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Compactness of Hardy-Type Operators over Star-Shaped Regions in $\mathbb{R}^N$

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Abstract. We study a compactness property of the operators between weighted Lebesgue spaces that average a function over certain domains involving a star-shaped region. The cases covered are (i) when the average is taken over a difference of two dilations of a star-shaped region in $\mathbb{R}^N$, and (ii) when the average is taken over all dilations of star-shaped regions in $\mathbb{R}^N$. These cases include, respectively, the average over annuli and the average over balls centered at origin.

1 Introduction

In [4], Heinig and Sinnamon have given a weight characterization for the boundedness of the so called "Hardy-Steklov operator" $T\colon L^p\big((0,\infty),U\big) \to L^q\big((0,\infty),V\big)$ defined by

$$(Tf)(x) = \int_{a(x)}^{b(x)} f(t)\,dt \tag{1.1}$$

where $p \in (1,\infty)$, $q \in (0,\infty)$ and $a = a(x)$, $b = b(x)$ are strictly increasing differentiable functions on $[0,\infty]$ satisfying $a(0) = b(0) = 0$, $a(x) < b(x)$ for $0 < x < \infty$ and $a(\infty) = b(\infty) = \infty$. Further, Sinnamon [10] has considered a higher dimensional Hardy-Steklov operator $T_E\colon L^p(E,u) \to L^q(E,v)$ defined by

$$(T_E f)(x) = \int_{b(\alpha_x)S\setminus a(\alpha_x)S} f(t)\,dt \tag{1.2}$$

that averages a function over a difference of two dilations of a star-shaped region S in $\mathbb{R}^N$ (the term star shaped region and other symbols in 1.2 are defined in Section 3). In fact, in a remarkable result (Theorem 2.1, [10]), Sinnamon has shown that the boundedness of the operator T_E can be characterized in terms of the boundedness of the one dimensional operator T.

In this paper, we complement Sinnamon's result by considering the compactness property of the operator T_E. In fact, we show (see Theorem 3.1) that as in the case of boundedness, the compactness of T_E can also be characterized in terms of the compactness of T. Then to obtain the precise necessary and sufficient conditions for the compactness of T_E, one can use any known criterion for the compactness of T, *e.g.*, the following result from [6] can be used:

Theorem A *Consider the operator* $T\colon L^p\big((0,\infty),U\big) \to L^q\big((0,\infty),V\big)$ *defined by* (1.1) *and let* U, V *be weight functions on* $(0,\infty)$.

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(a) *If* $1 < p \le q < \infty$, *then* T *is compact if and only if*

$$\sup_{\substack{0<t\le s\\ a(s)\le b(t)}} B(s,t) < \infty,$$

$$\lim_{t\to s^-} B(s,t) = \lim_{t\to b^{-1}(a(s))^+} B(s,t) = 0, \quad \textit{for every } s > 0$$

and

$$\lim_{s\to t^+} B(s,t) = \lim_{s\to a^{-1}(b(t))^-} B(s,t) = 0, \quad \textit{for every } t > 0$$

where

$$B(s,t) = \Big(\int_{a(s)}^{b(t)} U^{1-p'}\Big)^{\frac{1}{p'}} \Big(\int_t^s V\Big)^{\frac{1}{q}}. \tag{1.3}$$

(b) *If* $1 < q < p < \infty$, *then* T *is compact if and only if* $A = \max(A_1, A_2) < \infty$, *where*

$$A_1 = \bigg(\int_0^\infty \Big(\int_{b^{-1}(a(t))}^t \Big(\int_{a(t)}^{b(s)} W\Big)^{\frac{r}{p'}} \Big(\int_s^t V\Big)^{\frac{r}{p}} V(s)\,ds\Big)\,\sigma(t)\,dt\bigg)^{\frac{1}{r}} \tag{1.4}$$

and

$$A_2 = \bigg(\int_0^\infty \Big(\int_t^{a^{-1}(b(t))} \Big(\int_{a(s)}^{b(t)} W\Big)^{\frac{r}{p'}} \Big(\int_t^s V\Big)^{\frac{r}{p}} V(s)\,ds\Big)\,\sigma(t)\,dt\bigg)^{\frac{1}{r}} \tag{1.5}$$

with $W = U^{1-p'}$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ *and where* σ *is a normalizing function defined in Section 3 after the proof of Theorem 3.2.*

It is observed that for the case $1 < q < p < \infty$, the conditions for the boundedness and compactness of the operator T are same. In fact, this is quite expected, which can be seen from a general principle of Anto [1] in this regard. However, in [6], the authors gave a direct proof of Theorem A(b).

Further, as a special case of the operator T_E, one can obtain an operator that averages over the annuli. Consequently, we derive results (see Corollaries 3.4, 3.5 and 3.6) for such operators.

