9

Kinks in the laboratory

In this chapter we discuss two laboratory systems where kinks are known to exist. The first system is *trans* polyacetylene which has a broken Z_2 symmetry as in the $\lambda\phi^4$ model. The second system is a Josephson junction transmission line, which is a laboratory realization of the sine-Gordon system. Helium-3 is another laboratory system that contains a wide variety of topological defects and the reader is referred to [174] for a discussion. In the third section of this chapter we describe Scott Russell's solitons in water. These solitons are not topological like the others discussed in this book but we include the discussion anyway since the reader's curiosity may have been aroused by the story in the Preface.

9.1 Polyacetylene

Polyacetylene consists of a linear chain of CH bonds. A sequence of x units is written as $(CH)_x$. In the ground state of polyacetylene, the carbon atom forms three σ bonds, one of them is to the H in the CH unit, one to the unit on the left and one to the right. In addition, there is one more electron orbital that can cause bonding. This is called the π electron, and the π bond can form to the left or to the right. Then there are two possible sequences – first when the double (σ and π) bond is to the carbon on the right and the single to the left, the second when the double bond is to the left and the single to the right. These two possibilities are illustrated in Fig. 9.1 [149] in the *trans* configuration of polyacetylene.[1]

The average bond length $a \approx 1.22$ Å but the CH units are displaced so as to make double bonds (slightly) shorter than the single bonds. The physical displacements u_n along the horizontal axis in the two structures are depicted in Fig. 9.1. Qualitatively, the essential point is that the π electrons have to choose to either form the double

[1] In the *cis* configuration, there are also two states related by the left-right transformation but they are not degenerate in energy.

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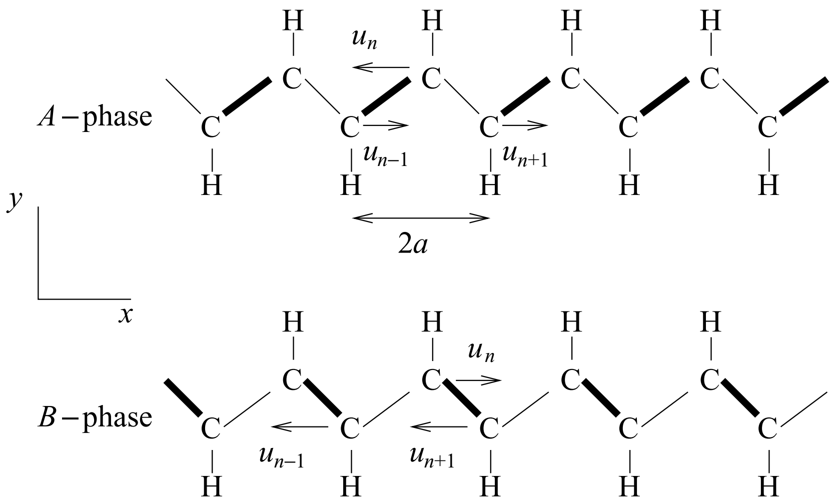

Figure 9.1 Structure of the two degenerate ground states of *trans* polyacetylene. The upper structure is denoted by A and the lower by B. Double bonds are denoted by heavy lines.

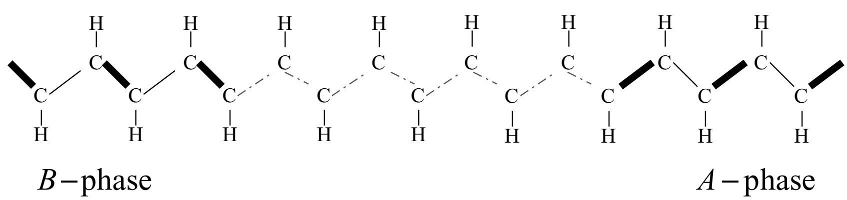

Figure 9.2 If the B state occurs on the left side of a chain and the A on the right, there is a kink in between where the simple alternating structure cannot be maintained. The kink is the region where the alternate single-double bonds do not exist.

bond to the left or to the right. Hence there is a Z_2 symmetry which is broken in the ("dimerized") ground state. Kinks form if different ground states are chosen at different locations (Fig. 9.2). The center of the kink is located at the CH unit where the π electron wavefunction is equally shared between the CH units to the left and right.

The Hamiltonian of the system depends on the displacement variables, u_n and on the locations of the π electrons

$$H = -\sum_{n,s}(t_{n+1,n}c^\dagger_{n+1,s}c_{n,s} + h.c.) + \sum_n \frac{K}{2}(u_{n+1} - u_n)^2 + \sum_n \frac{M}{2}\dot{u}_n^2 \qquad (9.1)$$

where

$$t_{n+1,n} = t_0 - \alpha(u_{n+1} - u_n) \qquad (9.2)$$

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is the hopping integral to leading order in displacements. The operators $c^\dagger_{n,s}$ and $c_{n,s}$ are creation and annihilation operators for electrons of spin s on the nth CH group. The parameter K is the effective spring constant of the σ bonds and M is the mass of the CH group.

