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DERIVATIONS ON A LIE IDEAL

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ABSTRACT. In this paper we prove the following result: let R be a prime ring with no non-zero nil left ideals whose characteristic is different from 2 and let U be a non central Lie ideal of R.

If $d \neq 0$ is a derivation of R such that $d(u)$ is invertible or nilpotent for all $u \in U$, then either R is a division ring or R is the 2×2 matrices over a division ring. Moreover in the last case if the division ring is non commutative, then d is an inner derivation of R.

In the last years, many results due to Herstein, Lanski, Bergen and others (see [1], [2], [3]) showed that some information on the structure of a ring can be obtained by examining the behavior of one of its derivations.

Recently in [3] Bergen studied rings with no non-zero nil left ideals, endowed with a derivation $d \neq 0$ with invertible or nilpotent values and proved that such a ring is either a division ring or the ring of 2×2 matrices over a division ring.

In this paper we generalize this result to the case of a Lie ideal, more precisely we shall prove the following

THEOREM. *Let R be a prime ring with no non-zero nil left ideals whose characteristic is different from* 2 *and let U be a non central Lie ideal of R. If $d \neq 0$ is a derivation of R such that $d(u)$ is invertible or nilpotent for all $u \in U$, then either R is a division ring or R is the ring of* 2×2 *matrices over a division ring D. Moreover in case D is not commutative, d is an inner derivation of R.*

We shall make use of the results in [4] and [5] where the authors study derivations with invertible and nilpotent values respectively on a Lie ideal.

Through this paper R will be a prime ring with 1 with no non-zero nil left ideals whose characteristic is different from 2, $Z = Z(R)$ will be the center of R, U a non central Lie ideal of R. We will assume that R is endowed with a derivation d satisfying the following condition: for all $u \in U$ either $d(u)$ is nilpotent or $d(u)$ is invertible.

Given two elements $a, b \in R$, the symbol $[a, b]$ will mean the element

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$ab - ba$; also, given two subsets U, V of R, then $[U, V]$ will be the additive subgroup of R generated by all $[a, b]$ for $a \in U$ and $b \in V$.

We start with the following:

LEMMA 1. *$U \supset [R, R]$ and R is a simple ring.*

PROOF. Let $J \neq 0$ be an ideal of R; by [1, Lemma 1] there exists an ideal $K \neq 0$ of R such that $[K, R] \subset U$ and $[K, R] \not\subset Z$. Let $I = K \cap J$; we note that $U \cap I^2$ is a Lie ideal containing $[I, I]$, hence $U \cap I^2 \not\subset Z$, otherwise $[I, I] \subset Z$ and this easily leads to $I \subset Z$, so the ring would be commutative contrary to the hypothesis that U is non central.

For every $x \in U \cap I^2$, $d(x) \in I$. Therefore, if $d(x)$ is invertible, for some $x \in U \cap I^2$, then $I = R$ and so $J = R$ and $U \supset [R, R]$, the desired conclusions. However, if $d(x)$ is nilpotent, for all $x \in U \cap I^2$, then by [5], $d(U \cap I^2) = 0$ resulting in the contradiction $d = 0$.

We remark that, since R is a simple ring with unity, then R is a primitive ring.

Our next goal is to prove that R is artinian.

By ([6], Lemma 1.2.2) it is enough to show that R contains a minimal right ideal or equivalently [7, page 75] R contains a non-zero transformation of finite rank. Since R is a primitive ring, R is a dense ring of linear transformations on a vector space V over a division ring D.

We begin with:

LEMMA 2. *Suppose R is not artinian. If $v \in V$ and $r \in R$ are such that $vr = 0$ then $vd(r) = 0$.*

PROOF. We will break the proof into three steps. First we will show that if $v, w \in V$ are linearly independent vectors and $vr = wr = 0$ for some $r \in R$, then $vd(r)$ and $wd(r)$ are linearly dependent over D.

Suppose this is not the case. Since R does not contain transformations of finite rank, $\dim Vr = \infty$, and we can choose $0 \neq v'' = v'r \in Vr$ such that v'', $vd(r)$, $wd(r)$ are linearly independent over D. By the density theorem, there exist $s, t \in R$ such that:

$$vd(r)s = v'$$

$$wd(r)s = 0$$

and

$$vd(r)t = 0$$

$$wd(r)t = 0$$

$$v''t = v.$$

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Then $vd(rsrt - rtrs) = vd(r)srt - vd(r)trs = v'rt = v''t = v$ and $wd(rsrt - rtrs) = wd(r)srt - wd(r)trs = 0$. This shows that the element $d(rsrt - rtrs)$ is neither nilpotent nor invertible. Since $d(rsrt - rtrs) \in d([R, R])$ and, by Lemma 1, $d([R, R]) \subset d(U)$, this is a contradiction.