Observe that the operator T is, in some sense, more general than the classical Hardy operator $(Hf)(x) = \int_0^x f(t)\,dt$. However, H can not be obtained from T since for this, the natural choice of the functions a and b in the operator T would be $a(x) = 0$ and $b(x) = x$, but then we would not have $a(\infty) = \infty$. Already necessary and sufficient conditions are available (see *e.g.* [8]) under which H is compact. Thus the natural question arises: if we consider an N-dimensional Hardy operator that averages over the dilations of a star-shaped region, can its compactness be obtained in terms of the compactness of H? The answer is affirmative and the corresponding results are discussed in Section 4. The boundedness of such operators has been characterized, again, by Sinnamon [9].

Finally, we also discuss the corresponding results for all the conjugate operators that we deal with in this paper.

The paper is organized in the following way. In Section 2, we collect certain preliminaries which are standard but will ease the reading of the paper. Section 3 contains the results involving the operator that averages over difference of two dilations of a star-shaped region, while in Section 4 we deal with the operator that averages over all dilations of a star-shaped region. Also, in that section, we discuss the compactness of conjugates of all the operators that we deal with in Sections 3 and 4.

2 Preliminaries

Let X be a normed linear space and X^* denote its conjugate space. We say that a sequence $\{x_n\}$ in X is strongly convergent (or simply convergent) to $x \in X$, written $x_n \to x$, if $\|x_n - x\| \to 0$ as $n \to \infty$. A sequence $\{x_n\}$ in X is said to converge weakly to $x \in X$, written $x_n \xrightarrow{w} x$, if $f(x_n) \to f(x)$, for each $f \in X^*$. A sequence $\{f_n\}$ in X^* is said to be *weak* convergent* to $f \in X^*$, written $f_n \xrightarrow{w^*} f$, if $f_n(x) \to f(x)$ for each $x \in X$. Note that the strong convergence implies the weak convergence which in turn implies the weak* convergence. The implications in the reverse direction do not hold in general. However, if X is a reflexive space then the weak* convergence implies the weak convergence.

Let $\Omega \subseteq \mathbb{R}^N$. A weight function on Ω is a function which is measurable and positive almost everywhere (a.e.) on Ω. For a weight function u, $L^p(\Omega, u)$, $1 \le p < \infty$, denotes the weighted Lebesgue space which is the set of all measurable functions f defined on Ω such that

$$\|f\|_{p,\Omega,u} := \Big(\int_\Omega |f(x)|^p u(x)\,dx\Big)^{\frac{1}{p}} < \infty.$$

Note that for $1 \le p < \infty$, $L^p(\Omega, u)$ is a Banach space and for $1 < p < \infty$, it is reflexive. If the duality on the weighted Lebesgue space $L^p(\Omega, u)$, $1 < p < \infty$, is defined by

$$\langle f, g\rangle = \int_\Omega f(x)g(x)\,dx, \quad g \in L^p(\Omega, u)$$

then we can identify the conjugate space of $L^p(\Omega, u)$ by $L^{p'}(\Omega, u^{1-p'})$, $p' = \frac{p}{p-1}$ being the conjugate index to p, *i.e.*,

$$[L^p(\Omega, u)]^* = L^{p'}(\Omega, u^{1-p'}).$$

For a bounded linear operator A between two normed linear spaces X and Y, we denote by A^* the conjugate of A acting between Y^* and X^*.

The proofs of the theorems presented in this paper require some well known assertions which are collected in the following:

Theorem B *Let X and Y be Banach spaces.*

(a) *A bounded linear operator $A\colon X \to Y$ is compact if and only if its conjugate A^* is compact.*

(b) *If* $A\colon X \to Y$ *is compact and* $\{x_n\}$ *is a sequence in* X *such that* $\{x_n\} \xrightarrow{w} x$, *for some* $x \in X$, *then* $Ax_n \to Ax$.
(c) *An operator* $A\colon X \to Y$ *is compact if* $A^*\colon Y^* \to X^*$ *is weak***-norm sequentially continuous* i.e., *for each sequence* $\{f_n\}$ *in* Y^* *with* $\{f_n\} \xrightarrow{w^*} f$, *for some* $f \in Y^*$, *we have* $A^*(f_n) \to A^* f$.

Finally, the word conjugate has been used in connection with the index, the space and the operator. But there should not be any ambiguity as it will be used according to the context. The preliminaries collected here can be found in any standard book on functional analysis *e.g.*, [2], [3], [5]. In particular, Theorem B(c) is taken from ([2], p. 15).