To connect with the discussion of Chapter 1, the displacement variable

$$\phi_n = (-1)^n u_n \tag{9.3}$$

can be viewed as a scalar field defined on a lattice interacting with a fermion (the π electron). The last two terms in Eq. (9.1) correspond to gradient and time derivative terms of a continuum field $\phi(x)$ that corresponds to the discrete variables, ϕ_n. The first term describes interactions between ϕ and the electrons. The effective interaction for the ϕ field, after integrating out the fermionic variables, must respect the Z_2 symmetry, and hence corresponds to a ϕ^4 interaction to lowest order. Therefore a non-relativistic version of the Z_2 model of Eq. (1.2) captures some of the gross features of polyacetylene.

The properties of kinks in polyacetylene (Fig. 9.2) have been studied in [150] using the Hamiltonian in Eq. (9.1) with the result that the kink width is approximately 14 lattice spacings and the mass is approximately six electron masses [152] in good agreement with experiments [75].

The quantum properties of polyacetylene kinks have also been studied. In Section 5.3 we discussed how kinks can carry fractional quantum numbers [83]. Polyacetylene kinks also carry fractional quantum numbers and electric charge, namely "half a bond" or $\pm(2e)/2$ charge since each bond consists of two electrons (one from each atom at either end of a bond) [150]. Indeed, in a chain where two single bonds are followed by a double bond (instead of the alternating single and double bonds in *trans* polyacetylene) the fractional charge can be shown to be one-third of a bond [151]. Reference [68] generalizes these ideas much further and shows that solitons may even carry irrational charges.

9.2 Josephson junction transmission line

We follow [134] in deriving the sine-Gordon equation for the Josephson transmission line.

Let us recall the basics of a transmission line, schematically shown in Fig. 9.3 [54]. A potential difference is applied to the ends of two elements of a transmission line e.g. the two cables of a coaxial cable. The potential, V, and current, I, in each of the wires are functions of the location on the transmission line, namely the x coordinate, and also of time. There is also a potential difference between the wires, and the current in the two wires can be different, but this is not shown in

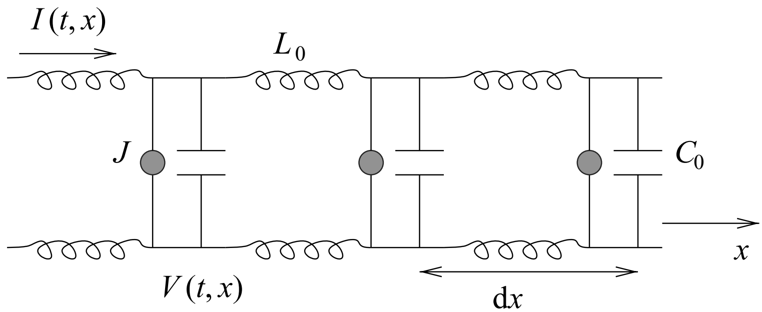

Figure 9.3 Schematics of a transmission line with inductance L_0 and capacitance C_0 per unit length. The symbols marked J represent a second coupling between the two transmission components. This coupling is absent in an ordinary transmission line but represents the tunneling current in the Josephson transmission line.

the figure. We will only be considering the potential and current distribution along a single wire. Let L_0 denote the inductance per unit length of the line, and C_0 the capacitance per unit length. Then Faraday's law of induction tells us that the induced e.m.f. between points $x + \mathrm{d}x$ and x is proportional to the rate of change of the current in the segment within those points

$$V(t, x + \mathrm{d}x) - V(t, x) = -(L_0 \mathrm{d}x)\frac{\partial I}{\partial t} \tag{9.4}$$

or

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} \tag{9.5}$$

Charge accumulates on the segment from x to $x + \mathrm{d}x$ in time $\mathrm{d}t$ owing to the different entering and exiting currents. The charge on the segment is also given by the capacitance times the potential. Hence

$$I(t, x + \mathrm{d}x) - I(t, x) = -(C_0 \mathrm{d}x)\frac{\partial V}{\partial t} \tag{9.6}$$

or

$$\frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t} \tag{9.7}$$

Equations (9.5) and (9.7) can be combined to obtain wave equations for the current and the potential.

A Josephson junction transmission line differs from the ordinary transmission line described above in that the two "wires" are superconductors and they are separated by a thin insulator. This set-up is shown in Fig. 9.4. Current can tunnel through the insulator and jump from one wire to the other. Hence the charge on a

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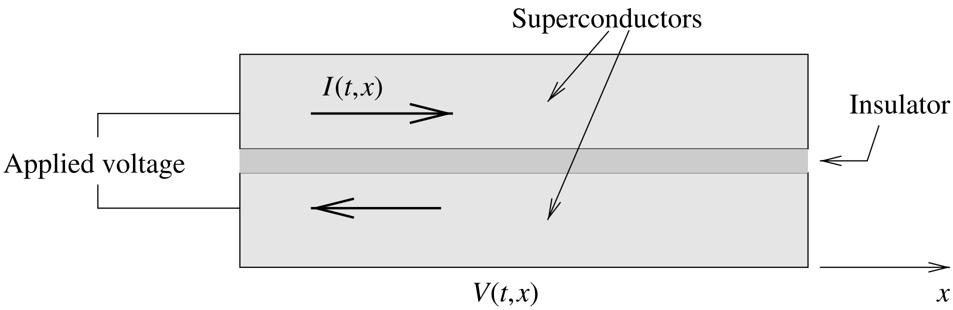