Consequently $vd(r)$ and $wd(r)$ are linearly dependent over D.

Next we show that, if $v, w \in V$ are linearly independent over D and $vr = wr = 0$ for some $r \in R$, then $vd(r) = wd(r) = 0$. Suppose, by contradiction, that $vd(r) \neq 0$. Since dim $Vr = \infty$, choose $0 \neq v'' = v'r \in Vr$ such that v'', $vd(r)$ are linearly independent over D. Then there exist $s, t, \in R$ such that

$$vd(r)s = v'$$

and

$$vd(r)t = v$$

$$v''t = v.$$

Then

$$vd(rsrt - rtrs) = vd(r)srt - vd(r)trs = v'rt - vrs = v''t = v.$$

This implies that $d(rsrt - rtrs)$ cannot be nilpotent, thus it has to be invertible. Set $a = rsrt - rtrs$. Since $vr = wr = 0$, then $va = wa = 0$. On the other side, v and w are linearly independent over D, so by the first step, we have that $vd(a)$ and $wd(a)$ are linearly dependent over D.

Let $\alpha, \beta \in D$ be such that $(\alpha v + \beta w)d(a) = 0$, as a consequence we get that $d(a)$ cannot be invertible and this is a contradiction; hence $vd(r) = 0$.

We are ready for the final step. Let $v \in V$ and $r \in R$ be such that $vr = 0$; suppose, by contradiction, that $vd(r) \neq 0$. Since dim $Vr = \infty$, there exists $w \in V$ such that wr and $vd(r)$ are linearly independent.

Let $s \in R$ be such that

$$wrs = 0$$

$$vd(r)s = v.$$

Since $vrs = wrs = 0$, it follows from the previous step that $vd(rs) = 0$. On the other side $vd(rs) = vd(r)s + vrd(s) = v$. Consequently $vd(r) = 0$ as claimed.

We proceed with the following:

LEMMA 3. *Suppose R is not artinian. If $v \in V$ and $r \in R$, then $vd(r) = \lambda vr$ where $\lambda \in D$ is independent on the choice of v.*

PROOF. Suppose by contradiction, that vr and $vd(r)$ are linearly independent over D. By the density of the action of R on V, there exists $s \in R$ such

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that $vrs = 0$ and $vd(r)s = v$. Since $vrs = 0$, it follows from Lemma 2 that $vd(rs) = 0$.

Hence we have $0 = vd(rs) = vrd(s)$ and $vd(r)s = v$, a contradiction. Thus vr and $vd(r)$ are linearly dependent over D.

We will show now that $vd(r) = \lambda vr$ where $\lambda \in D$ is independent on the choice of v.

Since $\dim Vr = \infty$, we can choose $w \in V$ in such a way that vr and wr are linearly independent over D. Then $vd(r) = \lambda_v vr$ and $wd(r) = \lambda_w wr$. Thus we get $(v + w)d(r) = (\lambda_v v + \lambda_w w)r$, on the other side $(v + w)d(r) = \lambda_{v+w}(v + w)r$. Since vr and wr are linearly independent, it follows that $\lambda_v = \lambda_w = \lambda_{v+w} = \lambda$.

We are finally able to prove the following:

THEOREM 1. *The ring R is artinian.*

PROOF. Suppose that the conclusion of the theorem is false.

We will first show that if $a \in [R, R]$ and $d(a)$ is nilpotent then either $d(a) = 0$ or a is nilpotent. Suppose that $a \in [R, R]$ and $d(a)$ is nilpotent. By Lemma 3, there exists $\lambda_a \in D$ such that $vd(a) = \lambda_a va$. If $\lambda_a = 0$, then $Vd(a) = 0$ which implies $d(a) = 0$. If $\lambda_a \neq 0$, then $0 = vd(a)^n = \lambda_a^n va^n$, hence $\lambda_a^n Va^n = 0$. Then $Va^n = 0$ and so $a^n = 0$. We have proved the claim.

Our next goal is to prove that for every $a \in [R, R]$, either $d(a) = 0$ or $d(a)$ is invertible.