3 Hardy-Steklov Operator

We call a region $S \in \mathbb{R}^N$ to be smoothly starshaped, if there exists a nonnegative, piecewise-C^1 function Ψ defined on the unit sphere in $\mathbb{R}^N$ with

$$S = \left\{ x \in \mathbb{R}^N \setminus \{0\} : |x| \le \Psi(x/|x|) \right\}.$$

Let S be a smoothly star-shaped region in $\mathbb{R}^N$ and

$$B = \left\{ x \in \mathbb{R}^N \setminus \{0\} : |x| = \Psi(x/|x|) \right\}.$$

We note that B is contained in the boundary of S and since Ψ is not assumed to be continuous B may not be the whole boundary of S. Let E be the union of all dilations of S, *i.e.*, $E = \bigcup_{\alpha>0} \alpha S$. Note that $E = \mathbb{R}^N$ whenever 0 is in the interior of S. For a non zero $x \in E$, since S is star-shaped, there is a least positive dilation $\alpha_x S$ which contains x. We write $S_x = \alpha_x S$ and note that $x/\alpha_x \in B$ so that x is on the boundary of S_x.

Throughout, a and b will denote strictly increasing differentiable functions on $[0, \infty]$ satisfying $a(0) = b(0) = 0$, $a(x) < b(x)$ for $0 < x < \infty$ and $a(\infty) = b(\infty) = \infty$. Clearly a^{-1} and b^{-1} exist and are also strictly increasing.

For weight functions u, v on E, define the higher dimensional Hardy-Steklov operator $T_E\colon L^p(E,u) \to L^q(E,v)$ by

$$(T_E f)(x) = \int_{b(\alpha_x)S \setminus a(\alpha_x)S} f(y)\,dy.$$

It can be seen that the operators T_E and

$$T_E^*\colon L^{q'}(E, v^{1-q'}) \to L^{p'}(E, u^{1-p'})$$

defined by

$$(T_E^* g)(x) = \int_{a^{-1}(\alpha_x)S \setminus b^{-1}(\alpha_x)S} g(y)\,dy$$

are mutually conjugate, *i.e.*,

$$\langle T_E f, g \rangle = \langle f, T_E^* g \rangle.$$

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Also, we shall be dealing with the one-dimensional Hardy-Steklov operator $T\colon L^p\big((0,\infty),U\big) \to L^q\big((0,\infty),V\big)$ defined by

$$(Tf)(x) = \int_{a(x)}^{b(x)} f(t)\,dt, \quad x \in (0,\infty).$$

Here U and V are the weight functions defined on $(0,\infty)$. As above the operators T and $T^*\colon L^{q'}\big((0,\infty),V^{1-q'}\big) \to L^{p'}\big((0,\infty),U^{1-p'}\big)$ defined by

$$(T^*f)(x) = \int_{b^{-1}(x)}^{a^{-1}(x)} f(t)\,dt$$

are also mutually conjugate.

Now, we prove our first main result that characterizes the compactness of the N-dimensional operator T_E in terms of the compactness of the one-dimensional operator T.

Theorem 3.1 *Let S be a smoothly star-shaped region in $\mathbb{R}^N$ and B, E, α_x be as defined above. Suppose $1 < p, q < \infty$ and u, v be weight functions on E. Then the operator $T_E\colon L^p(E,u) \to L^q(E,v)$ is compact if and only if the operator $T\colon L^p\big((0,\infty),U\big) \to L^q\big((0,\infty),V\big)$ is compact with*

$$U(t) = \Big(\int_B u^{1-p'}(t\tau)t^{N-1}\,d\tau\Big)^{1-p}, \quad t \in (0,\infty) \tag{3.1}$$

and

$$V(t) = \int_B v(t\sigma)t^{N-1}\,d\sigma, \quad t \in (0,\infty). \tag{3.2}$$

Proof First assume that $T\colon L^p\big((0,\infty),U\big) \to L^q\big((0,\infty),V\big)$ is compact. It suffices to show that $T_E^*\colon L^{q'}(E,v^{1-q'}) \to L^{p'}(E,u^{1-p'})$ is weak*-norm sequentially continuous since then, the result follows from Theorem B(c). Let $\{f_n\}$ be a sequence in $L^{q'}(E,v^{1-q'})$ such that $f_n \xrightarrow{w^*} 0$. Without any loss of generality, we may assume that each f_n is non negative. Define

$$F_n(t) = \int_B f_n(t\tau)t^{N-1}\,d\tau, \quad n \in \mathbb{N},\ t \in (0,\infty). \tag{3.3}$$

Then

$$\begin{aligned} F_n(t) &= \int_B f_n(t\tau)v^{-\frac{1}{q}}(t\tau)(t^{N-1})^{\frac{1}{q'}}v^{\frac{1}{q}}(t\tau)(t^{N-1})^{\frac{1}{q}}\,d\tau \\ &\le \Big(\int_B f_n^{q'}(t\tau)v^{1-q'}(t\tau)t^{N-1}\,d\tau\Big)^{\frac{1}{q'}}\Big(\int_B v(t\tau)t^{N-1}\,d\tau\Big)^{\frac{1}{q}}, \end{aligned}$$