Figure 9.4 A Josephson junction transmission line is constructed by separating two superconducting plates by a thin layer of an insulating material.

segment also changes owing to the Josephson current and Eq. (9.7) gets modified

$$\frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t} - j_{\mathrm{J}}(x, t) \tag{9.8}$$

where j_{J} is the Josephson current per unit length.

The charge carriers (Cooper pairs) in either superconductor are described by macroscopic wavefunctions

$$\psi_1 = \sqrt{\rho_1} \mathrm{e}^{\mathrm{i}\phi_1}, \qquad \psi_2 = \sqrt{\rho_2} \mathrm{e}^{\mathrm{i}\phi_2} \tag{9.9}$$

where ρ_1 and ρ_2 are the charge carrier number densities in the superconductors. The tunneling Josephson current per unit area is [55]

$$j_{\mathrm{J}} = j_0 \sin \phi \tag{9.10}$$

where j_0 is the maximum Josephson current and is proportional to $\sqrt{\rho_1 \rho_2}$, and

$$\phi = \phi_1 - \phi_2 \tag{9.11}$$

The Schrödinger equations for ψ_1 and ψ_2 imply

$$\frac{\partial \phi}{\partial t} = \frac{q}{\hbar} V \tag{9.12}$$

where V is the potential difference across the junction and $q = 2e$ is the electric charge of a Cooper pair.

From Eqs. (9.5) and (9.12) we obtain

$$I = \frac{-\hbar}{q L_0} \frac{\partial \phi}{\partial x} \tag{9.13}$$

Now we can insert this expression for I, and V as found from Eq. (9.12), in Eq. (9.8) to get

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{L_0 C_0} \frac{\partial^2 \phi}{\partial x^2} + \frac{j_0 q}{C \hbar} \sin \phi = 0 \tag{9.14}$$

Rescaling t and x by the Josephson time and length scales

$$\tau = \left(\frac{\hbar C_0}{qJ_0}\right)^{1/2}, \qquad l = \left(\frac{\hbar}{qJ_0L_0}\right)^{1/2} \tag{9.15}$$

gives the sine-Gordon equation as derived from Eq. (1.51) with $\alpha = 1 = \beta$.

9.3 Solitons in shallow water

The solitons discussed in this book have all had a topological origin. In contrast, the solitons first discovered by Scott Russell in a water channel, and mentioned in the Preface, have their origin in the non-linearities of hydrodynamics and do not have a topological origin.

The first step to show the existence of the water solitons is to derive the Korteweg-deVries (KdV) equation for waves of long wavelength moving in one direction in shallow water. We do not give this derivation here and instead refer the reader to, for example, Section 13.11 of [180].

The KdV equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta^2\frac{\partial^3 u}{\partial x^3} = 0 \tag{9.16}$$

where u is related to the height of the fluid surface and δ is a parameter. The soliton solution is [185]

$$u = u_\infty + (u_0 - u_\infty)\mathrm{sech}^2\left[\frac{X - X_0}{\Delta}\right] \tag{9.17}$$

where $X = x - vt$, X_0 is a constant, v is the velocity of the soliton,

$$\Delta = \delta\left[\frac{u_0 - u_\infty}{12}\right]^{-1/2} \tag{9.18}$$

The velocity of the soliton is given by

$$v = u_\infty + \frac{u_0 - u_\infty}{3} \tag{9.19}$$

in terms of the arbitrary constants u_0 and u_∞. Note that the amplitude of the soliton and the velocity are related.

9.4 Concluding remarks

There are a number of situations where solitons have been discussed in the particle physics literature. Most of these discussions, such as of domain walls in the $SU(5) \times Z_2$ model in Chapter 2, have been in the framework of Grand Unified

Theories. The attention has mostly focused on magnetic monopoles and strings because monopoles seem inevitable in this class of theories and strings are less constrained by cosmology. Similar topological structures also exist in the standard model of electroweak interactions but the monopoles are confined and the strings are unstable [2]. Domain wall and string solutions also exist in QCD in various external conditions, for example in high density matter such as might be present in the interiors of neutron stars [142]. Unlike solitons in the laboratory, however, solitons in particle physics and cosmology have not yet been discovered experimentally. Given the very similar underpinnings of laboratory and particle physics systems, there is hope that this situation will soon change.

9.5 Open questions

1. Is there a condensed matter system with spontaneously broken permutation symmetry? Discuss the domain walls in that system and whether a lattice can exist. Can the walls be observed experimentally?
2. If there are QCD domain walls in neutron stars, how might they be observed from Earth?

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