Let $a \in [R, R]$ be such that $d(a) \neq 0$ and suppose that $d(a)$ is nilpotent. By the first part of the proof we know that a is nilpotent. This implies the existence of three linearly independent vectors $v, w, u \in V$ such that

$$va = w$$

$$wa = 0$$

$$ua = z \neq 0 \quad z \in V.$$

Let now $v' \in V$ be such that v, w, u, v' are linearly independent over D. Then there exist $s, t, \in R$ such that

$$vs = -v$$

$$ws = -v$$

$$us = w$$

$$v's = z$$

and

$$vt = w$$

$$wt = 0$$

$$ut = v'$$

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We have

$$v(st - ts) = -vt - ws = -w + v = v - w$$

$$w(st - ts) = -vt = -w$$

$$u(st - ts) = wt - v's = -z$$

By setting $b = st - ts$, then $a + b \in [R, R]$ and

$$v(a + b) = w + v - w = v$$

$$w(a + b) = -w$$

$$u(a + b) = z - z = 0.$$

Since $a + b \in [R, R]$ then $d(a + b)$ is either invertible or nilpotent. From $u(a + b) = 0$ it follows that $ud(a + b) = 0$, hence $d(a + b)$ is not invertible. Thus $d(a + b)$ must be nilpotent. This implies that either $d(a + b) = 0$ or $a + b$ is nilpotent. Since $w(a + b) = -w$, then $a + b$ cannot be nilpotent.

It remains to examine the case $d(a + b) = 0$. In this case, $d(a) = -d(b)$ which implies that $d(b)$ is nilpotent. Again by the first part of the proof, either $d(b) = 0$ or b is nilpotent. The last possibility cannot occur since $wb = -w$. Thus $d(b) = 0$ and so $d(a) = 0$, a contradiction.

Hence we have proved that for every $a \in [R, R]$, either $d(a) = 0$ or $d(a)$ is invertible. By [4], R must be artinian contradicting our assumption.

We can better describe the ring R with the following

LEMMA 4. *$R \simeq D_n$, $n \leqq 2$, where D is a division ring.*

PROOF. By Wedderburn theorem $R \simeq D_n$ where D is a division ring. Suppose, by contradiction, that $n > 2$ and let e_{ij} be the usual matrix units. Since $e_{ij} = e_{ij}e_{jj} - e_{jj}e_{ij}$ is a commutator for $i \neq j$, it follows that $d(e_{ij})$ is either nilpotent or invertible. Since

$$d(e_{ij}) = d(e_{ii}e_{ij}) = e_{ii}d(e_{ij}) + d(e_{ii})e_{ij}, \text{ rank } d(e_{ij}) \leqq 2,$$

so $d(e_{ij})$ cannot be invertible. Therefore for every $i \neq j$, $d(e_{ij})$ is nilpotent. Let $A = (a_{ij}) \in D_n$, then, for $i \neq j$,

$$e_{ij}(Ae_{ij}) - (Ae_{ij})e_{ij} = e_{ij}Ae_{ij}$$

is a commutator; since

$$d(e_{ij}Ae_{ij}) = d(a_{ji}e_{ij}) = d(a_{ji}e_{ij}e_{jj}) = d(a_{ji}e_{ij})e_{jj} + a_{ji}e_{ij}d(e_{jj})$$

it follows that rank $d(e_{ij}Ae_{ij}) \leqq 2$ so $d(e_{ij}Ae_{ij})$ cannot be invertible, hence $d(e_{ij}Ae_{ij})$ is nilpotent for every $A \in D_n$. This implies that $0 = e_{ij}(d(e_{ij}Ae_{ij}))^n =$

$e_{ij}(d\ (e_{ij})Ae_{ij})^n$ and so $e_{ij}d(e_{ij})R$ is a nil right ideal of R. Consequently $e_{ij}d(e_{ij}) = 0$ for every $i \neq j$; similarly $d(e_{ij})e_{ij} = 0$.

If $i \neq j$, this gives us $0 = e_{ki}e_{ij}d(e_{ij}) = e_{kj}d(e_{ij})$.

Let $1 \neq i$, then

$$e_{k1}d(e_{ij}) = e_{k1}d(e_{i1}e_{1j}) = e_{k1}d(e_{i1})e_{1j} + e_{k1}e_{i1}d(e_{1j}).$$

Since both terms on the right hand side are equal to zero, it follows that $e_{k1}d(e_{ij}) = 0$ if $1 \neq i$ and similarly $d(e_{ij})e_{k1} = 0$ if $j \neq k$.