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and therefore using (3.2) and making change of variable $t\tau = y$, we have

$$\Big(\int_0^\infty F_n^{q'}(t)V^{1-q'}(t)\,dt\Big)^{\frac{1}{q'}} \le \Big(\int_0^\infty \int_B f_n^{q'}(t\tau)v^{1-q'}(t\tau)t^{N-1}d\tau\,dt\Big)^{\frac{1}{q'}}$$
$$= \Big(\int_E f_n^{q'}(y)v^{1-q'}(y)\,dy\Big)^{\frac{1}{q'}} < \infty,$$

which gives that $\{F_n\}$ is a sequence in $L^{q'}\big((0,\infty), V^{1-q'}\big)$. Next we note that if G is any function in $L^q((0,\infty), V)$ and $g\colon E \to \mathbb{R}$ is defined by

$$g(x) = G(t), \quad x = t\tau,$$

then $g \in L^q(E, v)$, since by using (3.2) and making change of variable $x = t\tau$, we have

$$\int_E g^q(x)v(x)\,dx = \int_0^\infty \int_B g^q(t\tau)v(t\tau)t^{N-1}d\tau\,dt$$
$$= \int_0^\infty G^q(t)V(t)\,dt < \infty.$$

Thus by using (3.3), we have

$$\int_0^\infty F_n(t)G(t)\,dt = \int_0^\infty \Big(\int_B f_n(t\tau)t^{N-1}\,d\tau\Big)G(t)\,dt$$
$$= \int_0^\infty \int_B f_n(t\tau)g(t\tau)t^{N-1}\,d\tau\,dt$$
$$= \int_E f_n(x)g(x)\,dx \to 0 \quad \text{as } n \to \infty,$$

i.e., $F_n \xrightarrow{w} 0$. Further, since T is compact, by Theorem B ((a) and (b)),

$$\|T^*F_n\|_{p',(0,\infty),U^{1-p'}} \to 0 \quad \text{as } n \to \infty.$$

Now, making change of variables $y = t\tau$, $x = s\sigma$ so that for $\sigma \in B$, $\alpha_x = s$, and using (3.1), (3.3), we have

$$\|T_E^* f_n\|_{p',E,u^{1-p'}} = \Big(\int_E \Big(\int_{a^{-1}(\alpha_x)S\setminus b^{-1}(\alpha_x)S} f_n(y)\,dy\Big)^{p'} u^{1-p'}(x)\,dx\Big)^{\frac{1}{p'}}$$
$$= \Big(\int_0^\infty \int_B \Big(\int_{b^{-1}(s)}^{a^{-1}(s)} \int_B f_n(t\tau)t^{N-1}\,d\tau\,dt\Big)^{p'}$$
$$\times u^{1-p'}(s\sigma)s^{N-1}\,d\sigma\,ds\Big)^{\frac{1}{p'}}$$
$$= \Big(\int_0^\infty \Big(\int_{b^{-1}(s)}^{a^{-1}(s)} F_n(t)\,dt\Big)^{p'} U^{1-p'}(s)\,ds\Big)^{\frac{1}{p'}}$$
$$= \|T^*F_n\|_{p',(0,\infty),U^{1-p'}},$$

and we are done.

Conversely, assume that $T_E \colon L^p(E,u) \to L^q(E,v)$ is compact. Let $\{F_n\}$ be a sequence in $L^{q'}\big((0,\infty), V^{1-q'}\big)$ such that $F_n \xrightarrow{w^*} 0$. Without any loss of generality we may assume that each F_n is non-negative. Define

$$f_n(t\tau) = F_n(t)v(t\tau)V^{-1}(t), \quad n \in \mathbb{N},\ t \in (0,\infty),\ \tau \in B. \tag{3.4}$$

Then

$$\int_B f_n(t\tau)t^{N-1}\,d\tau = F_n(t), \quad n \in \mathbb{N},\ t \in (0,\infty). \tag{3.5}$$

Now using (3.2) and (3.4), we have

$$\begin{aligned}
\Big(\int_E f_n^{q'}(x)v^{1-q'}(x)\,dx\Big)^{\frac{1}{q'}} &= \Big(\int_0^\infty \int_B f_n^{q'}(t\tau)v^{1-q'}(t\tau)t^{N-1}\,d\tau\,dt\Big)^{\frac{1}{q'}} \\
&= \Big(\int_0^\infty F_n^{q'}(t)\Big(\int_B v^{q'}(t\tau)v^{1-q'}(t\tau)t^{N-1}\,d\tau\Big) \\
&\qquad \times V^{-q'}(t)\,dt\Big)^{\frac{1}{q'}} \\
&= \Big(\int_0^\infty F_n^{q'}(t)V^{1-q'}(t)\,dt\Big)^{\frac{1}{q'}} \\
&< \infty,
\end{aligned}$$

