_k$.

Observe that Σ_k is an analytic map. Let $p_i \in M_i$, $i = 1, \ldots, k$ be such that $T_{p_1}M_1 + \cdots + T_{p_k}M_k = \mathbb{R}^{2n}$. Then, the rank of Σ_k is $2n$ at the point $(p_1, \ldots, p_k)$.

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Therefore, the critical set of Σ_k is a proper analytic subvariety of $M_1 \times \cdots \times M_k$ and hence has $\tau_{(M_1\times\cdots\times M_k)}$ measure zero.

If μ_i is a smooth measure on M_i, $i = 1, \ldots, k$, then $\mu_1 \times \cdots \times \mu_k$ is a smooth measure on $M_1 \times \cdots \times M_k$ and so $\varphi_k \mu_1 \times \cdots \times \mu_k$ is a smooth measure on $M_1 \times \cdots \times M_k$.

The proof of Theorem 3.1 now follows from the following lemma. □

LEMMA 3.3. *Let M and N be Riemannian manifolds. Let τ and ν denote the Riemannian measures on M and N, respectively. Let $\mu = \psi\tau$ be a smooth measure on M. Suppose $f : M \to N$ is a differentiable map. If f is a submersion everywhere except on a set of τ-measure zero, then the push-forward of μ by f is absolutely continuous with respect to ν.*

PROOF. Let $Z = \{x \in M \mid f \text{ is not a submersion at } x\}$. Then, $\tau(Z) = 0$ and so $\mu(Z) = 0$. Let $\mathcal{J}_f$ denote the normal Jacobian of f, that is, the absolute value of the determinant of df restricted to the orthogonal complement of its kernel. Then, $\mathcal{J}_f$ is strictly positive on the set of regular points of f, that is, on $M \setminus Z$.

Let $f_*\mu$ denote the push-forward of μ by f. For $x \in N$, let σ_x denote the Riemannian measure on the manifold $f^{-1}(x)$. Let $U \subseteq N$ be a Borel set. By the co-area formula (see [1, page 159]),

$$\begin{aligned} f_*\mu(U) &= \mu(f^{-1}(U)) = \mu(f^{-1}(U) \setminus Z) \\ &= \int_{f^{-1}(U)\setminus Z} d\mu = \int_{f^{-1}(U)\setminus Z} \psi \, d\tau \\ &= \int_{U\setminus f(Z)} \int_{f^{-1}(x)} \frac{\psi(y)}{\mathcal{J}_f(y)} \, d\sigma_x(y) \, d\nu(x), \end{aligned}$$

that is,

$$\frac{df_*\mu}{d\nu}(x) = \int_{f^{-1}(x)} \frac{\psi(y)}{\mathcal{J}_f(y)} \, d\sigma_x(y),$$

and so $f_*\mu$ is absolutely continuous with respect to ν. □

We need the following lemma to prove Theorem 1.2.

LEMMA 3.4. *Let M be a connected real-analytic submanifold of $\mathbb{R}^{2n}$. Assume that M is not contained in any affine hyperplane. Then, there exist points $p_1, \ldots, p_k \in M$ such that $T_{p_1}M + \cdots + T_{p_k}M = \mathbb{R}^{2n}$.*

PROOF. Suppose $\sum_{p\in M} T_pM \neq \mathbb{R}^{2n}$. Then, there exists a hyperplane Y in $\mathbb{R}^{2n}$ such that $\sum_{p\in M} T_pM \subseteq Y$.

Fix $p \in M$. Let $q \in M$. Since M is connected, there exists a differentiable map $\xi : [0, 1] \to M$ such that $\xi(0) = p$ and $\xi(1) = q$. By the fundamental theorem of calculus,

$$q = p + \int_0^1 \dot{\xi}(t) \, dt.$$

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Since $\dot{\xi}(t) \in Y$ for $t \in [0,1]$ and Y is closed, it follows that $q \in p + Y$. This implies that M is contained in the affine hyperplane $p + Y$, which is a contradiction.

Hence, $\sum_{p \in M} T_p M = \mathbb{R}^{2n}$. By the Steinitz exchange lemma, there exist finitely many points $p_1, \ldots, p_k \in M$ such that $T_{p_1}M + \cdots + T_{p_k}M = \mathbb{R}^{2n}$. □

By possibly adding one more point, we can assume that the integer k obtained in Lemma 3.4 is even.

PROOF OF THEOREM 1.2. Let $\widetilde{M} = \{-p \mid p \in M\}$. Then, $\widetilde{M}$ is a connected real-analytic submanifold of $\mathbb{R}^{2n}$. Observe that if $p \in M$, $T_p M = T_{(-p)}\widetilde{M}$. Therefore, by Lemma 3.4,

$$T_{p_1}M + T_{(-p_2)}\widetilde{M} + \cdots + T_{p_{k-1}}M + T_{(-p_k)}\widetilde{M} = \mathbb{R}^{2n}. \tag{3.2}$$

Let μ be a smooth measure on M. Let $\tilde{\mu}$ denote the push-forward of μ by the map which sends $p \in M$ to $-p$. Then, by Theorem 3.1 and (3.2), it follows that $(\mu \natural \tilde{\mu})^{k/2}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2n}$. Therefore, $W((\mu \natural \tilde{\mu})^{k/2})$ is a compact operator.