Now, let $d(e_{ij}) = (a_{uv})$. Then for $1 \neq i$,

$$e_{k1}(a_{uv}) = 0 = e_{k1} \sum a_{uv}e_{uv} = \sum a_{1v}e_{kv}$$

and so $a_{1v} = 0$ for $v = 1, \ldots, n$ and $1 \neq i$; on the other hand, for $k \neq j$,

$$(a_{uv})e_{k1} = 0 = (\sum a_{uv}e_{uv})e_{k1} = \sum a_{uk}e_{u1};$$

hence $a_{uk} = 0$ for $k = 1, \ldots, n$ and $k \neq j$.

Therefore $d(e_{ij}) = a_{ij}e_{ij} = \alpha e_{ij}$, for every $i \neq j$, where $\alpha = \alpha(i, j) \in D$. Moreover

$$d(e_{ii}) = d(e_{ij}e_{ji}) = d(e_{ij})e_{ji} + e_{ij}d(e_{ji}) = \alpha e_{ij}e_{ji} + \beta e_{ij}e_{ji} = (\alpha + \beta)e_{ii}.$$

Thus we have shown that for every $i \neq j$, $d(e_{ij}) = \alpha e_{ij}$, where $\alpha = \alpha(i, j) \in D$.

Now, e_{ij} and e_{ji} are both commutators, hence $d(e_{ij} + e_{ji}) = \alpha e_{ij} + \beta e_{ji}$ and $d(e_{ij} - e_{ji}) = \alpha e_{ij} - \beta e_{ji}$ should be nilpotent. It follows that raising both to the $2m$ power, where m is a suitable integer, we get $(\alpha\beta)^m e_{ii} + (\beta\alpha)^m e_{jj} = 0$, hence either $\alpha = 0$ or $\beta = 0$.

Since

$$d([e_{ij}, e_{ji}]) = (\alpha + \beta)[e_{ij}, e_{ji}] = (\alpha + \beta)(e_{ii} - e_{jj})$$

is nilpotent, then

$$d([e_{ij}, e_{ji}])^{2n} = (\alpha + \beta)^{2n}(e_{ii} + e_{jj}) = 0$$

and so $\alpha + \beta = 0$. Combining this with the previous fact that either $\alpha = 0$ or $\beta = 0$, we obtain $\alpha = \beta = 0$. Hence, for every $i \neq j$, $d(e_{ij}) = 0$.

Let now $A = (a_{ij}) = \sum a_{ij}e_{ij}$; then

$$d(A) = d(\sum a_{ij}e_{ij}) = \sum d(a_{ij})e_{ij} = \sum \bar{d}(a_{ij})e_{ij}$$

where $\bar{d}$ is a suitable derivation defined on D.

Take $\alpha \in D$ such that $d(\alpha) \neq 0$. Since

$$[(\alpha e_{11}), (e_{12} + e_{21})] = \alpha e_{12} - \alpha e_{21} = \alpha(e_{12} - e_{21}),$$

we have that

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$$d([(\alpha e_{11}), (e_{12} + e_{21})]) = d(\alpha)(e_{12} - e_{21})$$

and so either

$$d([(\alpha e_{11}), (e_{12} + e_{21})])^m = d(\alpha)^m(e_{12} \pm e_{21}) \neq 0 \text{ or}$$

$$d([(\alpha e_{11}), (e_{12} + e_{21})])^m = d(\alpha)^m(e_{11} \pm e_{22}) \neq 0$$

according as m is odd or even; moreover rank $d([(\alpha e_{11}), (e_{12} + e_{21})]) = 2$, but $n > 2$ so it cannot be invertible.

Since we have shown that the ring R under our hypotheses is either a division ring D or D_2, the ring of 2×2 matrices over a division ring, we wish to examine which derivations d in D_2 satisfy the condition: $d(u)$ invertible or nilpotent for all $u \in U$, U a non central Lie ideal.

As by the proof of Lemma 8 of [2] and Lemma 10 of [4], we can conclude that if D is non commutative and char $D \neq 2$ then the derivation d must be inner.

By combining Lemma 4 and the above remark, we have the final result:

THEOREM 2. *Let R be a prime ring with no non-zero nil left ideals and* char $R \neq 2$. *Let U be a non central Lie ideal of R; if $d \neq 0$ is a derivation of R such that $d(u)$ is either invertible or nilpotent for all $u \in U$, then either*
(1) *$R \simeq D$, D a division ring or*
(2) *$R \simeq D_2$, the 2×2 matrices over a division ring D.*
Moreover, if D is non commutative, d is an inner derivation.

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