which means that $\{f_n\}$ is a sequence in $L^{q'}(E, v^{1-q'})$. Thus (3.1) and (3.5) yield

$$\|T^*F_n\|_{p',(0,\infty),U^{1-p'}} = \|T_E^* f_n\|_{p',E,u^{1-p'}}.$$

We now show that $f_n \xrightarrow{w} 0$. For any function $g \in L^q(E,v)$, using (3.4) we have

$$\begin{aligned}
\int_E f_n(x)g(x)\,dx &= \int_0^\infty \int_B F_n(t)v(t\tau)V^{-1}(t)g(t\tau)t^{N-1}\,d\tau\,dt \\
&= \int_0^\infty F_n(t)\Big(\int_B v(t\tau)g(t\tau)t^{N-1}\,d\tau\Big)V^{-1}(t)\,dt \\
&= \int_0^\infty F_n(t)G(t)\,dt \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$G(t) = \Big(\int_B v(t\tau)g(t\tau)t^{N-1}\,d\tau\Big)V^{-1}(t), \quad t \in (0,\infty),$$

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and it can be easily verified that $G \in L^q\big((0,\infty), V\big)$. Indeed, using (3.2), we have

$$\begin{aligned}\int_0^\infty G^q(t)V(t)\,dt &= \int_0^\infty \Big(\int_B v(t\tau)g(t\tau)t^{N-1}\,d\tau\Big)^q V^{1-q}(t)\,dt\\ &= \int_0^\infty \Big(\int_B g(t\tau)v^{\frac{1}{q}}(t\tau)(t^{N-1})^{\frac{1}{q}}v^{\frac{1}{q'}}(t\tau)(t^{N-1})^{\frac{1}{q'}}\,d\tau\Big)^q V^{1-q}(t)\,dt\\ &\leq \int_0^\infty \Big(\int_B g^q(t\tau)v(t\tau)t^{N-1}\,d\tau\Big)\\ &\qquad\times \Big(\int_B v(t\tau)t^{N-1}\,d\tau\Big)^{q-1} V^{1-q}(t)\,dt\\ &= \int_E g^q(x)v(x)\,dx < \infty.\end{aligned}$$

Now as T_E is compact, by Theorem B((a) and (b)), $\|T_E^* f_n\|_{p',E,u^{1-p'}}$ and hence $\|T^* F_n\|_{p',(0,\infty),U^{1-p'}}$ converges to 0 as $n \to \infty$. Now the compactness of T follows from Theorem B(c).

As remarked in Section 1, to give the precise necessary and sufficient conditions for the compactness of the operator T_E, one can use any known criterion for the compactness of the operator T, *e.g.,* Theorem A. We give below two theorems respectively, for $1 < p \leq q < \infty$ and $1 < q < p < \infty$ to characterize the compactness of T_E precisely.

Theorem 3.2 *Let all the hypothesis of Theorem 3.1 be satisfied with the additional condition that* $1 < p \leq q < \infty$. *Then the operator* $T_E \colon L^p(E,u) \to L^q(E,v)$ *is compact if and only if*

$$\sup_{\substack{0<t\leq s\\ a(s)\leq b(t)}} \mathcal{S}(s,t) < \infty, \tag{3.6}$$

$$\lim_{s\to t^+} \mathcal{S}(s,t) = \lim_{s\to a^{-1}(b(t))^-} \mathcal{S}(s,t) = 0, \quad \textit{for every } t > 0 \tag{3.7}$$

and

$$\lim_{t\to s^-} \mathcal{S}(s,t) = \lim_{t\to b^{-1}(a(s))^+} \mathcal{S}(s,t) = 0, \quad \textit{for every } s > 0 \tag{3.8}$$

where

$$\mathcal{S}(s,t) = \Big(\int_{b(t)S\setminus a(s)S} u^{1-p'}\Big)^{\frac{1}{p'}} \Big(\int_{sS\setminus tS} v\Big)^{\frac{1}{q}}.$$

Proof The proof is immediate in view of Theorems 3.1 and A(a), once we show that

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$\mathcal{S} = B$. Indeed, using (3.1) and (3.2), we have

$$\begin{aligned} B(s,t) &= \Big(\int_{a(s)}^{b(t)} U^{1-p'}(\xi)\, d\xi \Big)^{\frac{1}{p'}} \Big(\int_t^s V(\xi)\, d\xi \Big)^{\frac{1}{q}} \\ &= \Big(\int_{a(s)}^{b(t)} \int_B u^{1-p'}(\xi\tau)\xi^{N-1}\, d\tau\, d\xi \Big)^{\frac{1}{p'}} \Big(\int_t^s \int_B v(\xi\tau)\xi^{N-1}\, d\tau\, d\xi \Big)^{\frac{1}{q}} \\ &= \Big(\int_{b(t)S\setminus a(s)S} u^{1-p'}(x)\, dx \Big)^{\frac{1}{p'}} \Big(\int_{sS\setminus tS} v(x)\, dx \Big)^{\frac{1}{q}} \\ &= \mathcal{S}(s,t). \end{aligned}$$