Observe that $W(\tilde{\mu}) = W(\mu)^*$, where $W(\mu)^*$ is the adjoint of the operator $W(\mu)$. By Theorem 2.2,

$$W((\mu \natural \tilde{\mu})^{k/2}) = (W(\mu)W(\tilde{\mu}))^{k/2} = (W(\mu)W(\mu)^*)^{k/2}.$$

Since $W(\mu)W(\mu)^*$ is self-adjoint and $(W(\mu)W(\mu)^*)^{k/2}$ is compact, it follows that $W(\mu)W(\mu)^*$ is compact. By the polar decomposition, it follows that $W(\mu)$ is compact. □

4. Curve in $\mathbb{R}^2$

In this section, we prove the following result, which states that if $n = 1$, the assumption of real analyticity can be removed from Theorem 1.2.

THEOREM 4.1. *Suppose $M \subseteq \mathbb{R}^2$ is a connected smooth hypersurface of finite type. Let μ be a smooth measure on M. Then, $W(\mu)$ is a compact operator.*

We need the following lemma to prove this theorem.

LEMMA 4.2. *Let $\gamma : [a,b] \to \mathbb{R}^2$ be a finite type unit-speed simple smooth curve. Let $\delta : [c,d] \to \mathbb{R}^2$ be a unit-speed simple smooth curve. Let μ and ν be smooth measures on* $\mathrm{Im}(\gamma)$ *and* $\mathrm{Im}(\delta)$*, respectively. Then, $\mu \natural \nu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.*

PROOF. Let $E \subseteq \mathbb{R}^2$ be a Borel set. Then, by (3.1),

$$\mu \natural \nu(E) = (\varphi_2 \mu \times \nu)(\Sigma_2^{-1}(E)),$$

where $\Sigma_2 : \mathrm{Im}(\gamma) \times \mathrm{Im}(\delta) \to \mathbb{R}^2$ is the map given by $\Sigma_2(\gamma(s), \delta(t)) = \gamma(s) + \delta(t)$. Define $S : [a,b] \times [c,d] \to \mathbb{R}^2$ by

$$S(s,t) = \gamma(s) + \delta(t).$$

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Let $T = \{(s,t) \in [a,b] \times [c,d] \mid \dot{\gamma}(s) = \pm\dot{\delta}(t)\}$. Observe that T is the critical set of S. We claim that T has area zero. Suppose T has a positive area. Then, by Fubini's theorem, there exists $t \in [c,d]$ such that the set $Z = \{s \in [a,b] \mid \dot{\gamma}(s) = \pm\dot{\delta}(t)\}$ has positive length. By the Lebesgue differentiation theorem [4, Theorem 3.21], there exists $s' \in [a,b]$ such that s' is a point of density of Z. Then, there exist distinct $s_j \in Z$ such that the sequence $\{s_j\}$ converges to s', and so the sequence $\{\dot{\gamma}(s_j)\}$ converges to $\dot{\gamma}(s')$. Let $\vec{v}$ be a unit vector in $\mathbb{R}^2$ perpendicular to $\dot{\delta}(t)$. Then,

$$\langle \dot{\gamma}(s_j)\,,\, \vec{v}\rangle = \langle \pm\dot{\delta}(t)\,,\, \vec{v}\rangle = 0, \quad j = 1, 2, \ldots.$$

Hence, all the coefficients in the Taylor expansion of $\langle \dot{\gamma}(s)\,,\, \vec{v}\rangle$ about $s = s'$ are zero. This contradicts the fact that γ is a curve of finite type. Therefore, T has area zero.

Since $\mathrm{Im}(\gamma)$ and $\mathrm{Im}(\delta)$ are smooth submanifolds of $\mathbb{R}^2$, it follows that $\mathrm{Im}(\gamma) \times \mathrm{Im}(\delta)$ is a smooth submanifold of $\mathbb{R}^4$. Let τ denote the Riemannian measure on $\mathrm{Im}(\gamma) \times \mathrm{Im}(\delta)$. Since $\gamma \times \delta$ is a smooth map and the set T has area zero, it follows that $(\gamma \times \delta)(T)$ has τ-measure zero.

Observe that $(\gamma \times \delta)(T)$ is the critical set of Σ_2. The proof now follows from Lemma 3.3. □

Proof of Theorem 4.1. Suppose $M \subseteq \mathbb{R}^2$ is a hypersurface of finite type. Let μ be a smooth measure on M. Let $\tilde{\mu}$ denote the push-forward of μ by the map which sends $p \in M$ to $-p$. It follows from Lemma 4.2 that $\mu \natural \tilde{\mu}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. Therefore, $W(\mu \natural \tilde{\mu})$ is compact. Observe that $W(\tilde{\mu}) = W(\mu)^*$. By Theorem 2.2,

$$W(\mu \natural \tilde{\mu}) = W(\mu)W(\tilde{\mu}) = W(\mu)W(\mu)^*.$$

Since $W(\mu)W(\mu)^*$ is compact, it follows by the polar decomposition that $W(\mu)$ is compact. □

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