To give the corresponding result for the case $1 < q < p < \infty$, we define a normalizing function σ as

$$\sigma(t) = \sum_{k\in\mathbb{Z}} \chi_{(M_k, M_{k+1})}(t) \frac{d}{dt} (b^{-1} oa)^k(t), \quad t \in (0,\infty).$$

Here $(b^{-1}oa)^k$ denotes k times repeated composition and the numbers M_k come from the sequence $\{M_k\}_{k\in\mathbb{Z}}$ which is defined as follows:

$$\begin{aligned} M_0 &= b^{-1}(1) \\ M_{k+1} &= a^{-1}\big(b(M_k)\big), \qquad \text{if } k \geq 0 \\ M_k &= b^{-1}\big(a(M_{k+1})\big), \quad \text{if } k < 0. \end{aligned}$$

Note that $a(M_{k+1}) = b(M_k)$, $k \in \mathbb{Z}$.

Theorem 3.3 *Let $1 < q < p < \infty$ and assume all the hypothesis of Theorem 3.1 are satisfied. Then the operator $T_E \colon L^p(E,u) \to L^q(E,v)$ is compact if and only if $B = \max(B_1, B_2) < \infty$, where*

$$B_1 = \Big(\int_0^\infty \int_{tS\setminus b^{-1}(a(t))S} \Big(\int_{b(\alpha_x)S\setminus a(t)S} u^{1-p'} \Big)^{\frac{r}{p'}} \Big(\int_{tS\setminus \alpha_x S} v \Big)^{\frac{r}{p}} v(x)\, dx \sigma(t)\, dt \Big)^{\frac{1}{r}}$$

and

$$B_2 = \Big(\int_0^\infty \int_{a^{-1}(b(t))S\setminus tS} \Big(\int_{b(t)S\setminus a(\alpha_x)S} u^{1-p'} \Big)^{\frac{r}{p'}} \Big(\int_{\alpha_x S\setminus tS} v \Big)^{\frac{r}{p}} v(x)\, dx \sigma(t)\, dt \Big)^{\frac{1}{r}},$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and σ the normalizing function defined above.

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Proof Using (1.4), (3.1), (3.2) and making change of variable $\xi\tau = z$ and $s\tau = x$, we have

$$A_1 = \Big(\int_0^\infty \int_{b^{-1}(a(t))}^t \Big(\int_{a(t)}^{b(s)} U^{1-p'}(\xi)\,d\xi \Big)^{\frac{r}{p'}} \Big(\int_s^t V(\xi)\,d\xi \Big)^{\frac{r}{p}} V(s)\,ds\sigma(t)\,dt \Big)^{\frac{1}{r}}$$
$$= \Big(\int_0^\infty \int_{b^{-1}(a(t))}^t \Big(\int_{a(t)}^{b(s)} \int_B u^{1-p'}(\xi\tau)\xi^{N-1}\,d\tau\,d\xi \Big)^{\frac{r}{p'}}$$
$$\times \Big(\int_s^t \int_B v(\xi\tau)\xi^{N-1}\,d\tau\,d\xi \Big)^{\frac{r}{p}} \int_B v(s\tau)s^{N-1}\,d\tau\,ds\sigma(t)\,dt \Big)^{\frac{1}{r}}$$
$$= \Big(\int_0^\infty \int_{tS\setminus b^{-1}(a(t))S} \Big(\int_{b(\alpha_x)S\setminus a(t)S} u^{1-p'}(z)\,dz \Big)^{\frac{r}{p'}}$$
$$\times \Big(\int_{tS\setminus \alpha_x S} v(z)\,dz \Big)^{\frac{r}{p}} v(x)\,dx\sigma(t)\,dt \Big)^{\frac{1}{r}}$$
$$= B_1.$$

Similarly, it can be shown that $A_2 = B_2$. The result now follows by Theorems A(b) and 3.1.

Now, as mentioned in Section 1, we can derive the special case of the operator T_E when the star-shaped region S is replaced by the unit ball in $\mathbb{R}^N$ with center as origin. In such a situation, the differences of the star-shaped regions become anulli, $E = \mathbb{R}^N$ and $\alpha_x = |x|$. The operator T_E takes the shape of

$$(T_N f)(x) = \int_{a(|x|)<|y|<b(|x|)} f(y)\,dy, \quad x, y \in \mathbb{R}^N.$$

With this special case, we immediately obtain the following corollaries of Theorems 3.1, 3.2 and 3.3 respectively:

Corollary 3.4 *Let $1 < p, q < \infty$ and let u, v be weight functions on $\mathbb{R}^N$. Then the operator $T_N\colon L^p(\mathbb{R}^N, u) \to L^q(\mathbb{R}^N, v)$, defined above is compact if and only if $T\colon L^p\big((0,\infty), U\big) \to L^q\big((0,\infty), V\big)$ is compact, with*

$$U(t) = \Big(\int_{\sum_N} u^{1-p'}(t\tau), t^{N-1}\,d\tau \Big)^{1-p}, \quad t \in (0,\infty),$$

and

$$V(t) = \int_{\sum_N} v(t\tau)t^{N-1}\,d\tau, \quad t \in (0,\infty),$$

where $\sum_N$ is the surface of unit ball in $\mathbb{R}^N$.

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Corollary 3.5 *Let $1 < p \le q < \infty$ and u, v be weight functions on $\mathbb{R}^N$. Then $T_N\colon L^p(\mathbb{R}^N, u) \to L^q(\mathbb{R}^N, v)$ is compact if and only if*

$$\sup_{\substack{0<t\le s\\ a(s)\le b(t)}} \mathcal{B}(s,t) < \infty,$$

$$\lim_{s\to t^+} \mathcal{B}(s,t) = \lim_{s\to a^{-1}(b(t))^-} \mathcal{B}(s,t) = 0, \quad \textit{for every } t > 0,$$

and

$$\lim_{t\to s^-} \mathcal{B}(s,t) = \lim_{t\to b^{-1}(a(s))^+} \mathcal{B}(s,t) = 0, \quad \textit{for every } s > 0,$$

where

$$\mathcal{B}(s,t) = \Big(\int_{a(s)<|z|<b(t)} u^{1-p'}(z)\,dz\Big)^{\frac{1}{p'}} \Big(\int_{t<|z|<s} v(z)\,dz\Big)^{\frac{1}{q}}.$$

Corollary 3.6 *Let $1 < q < p < \infty$ and u, v be weight functions on $\mathbb{R}^N$. Then $T_N\colon L^p(\mathbb{R}^N, u) \to L^q(\mathbb{R}^N, v)$ is compact if and only if $K = \max(K_1, K_2) < \infty$, where*

$$K_1 = \Bigg(\int_0^\infty \int_{b^{-1}(a(t))<|x|<t} \Big(\int_{a(t)<|z|<b(|x|)} u^{1-p'}(z)\,dz\Big)^{\frac{r}{p'}} \times \Big(\int_{|x|<|z|<t} v(z)\,dz\Big)^{\frac{r}{p}} v(x)\,dx\sigma(t)\,dt\Bigg)^{\frac{1}{r}}$$

and

$$K_2 = \Bigg(\int_0^\infty \int_{t<|x|<a^{-1}(b(t))} \Big(\int_{a(|x|)<|z|<b(t)} u^{1-p'}(z)\,dz\Big)^{\frac{r}{p'}} \times \Big(\int_{t<|z|<|x|} v(z)\,dz\Big)^{\frac{r}{p}} v(x)\,dx\sigma(t)\,dt\Bigg)^{\frac{1}{r}},$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and σ is the normalizing function as defined earlier.

4 Final Results and Remarks

For $x \in E$, write $S_x = \alpha_x S$ and define the operator $H_E\colon L^p(E, u) \to L^q(E, v)$ by

$$(H_E f)(x) = \int_{S_x} f(y)\,dy.$$

Clearly, the operator H_E is N-dimensional analogue of the classical Hardy operator $(Hf)(x) = \int_0^x f(t)\,dt$, where the average is taken over the dilations of a star-shaped region S. As mentioned already, the operator H can not be obtained from the operator T. Similarly, H_E can not be obtained from T_E. However, the compactness of H_E

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can be characterized in terms of the compactness of H. Since the proof is very similar to the proof of Theorem 3.1 (with some obvious modifications), we state the result only.

Theorem 4.1 *Under the hypothesis of Theorem 3.1, the operator* $H_E\colon L^p(E,u) \to L^q(E,v)$ *is compact if and only if the classical Hardy operator* $H\colon L^p\big((0,\infty),U\big) \to L^q\big((0,\infty),V\big)$ *is compact with* U *and* V *as defined respectively by* (3.1) *and* (3.2).

Remark To obtain the precise necessary and sufficient conditions for the compactness of H_E (results corresponding to Theorems 3.2 and 3.3), one can use any known criterion for the compactness of H, see *e.g.*, ([8], Theorems 7.3 and 7.5).

Remark Analogous to the discussion after Theorem 3.3, we can derive special case when, in H_E, the star-shaped region S is replaced by the unit ball in $\mathbb{R}^N$.

It is natural to study the compactness of the conjugate operators T_E^*, T_N^* and H_E^*. For the sake of conciseness, we only deal with T_E^*. The remaining can be dealt with exactly on the similar lines.

For the case $1 < p \le q < \infty$, we have the following result

Theorem 4.2 *Let the hypothesis of Theorem 3.2 are satisfied. Then the operator* $T_E^*\colon L^p(E,u) \to L^q(E,v)$ *is compact if and only if*

$$\sup_{\substack{0<t\le s\\ a(s)\le b(t)}} \mathcal{S}^*(s,t) < \infty,$$

$$\lim_{s\to t^+} \mathcal{S}^*(s,t) = \lim_{s\to a^{-1}(b(t))^-} \mathcal{S}^*(s,t) = 0, \quad \textit{for every } t > 0,$$

$$\lim_{t\to s^-} \mathcal{S}^*(s,t) = \lim_{t\to b^{-1}(a(s))^+} \mathcal{S}^*(s,t) = 0, \quad \textit{for every } s > 0,$$

where

$$\mathcal{S}^*(s,t) = \Big(\int_{b(t)S\setminus a(s)S} v\Big)^{\frac{1}{q}} \Big(\int_{sS\setminus tS} u^{1-p'}\Big)^{\frac{1}{p'}}.$$

Proof By Theorems B(a) and 3.2, $T_E^*\colon L^{q'}(E,v^{1-q'}) \to L^{p'}(E,u^{1-p'})$ is compact if and only if (3.6), (3.7) and (3.8) hold. Now the result is obtained by replacing $q', p', v^{1-q'}, u^{1-p'}$ respectively by p, q, u and v.

On the similar lines we have the result for the case $1 < q < p < \infty$ also.

Theorem 4.3 *Let the hypothesis of Theorem 3.3 are satisfied. Then the operator* $T_E^*\colon L^p(E,u) \to L^q(E,v)$ *is compact if and only if*

$$\Big(\int_0^\infty \int_{tS\setminus b^{-1}(a(t))S} \Big(\int_{b(\alpha_x)S\setminus a(t)S} v\Big)^{\frac{r}{q}} \Big(\int_{tS\setminus \alpha_x S} u^{1-p'}\Big)^{\frac{r}{q'}} u^{1-p'}(x)\, dx\sigma(t)\, dt\Big)^{\frac{1}{r}} < \infty$$

and

$$\Big(\int_0^\infty \int_{a^{-1}(b(t))S\setminus tS} \Big(\int_{b(t)S\setminus a(\alpha_x)S} v\Big)^{\frac{r}{q}} \Big(\int_{\alpha_x S\setminus tS} u^{1-p'}\Big)^{\frac{r}{q'}} u^{1-p'}(x)\, dx\sigma(t)\, dt\Big)^{\frac{1}{r}} < \infty$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ *and* σ*, the normalizing function as defined in Section 3.*

Remark The results in Theorems 4.2 and 4.3 can also be obtained by a different approach. One can first formulate a result corresponding to Theorem 3.1 relating the compactness of T_E^* and T^* and then apply Theorems 5.1, 5.2 from [6], which give necessary and sufficient conditions for the compactness of T^* for the cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$, respectively.

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References

[1] T. Ando, *On the compactness of integral operators*. Proc. Ned. Akad. Wetensch. **62**(1962), 235–239.
[2] J. Diestel, *Sequences and Series in Banach Spaces*. Springer-Verlag, Berlin, Heidelberg, New York, 1986.
[3] N. Dunford and J. T. Schwartz, *Linear Operators I. General Theory*. Interscience Publishers, Inc., New York-London, 1958.
[4] H. P. Heinig and G. Sinnamon, *Mapping properties of integral averaging operators*. Studia Math. **129**(1998), 157–177.
[5] P. K. Jain, O. P. Ahuja and K. Ahmad, *Functional Analysis*. New Age International, New Delhi, 1995.
[6] P. Jain and B. Gupta, *Compactness of Hardy-Steklov operator.* J. Math. Anal. Appl. **228**(2003), 680–691.
[7] A. Kufner and L. E. Persson, *Weighted Inequalities of Hardy Type*. World Scientific, 2003.
[8] B. Opic and A. Kufner, *Hardy-Type Inequalities*. Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow, 1990.
[9] G. Sinnamon, *One dimensional Hardy-type inequalities in many dimensions*. Proc. Roy. Soc. Edinburgh A **128**(1998), 833–848.
[10] ———, *Hardy-type inequalities for a new class of integral operators*. In: Analysis of Divergence, Control and Management of Divergent Processes (eds., W. O. Bray and C. V. Stanojevic), Birkhauser, Boston, 1999, 297–307.

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